Advanced Digital Signal Processing – Wavelets and Multirate
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Module No. # 01
Lecture No. # 02
The Haar Wavelet

Today, we shall begin with the second lecture on the subject of wavelets and multi-rate digital signal processing, in which our objective would be to introduce the Haar multi-resolution analysis, about which we had very briefly talked in the previous lecture.

Before I go on to the analytical and mathematical details of the Haar Multi-Resolution Analysis or MRA, as it is called in short, let me once again review the idea behind the Haar form of analysis or functions. We call that Haar, was a mathematician or mathematician scientist if you would like to call him that, and the very radical idea that he gave was that one could think of continuous functions in terms of discontinuous ones, and do so, to the limit of reaching any degree of continuity that you desire.

What I mean is, start from a very discontinuous function, and then, make it smoother and smoother, all the while adding discontinuous functions until you go arbitrarily close to the continuous function that you are trying to approximate. This is the central idea in the Haar way of representing functions.

We also briefly discussed why this was something important. It seems like something silly to do, at first glance but actually is very important, and the reason why it is important has been mentioned was, if you think about digitally communicating. Say for example, an audio piece you are doing exactly that. The beautiful smooth audio pattern is being converted, into a highly discontinuous stream of bits. What I mean by discontinuous is, when you transmit that stream of bits, on a communication channel, you are in fact, introducing discontinuity every time a bit changes. So after, every bit interval, there is a change of waveform and therefore, discontinuity at some level, even if not in the function, in its derivative or in a second derivative whatever be.

Whatever it is, the idea of representing continuous functions in terms of discontinuous ones has its place, in practical communication and therefore, what Haar did was something very useful to us today.

What we are going to do today is to build up the idea of wavelets. In fact, more specifically what are called, Dyadic wavelets, starting from the Haar wavelet, and to do that, let us first consider how we represent a picture on a screen. And I am going to show that schematically in the drawing here.

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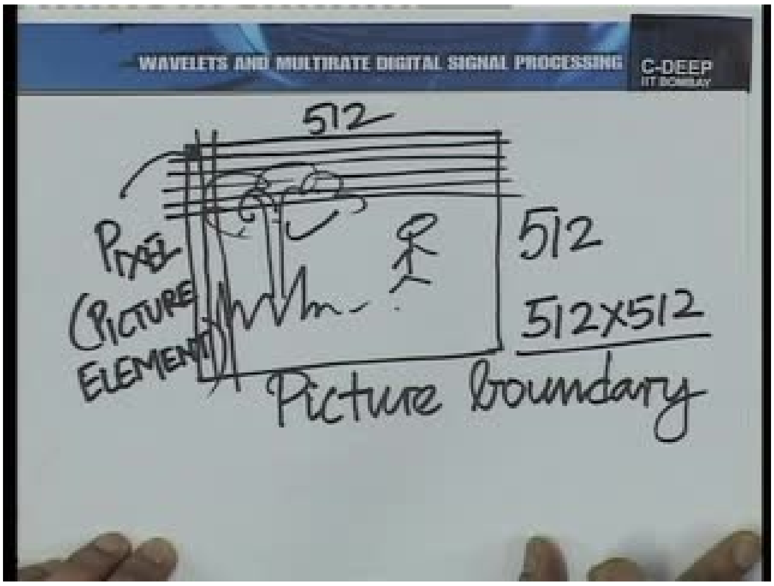

So you see, let us assume, that this is the picture boundary. And, I am trying to represent this picture on a screen whatever that picture might be. So, just for the sake of drawing, let me draw some kind of a pattern that let us say, you have a tree and some person standing there. I mean forgive my drawing, but put away some grass, may be here.

Now this is, inherently a continuous picture. How do I represent it on the computer I divide, this entire area into very small sub areas so, I visualize this in divided into tiny what are called 'picture elements' or 'pixels'.

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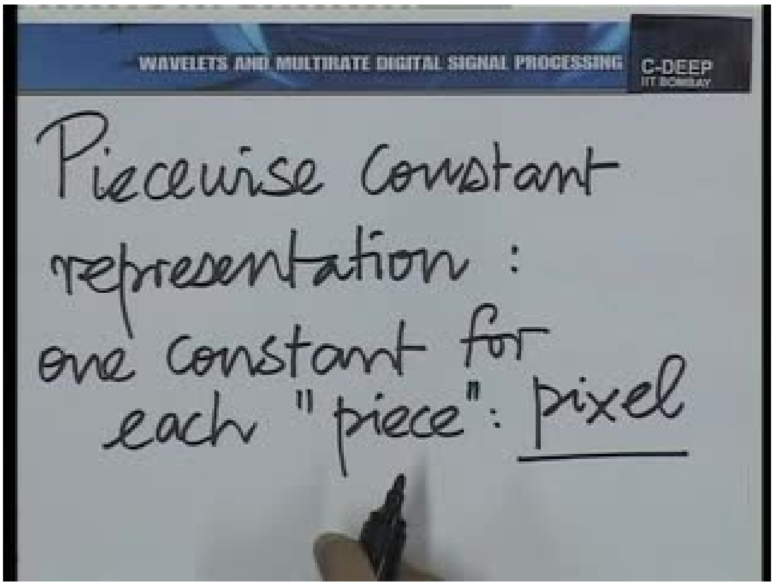

So each small area here is a pixel. A picture element, so to speak, and there are for example, suppose I make 512 divisions on the vertical and 512 divisions on the horizontal. I say that I have a 512 x 512 image, that many pixels and in each pixel region, I represent the image by a Constant. So, the first thing to understand is, there is a Piece wise Constant representation. Let us write that down, there is a 'Piecewise Constant' representation of the image, one constant for each 'piece' and that 'piece' is the pixel or the picture element.

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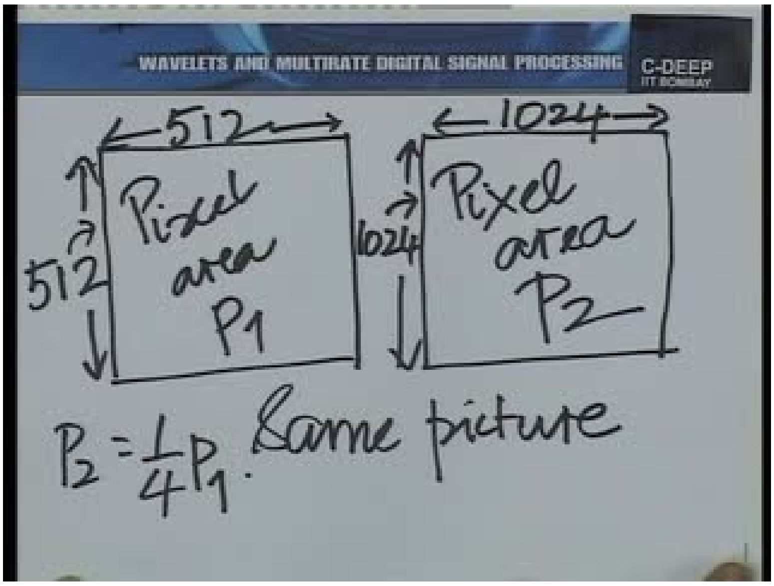

Now, suppose I increase the resolution, so I go from a resolution of 512. So I take the same, what I mean is, I take the same picture. In this case, I make a division 512 x 512. In this case, I make a division 1024 x 1024.

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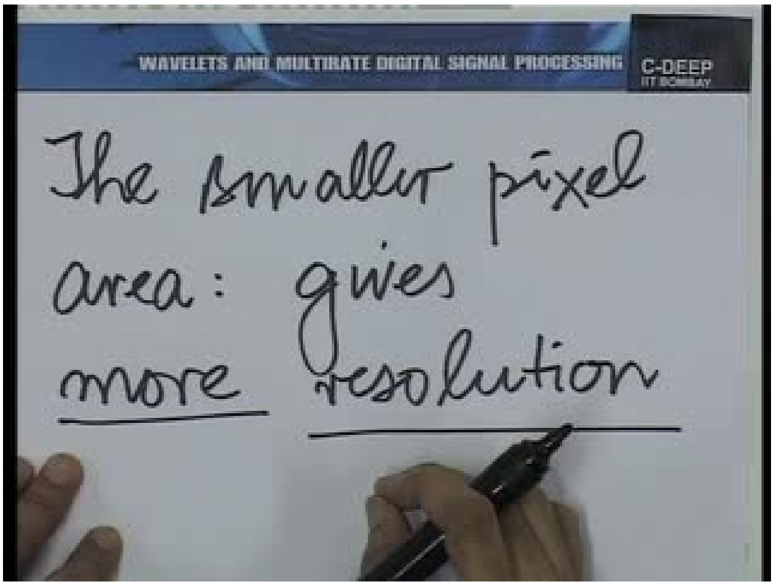

Now obviously, the pixel area here, let us say, the pixel area here is P_2 and the pixel area here is P_1. It is very easy to see that, $P_2 = \frac{1}{4}(P_1)$ and therefore, I have reduced the area by a factor of four. Naturally, if I use a Constant to represent the value or the intensity of the picture on each pixel here and do the same here, what you see in this picture is going to be closer to the original picture in some sense and what you see here.

So in other words, we can capture this by saying, the smaller the pixel area, the larger the resolution. Now this is the beginning, of the Haar multi-resolution analysis. The more we reduce the pixel area, the closer we are going to go to the original image. Even though this captures the idea that we are trying to build, it is not quite the idea of the Haar MRA. The Haar MRA does something deeper and that is what I am now going to explain mathematically in some depth.

Now here, I gave the example of a two dimensional situation, which apparently is more difficult than one-dimensional, but it is easier for us to understand physically. We can more easily relate to the idea of a Piecewise Constant representation in the context of images or pictures, but the same thing could be true of audio. For example, So you could visualize a situation, though seemingly more and natural where you record an audio piece, by dividing the time over which the audio is recorded into small segments. Now let me show that pictorially, it could be easier to understand.

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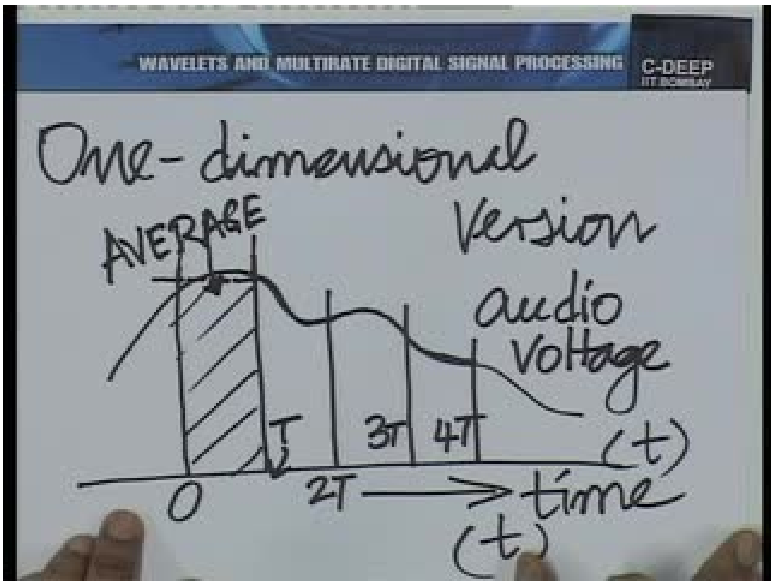

So suppose for example, you had this waveform here, so the one-dimensional version. So suppose I have, this is the time axis and I have this waveform here, assume that this is the audio waveform audio voltage recording.

Let us without any loss of generality; assume that this is the zero point in time. So let time be represented by‘t’ and let this be the zero point in time.

Now let me assume that I divide this, time axis into small intervals of size T. Here, this point is T, this point is 2T and so on, and I make a, Piecewise Constant approximation that means, I represent the audio voltage in each of these regions of size T by one number.

Now, what is the most obvious number or what are the set of most obvious numbers that one can use, to represent this waveform in each of these time intervals? for example in this time interval, or for that matter any of the time intervals, it makes sense to take the area under the curve, and divide by the time interval, to get the average of the waveform in the time interval and use that as the number to represent the function.

So here for example, you can visualize, that the average will lie somewhere here. I am just showing it in dotted **(…)** so average so intuitively it makes sense to represent the voltage waveform in each of these intervals of size t by the average of that waveform in that interval is that right.

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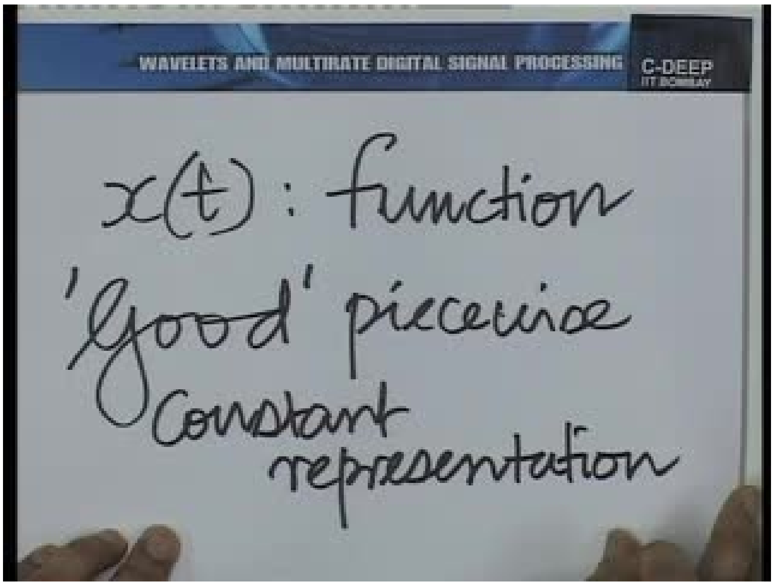

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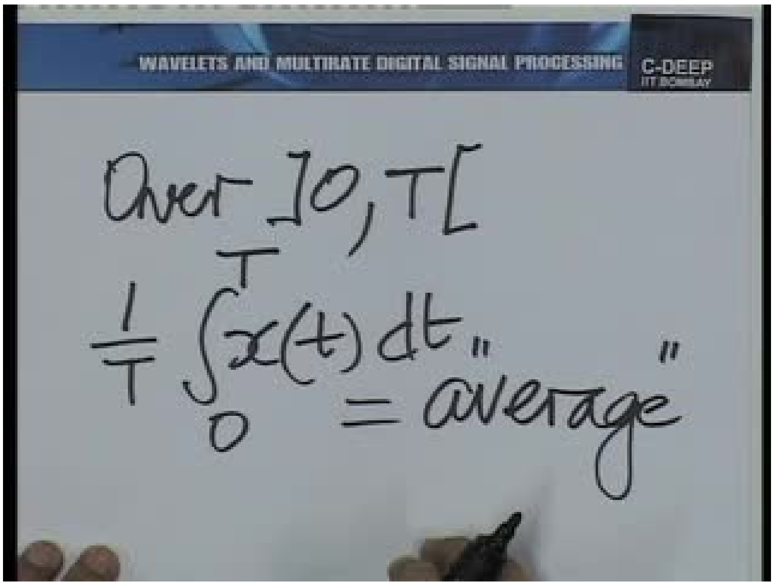

Let us write that down mathematically, so what we are saying if you have a function x of time x (t) a 'good' Piecewise Constant representation is the following. Over the interval of T, over the interval from say]0 , T[, now you know strictly it is]0, T[(the open interval) between 0 and T, the representation would be $1/T \int_0^T x(t)dt$ = average

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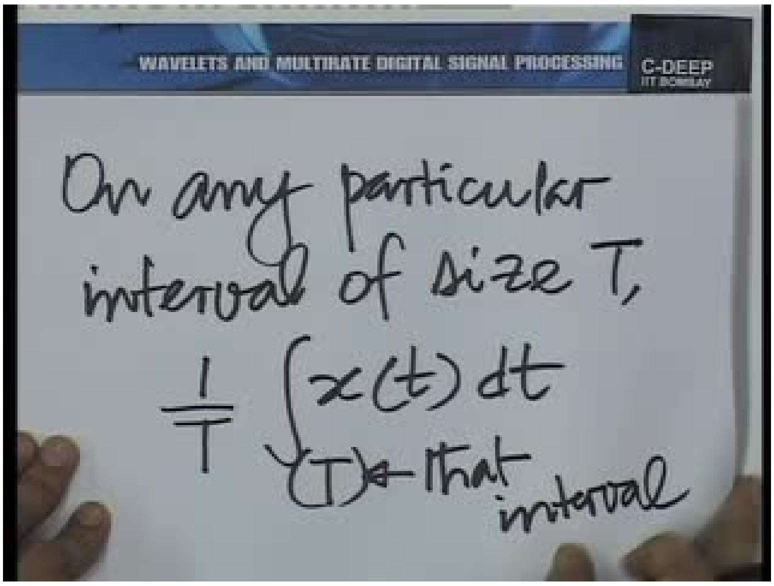

Now of course, on any particular interval of T, the same holds. So we say that on every interval of T, on any particular interval of T, of size T, the average would be obtained by 1/T integral, over that interval of T when you write it like this $1/T \int_T$ you mean that, particular interval of T $_{(T)}\int^{x(t)}$ with respect to small t. This is a Piecewise Constant representation of the function on that interval of size T.

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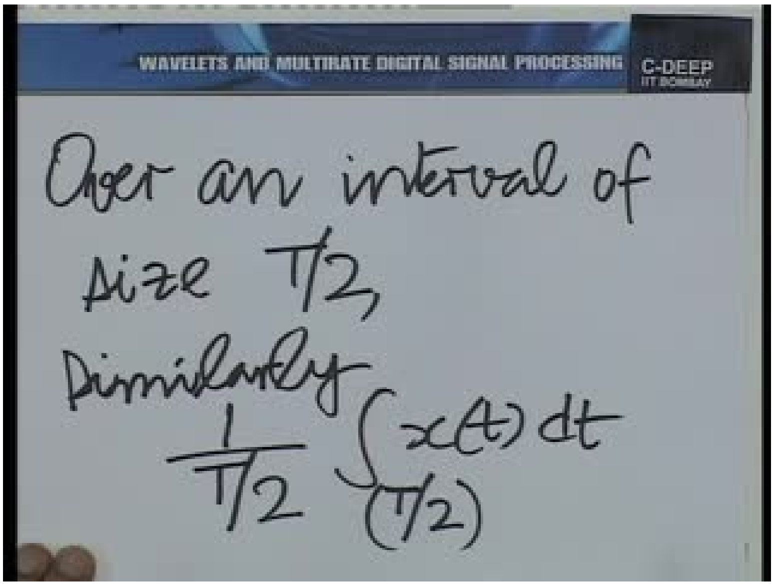

Now the same thing could be done for an interval of size T/2 .So over an interval of size T /2, you would similarly have $1/_{T/2}\ ^{x(t)\,dt}\int_{(T/2)}$.

Now we are going closer to the idea of wavelets. Let us pick a particular interval of size T, in fact, again without any loss of generality, let us choose the interval from] 0, T [and divide it into 2 subintervals of size T/2.

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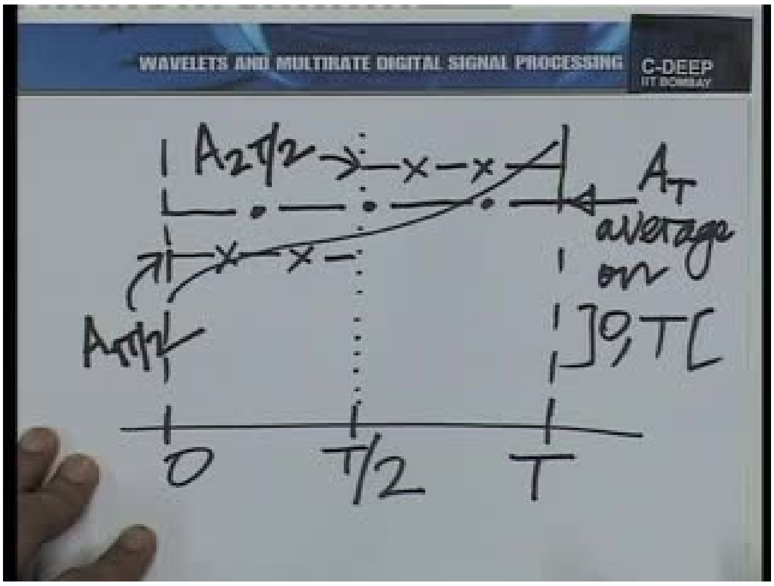

So what I mean is, take this interval of size T,] 0, T [and expanding it, so you have this function here, over that interval, divide this into 2 subintervals of size T/2. First take the piecewise constant approximation, on the entire interval of T and I will show that, by a (●) and (▬) line. You can visualize the average would be somewhere here. So this is the average on the entire interval] 0, T [.

Now I take the sub intervals of size T/2. So I have this subinterval of size T/2, I use a (▬) and (**x**) to write down the average there; so I have (▬) dash and (**x**) cross. Here you can visualize that, in this subinterval, the average would be somewhere here, and similarly in this subinterval you could write down an average something like this.

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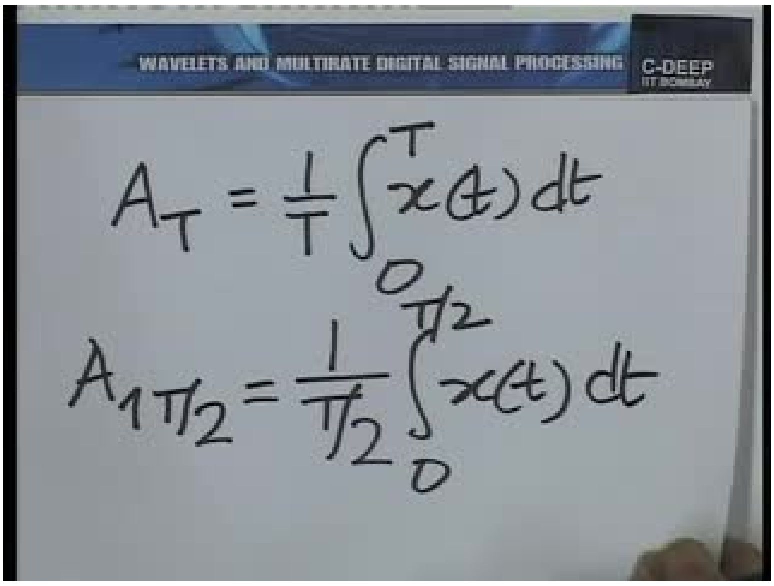

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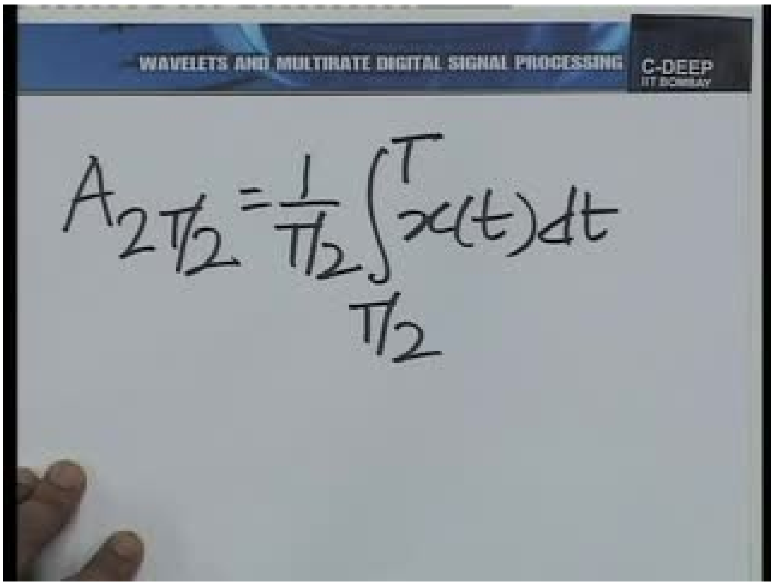

Now let us give this a name. Let us call this average A_T, let us call this average $A_{1\,T/2}$ and let us call this average A_2, on an interval of size T/2 and let us write down the expressions for each of these averages. What are the expressions? A_T is obviously $1/T\ {}_0\int^T x(t)dt$, $A_{1T/2}$ is $1/_{T/2}\ {}_0\int^{T/2} x(t)dt$ and similarly $A_{2\ T/2}$ is $1/_{T/2}\ {}^T\int_{T/2} x(t)dt$ for convenience, let me flash all the three expressions before you once again.

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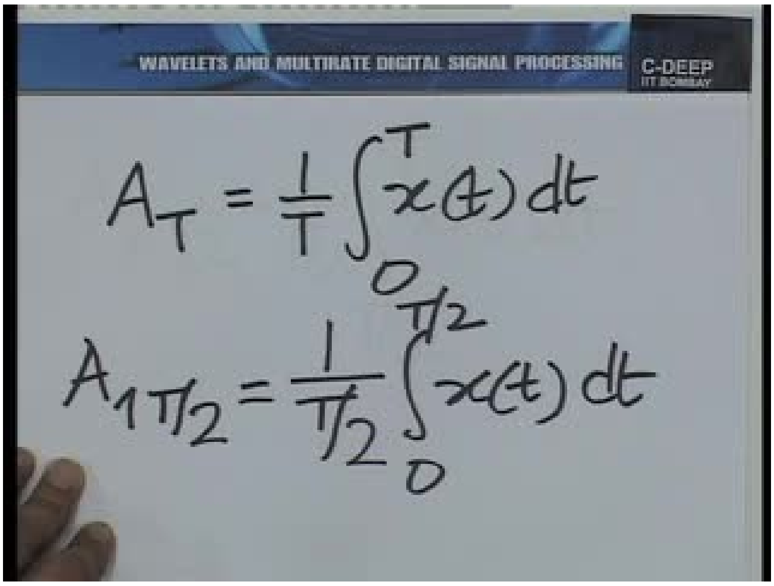

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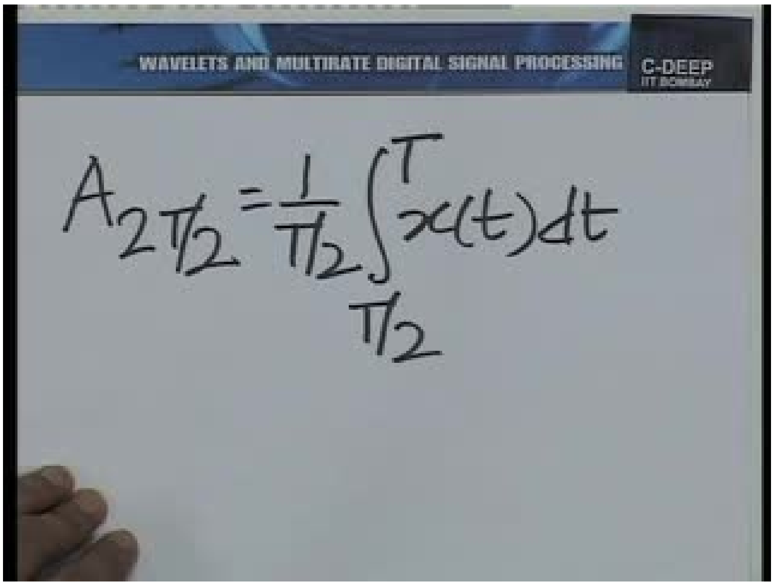

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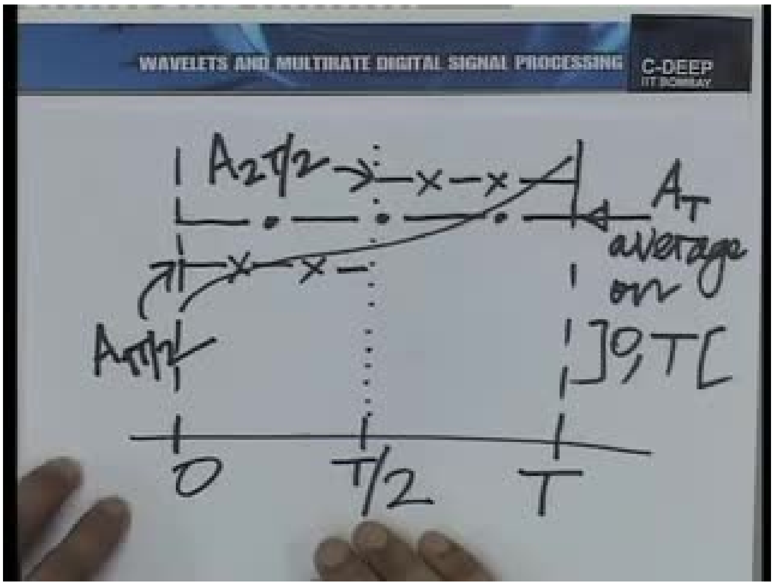

A_T is the average over the entire interval of T, A1 $_{T/2}$ the average over the first interval of T/2 with this expression and $A_{2\ T/2}$ the average from T/2 to T, the second subinterval of size T/2 with this expression, and just to get our idea straight, here again is the picture.

Now the key idea, in the Haar multi-resolution analysis is to try and relate these three terms. So to relate A_T , $A_{1\ T/2}$ and $A_{2\ T/2}$ and it is in that relationship that the Haar wavelet is hidden.

So what is the relationship? Now the relationship is very simple I mean all that we need to do is to notice that we have divided $_0\int^T$ into 2 integrals over$_0\int^{T/2}$ and $_{T/2}\int^{\ T}$ and then remember there is a slight difference in the constant associated.

So we have a constant of 1/T in A_T and a constant of $1/_{T/2}$ in $A_{1T/2}$ and in A $_{2\ T/2}$ where upon we have this very simple relationship between the three quantities.

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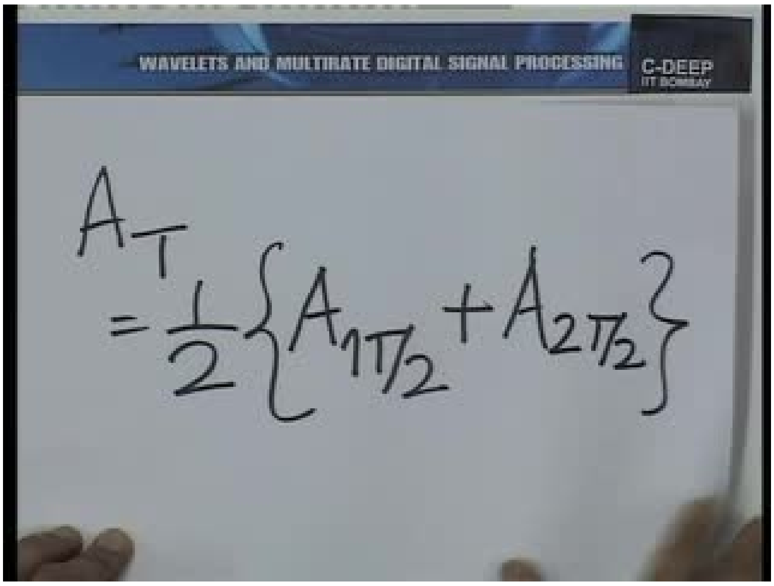

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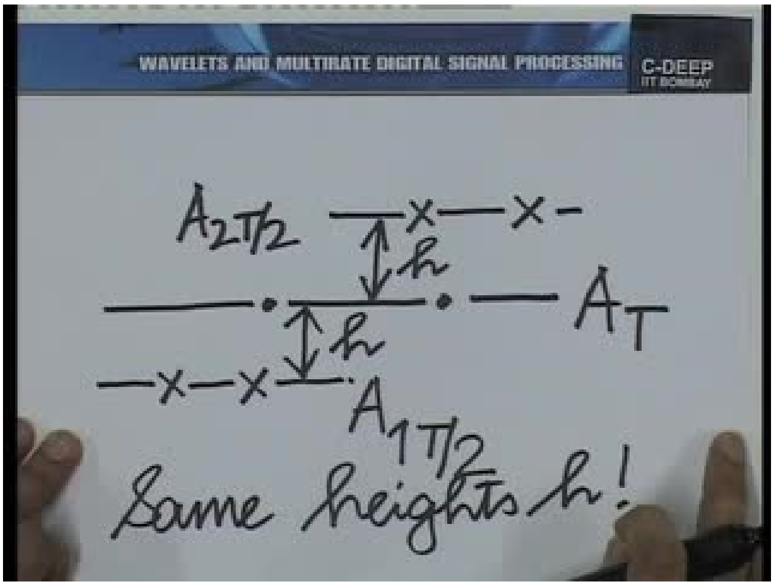

A_T is half I leave it to you to verify, $A_T = 1/2 \{ A_{1T/2} + A_{2\,T/2} \}$ and how do we interpret this, let me try and you know kind of focus just on this relationship in other words. let us just focus on these three constants and make a drawing there. So, what we are saying is something like this, I have this $A_{T,}$ I have this $A_{1\,T/2}$ here and I have this $A_{2\,T/2}$ there and we are seeing this plus this by 2 gives you this. In other words, this is as much higher above A_T as this is low, what we are saying is these two heights are the same. That is what this relationship implies.

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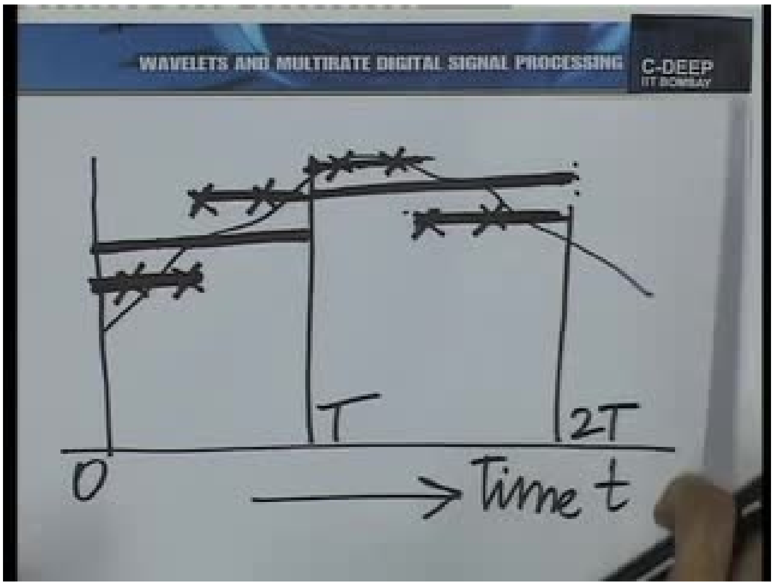

Now another way of saying it is, if I were to make, a Piecewise constant approximation on intervals of size T, how would they look? So let me just sketch them, so I take this function once again. Here, I have this function, here I have divided to intervals of size T let me show just two intervals for the moment.

So this is how the function would look, when you make a piecewise constant approximation, on intervals of size T and when you do it on intervals of size T/2 it would look like this, something like this.

Now, this is a function, so let me highlight it, now let me darken, it this is in its own right a function a piecewise constant function .The one which I have darkened here, and this is in its own right the darkened part is in its own right an approximation to the original function here.

Similarly let me now darken this and put some other mark on it. Let us keep the crosses (**xxx**) so I will darken this, I will put (**xxx**) on it, this is also another function.

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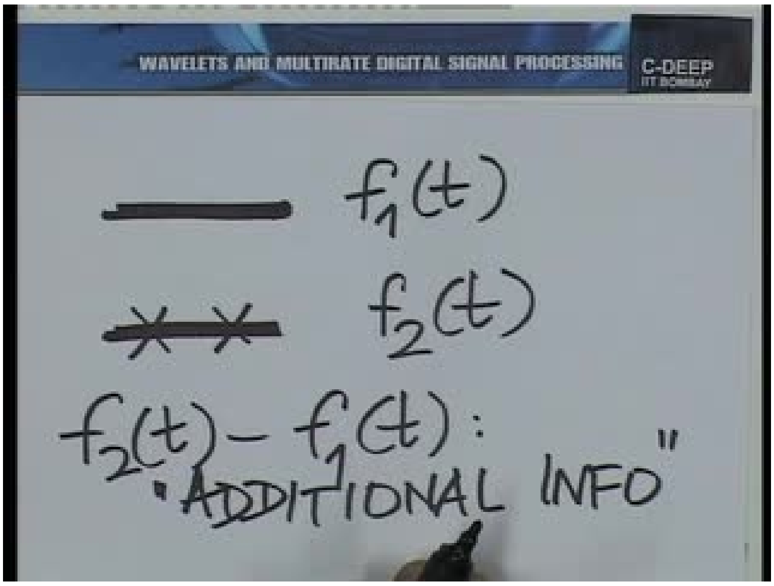

So (▬) dark and (**x**) cross function is another function that is in its own right and approximation too. So let us give the names, let us call this function just the (▬) dark one as $f_1(t)$ and let us call this function the one which we have shown with (▬) and cross as $f_2(t)$ $f_2(t)-f_1(t)$ is like additional information. What we are saying is, instead of a Piecewise constant approximation on an interval of size T when we try and make a Piecewise constant approximation on intervals of size T/2 you are bringing in something more. Go back to the original case of the picture.

We have inherently underlying, a continuous two-dimensional picture a continuous two-dimensional scene. When we make an approximation with a 512 x 512 resolution then we have actually brought in, one level of detail, when we go to a 1024 x 1024 representation, the level of detail is 4 times more.

What is the additional detail that we have got , in going from 512 x 512 to 1024 x 1024 in effect when we take this difference $f_2(t) – f_1(t)$ we are answering that question.

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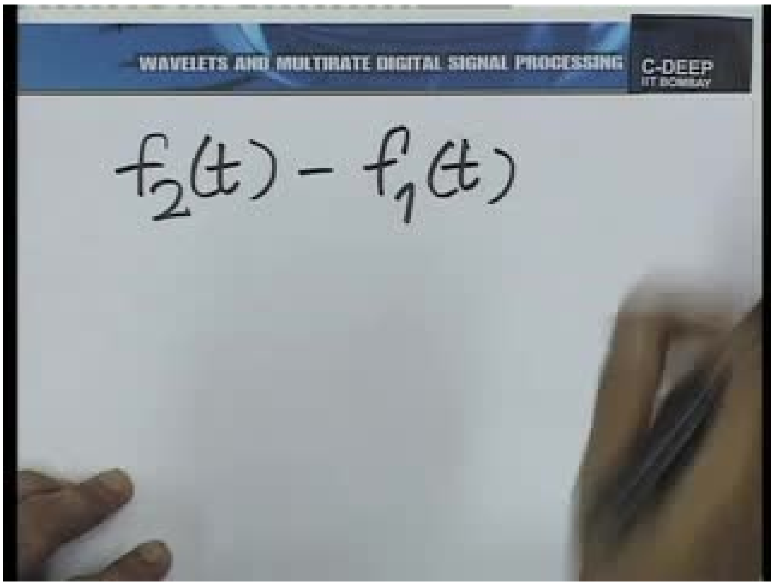

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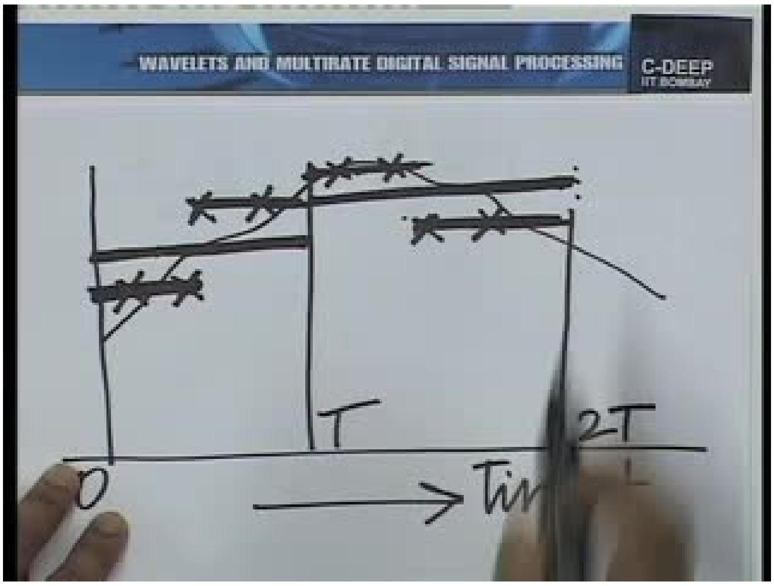

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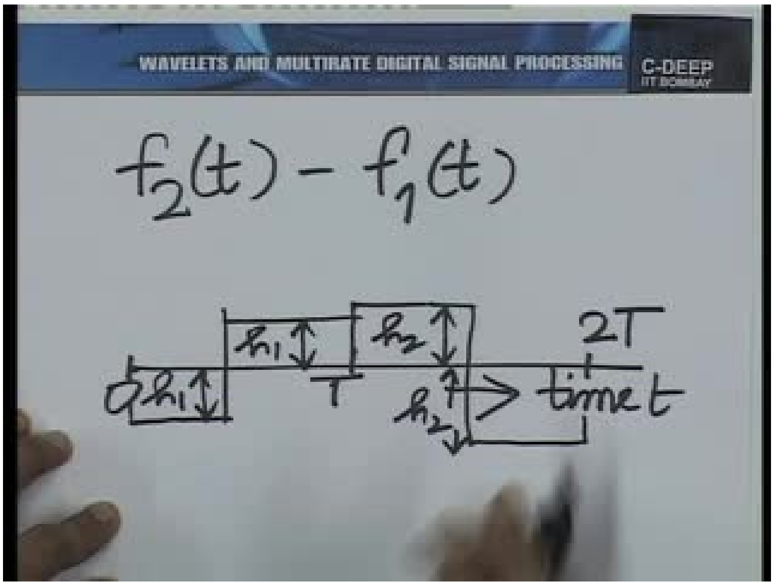

So now let us see how, f_2 (t) - f_1 (t) would look. It is very easy to see that f $_2$(t) - f $_1$(t) has an appearance like this. Let me flash them before you, f_2(t) and f_1(t) just for a second here that you get a feel this is f_2 and this is f_1 and visualize subtracting this from this what would you get a function that look something like this

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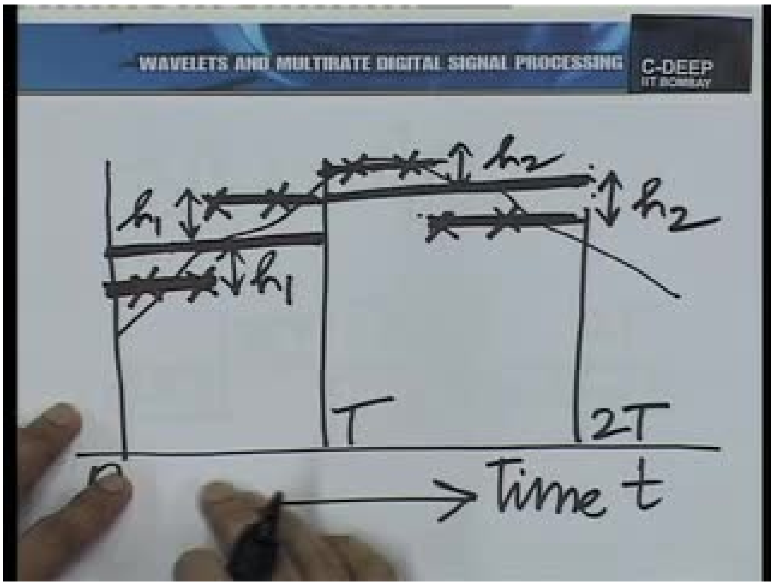

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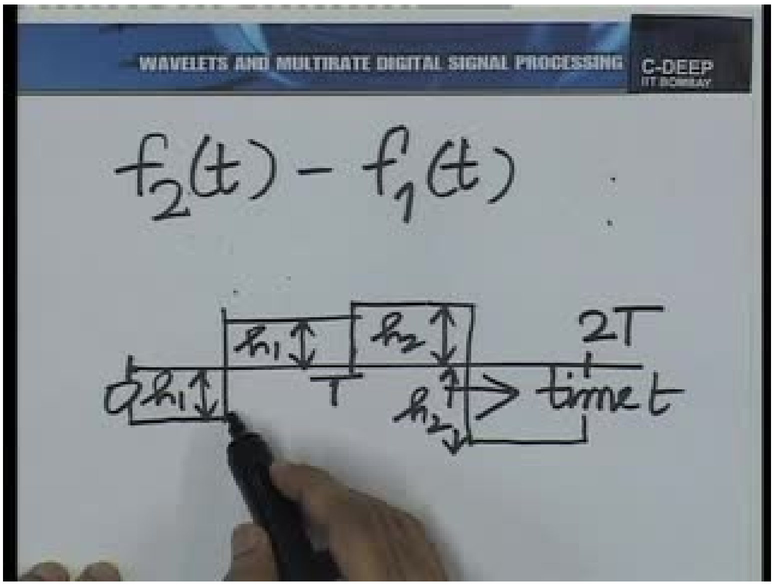

So, I have the time axis here, so if I mark the intervals of size T there, something like this. May be this has height h_1 and this has height $h_{2.}$ Let me mark $h_1$and h_2 on this diagram 2. So, this is h_1 and this is $h_{2,}$ of course, so this is simple enough.

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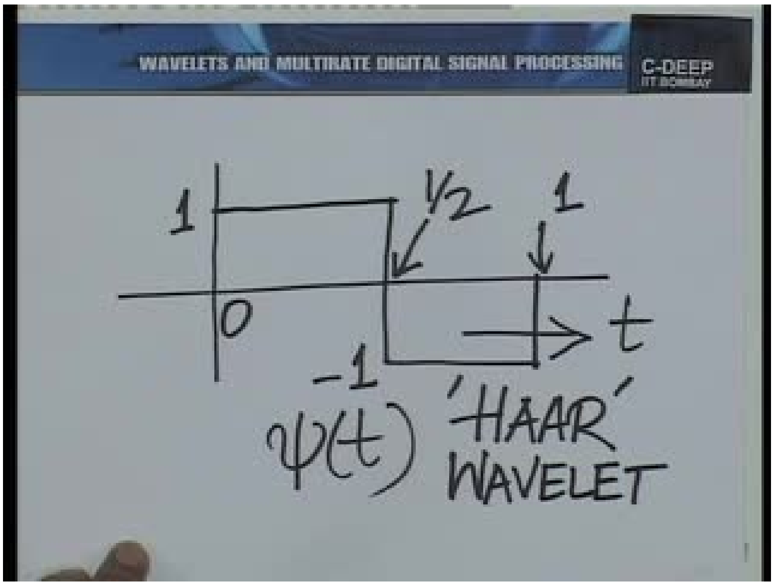

Now if we look carefully, we can construct all of this, by using just one function. And what is that function. Suppose I go to visualize, a function like this, 1 over the interval from 0 to 1½ and -1 over the next ½ interval. This is the point ½, this is the point 1, point

0, 1 here and -1 and let us give this function a name, let us call it ψ (t). In fact, this is indeed what is called the Haar wavelet. Haar, again the name of the mathematician.

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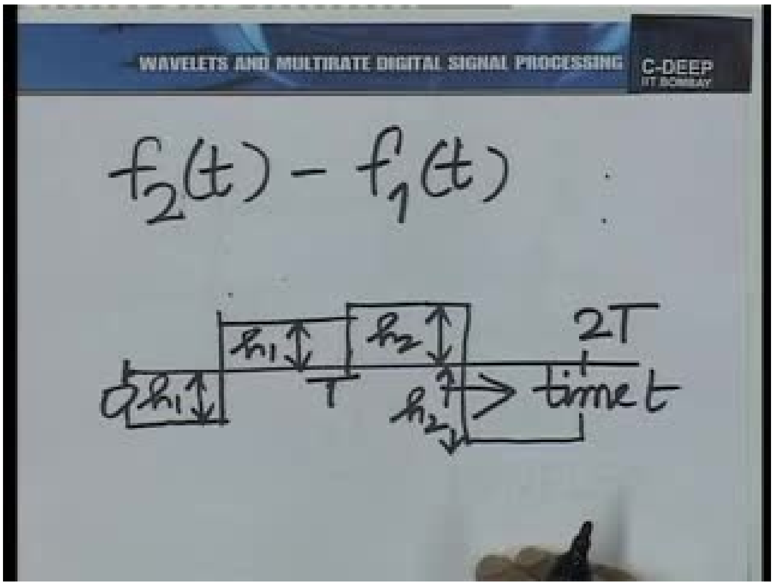

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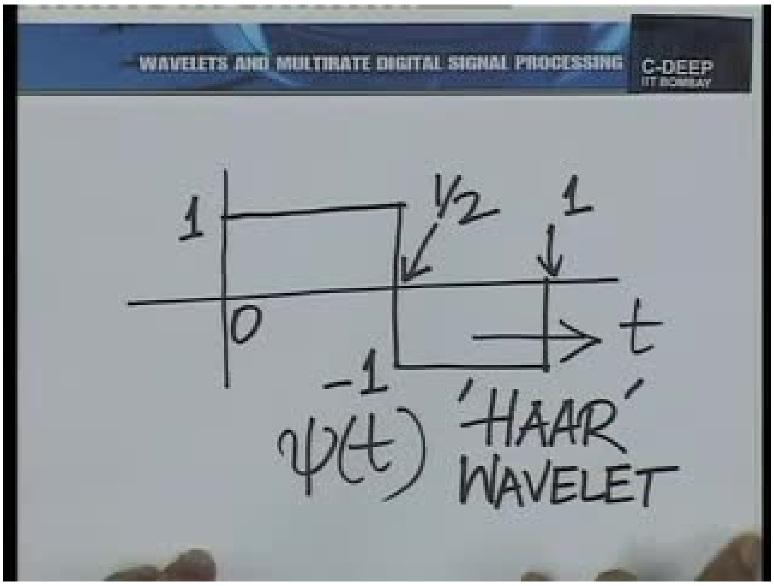

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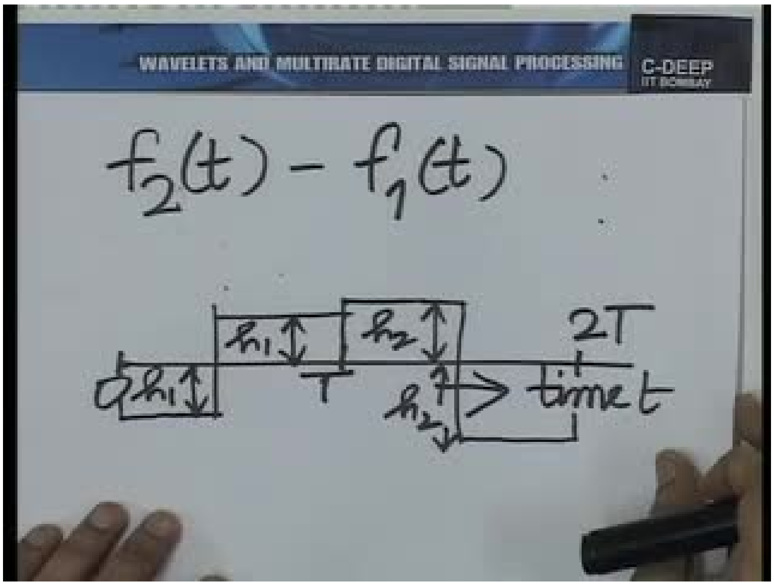

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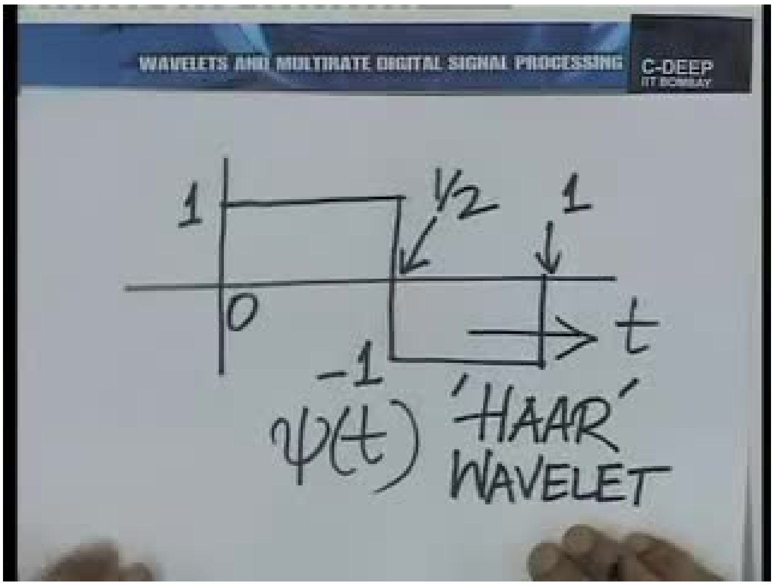

It is very easy to see that, using this function I can construct any such f_2 (t) - f $_1$(t). Indeed, if I were to take this function stretch it or compress, whatever might be the case depending on the value of capital T, dilate is the more general word. So if I were to dilate this function, to occupy an interval of T and bring it to this particular interval of T, so I dilate that function ψ(t) and bring it to this interval of T. And then I multiply ψ(t) so dilated by the constant $h_{1,}$ of course, h_1 should be an algebraic constant. It should be given a sign here, for example, h_1 should be given a negative value, because we have started ψ (t) with a positive -1 here.

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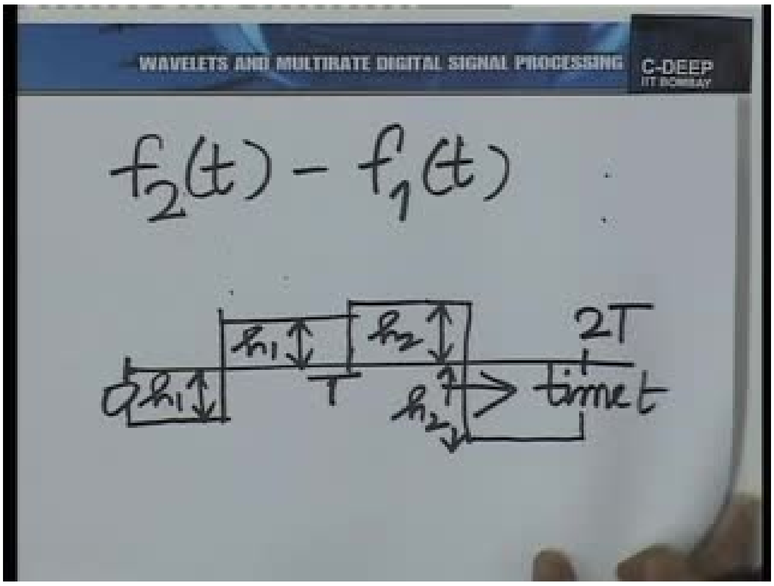

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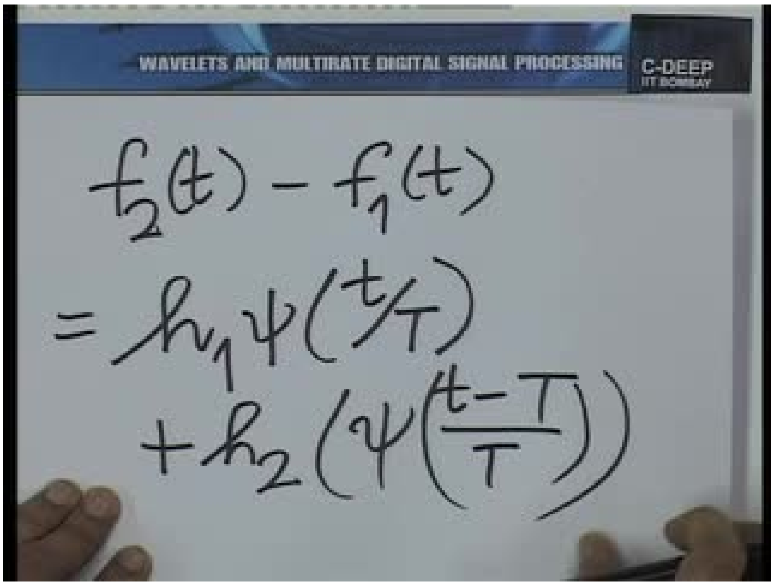

Similarly, h_2 has a positive value. Here so in other words, this segment of $f_2(t) - f_1(t)$ is of the following form some $h_1\ \psi(t/T) + h_2\,(\psi(t - T/T))$.

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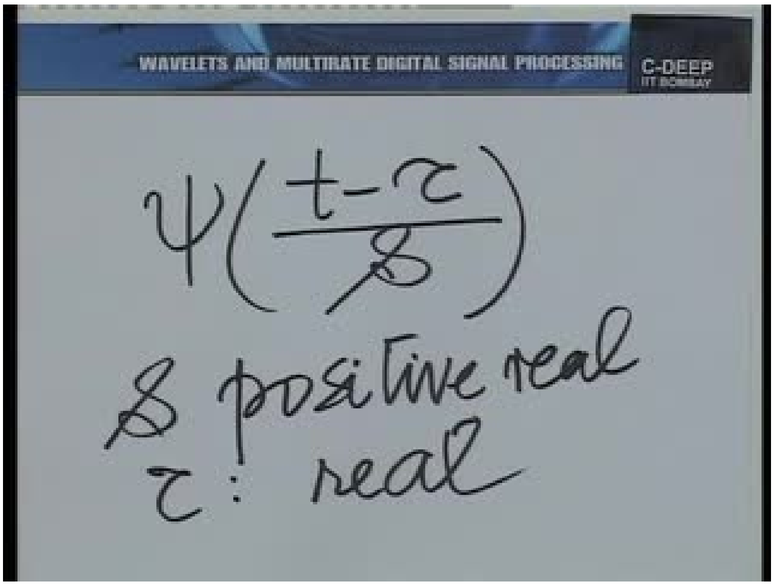

So here, this is both dilated and translated. In other words in general, when we start from the function $\psi(t)$,we are constructing functions of the form $\psi(t - \tau / s)$where of course, **S** is a positive real number and **τ** is real. This is the general function, which we are using as a built in block. different values of **τ** and different values of **S** of course here at a particular resolution at a particular level of detail the value of s is only 1 for example when we are representing the function on intervals of size T we take, **S** = T . If we were to represent, the function on, intervals of size T/2 then s would become T/2 and so on. Then, what we are doing in effect is dilating and translating, now we introduce those terms.

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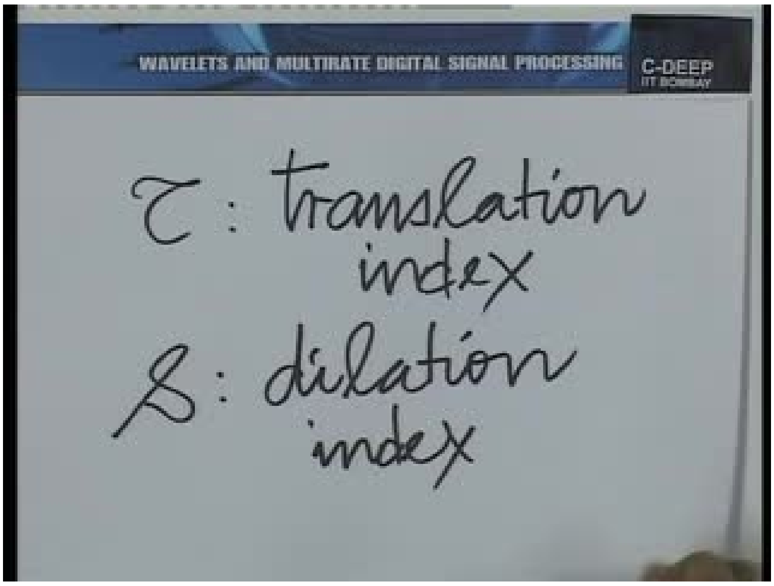

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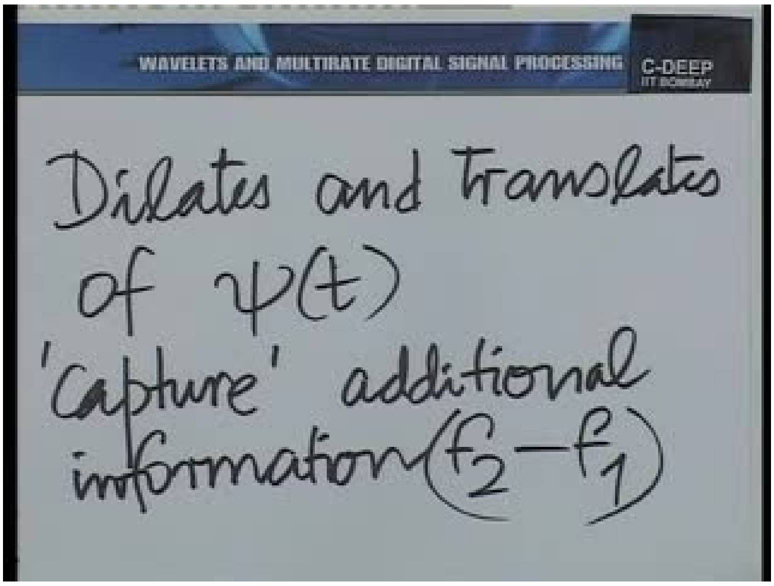

τ is called a translation index or translation variable and **S** is called a dilation index or dilation variable, and we are dilating and translating, or we are constructing dilates and translates of a basic function. Dilates and translates of ψ (t) 'capture' the additional information in (f_{2T} - f_1).

Let us spend a minute in reflecting about, why this is so important. What have we done so far, just looks like very simple functional analysis or just a very simple transformation or algebra of functions?

What is so striking in what we have just said, what is striking is that, what we have done to go from T to T/2 can also be done to go from T/2 to T/4, not only that; what we have done to go from T to T/2. In other words, for intervals of length T, to intervals of length T/2 all over the time axis can be done all over the time axis. To go from intervals of size T/2 to intervals of size T/4 and then, you could go from intervals of size T/4 to intervals of size T/8, T/16, T/32, t/64 and what have you to as small an interval as you desire. Each time what you add in terms of information, is going to get captured, by these dilates and translates of the single function ψ (t).

A very serious statement if we think about it deeply enough that one single function ψ (t) allows you to bring in resolution step by step to any level of detail. In fact, in formal language in functional analysis we would put it something like this.

In mathematics, in these arguments of limits and continuity and so on or in some of these proof related to convergence, there is this notion of the adversary and the defendant. So here the defendant is trying to show the one who makes the proposition ,is trying to show that by this process, you can go arbitrarily close to a continues function as close as you desire.

Now as close in what sense, well it could be in terms of, what is called the mean squared error or the squared error. So let us formulate that adversary 'proponent' kind of argument here.

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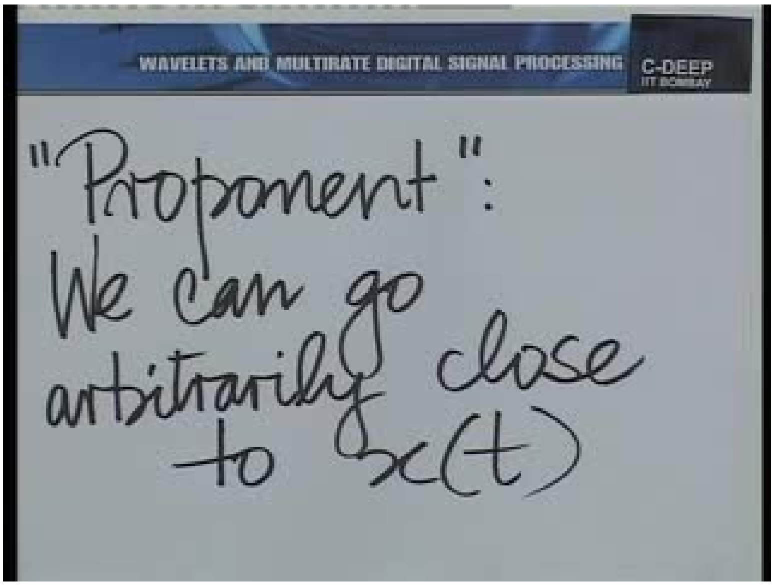

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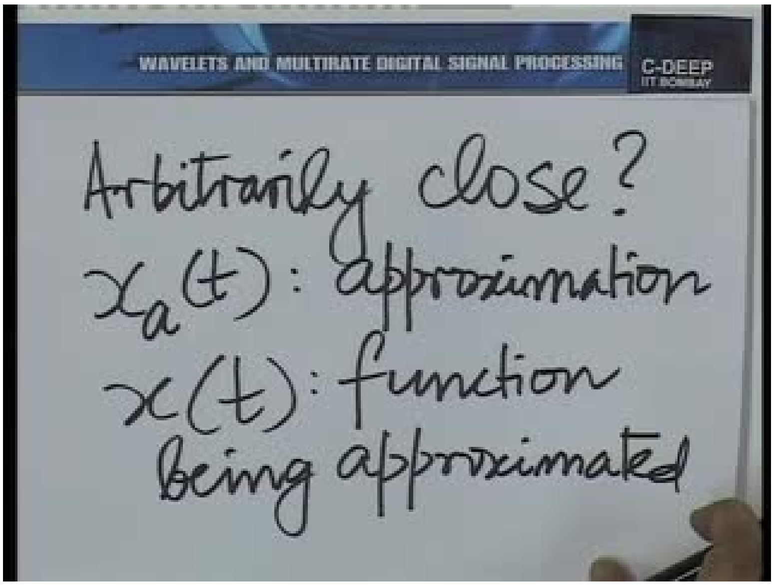

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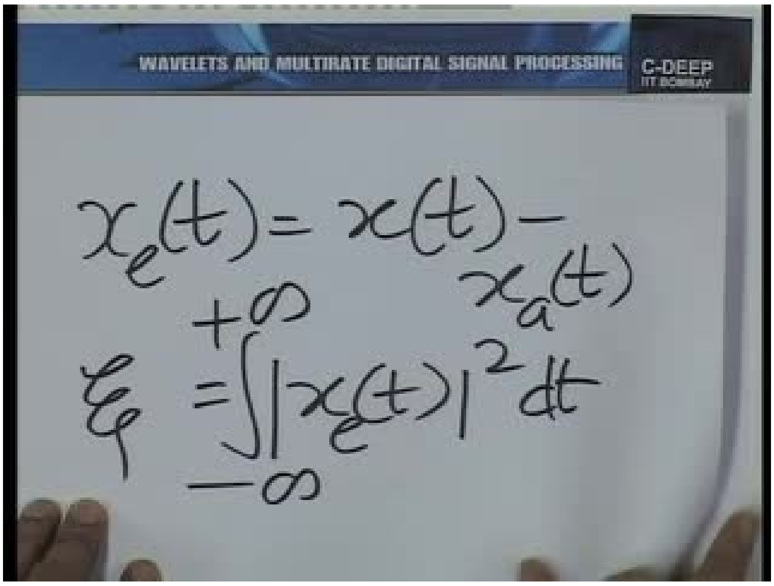

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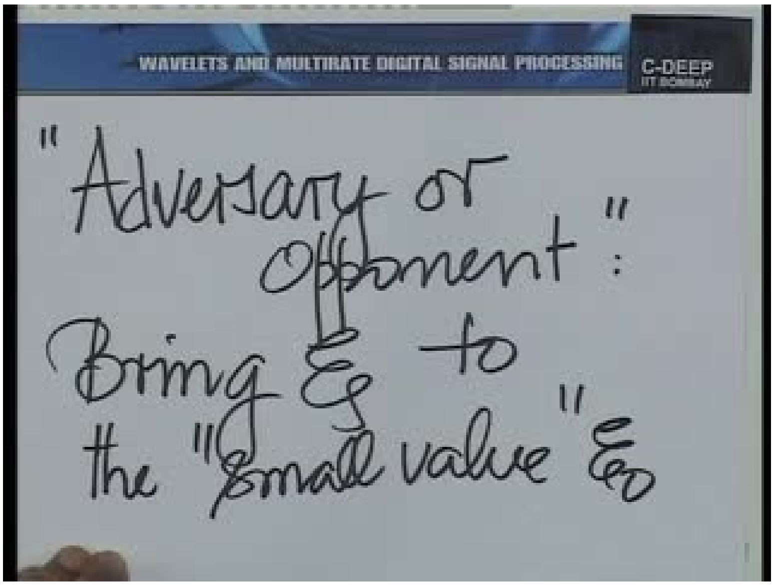

So, what we are saying is, the proponent says, we can go arbitrarily close, to x (t) to a continuous function x (t) by this mechanism. Arbitrarily close in what sense. In the sense, if x_a is the approximation, approximation at a particular resolution and if x (t) is the original function. Then if we take, what is called the squared error, so we look at x_e (t) that is x(t) - x_a (t) and integrate x_e(t) the whole squared the modulus whole squared actually overall t. we call this the squared error script e. Then the "Adversary or Opponent" says bring E to the small value

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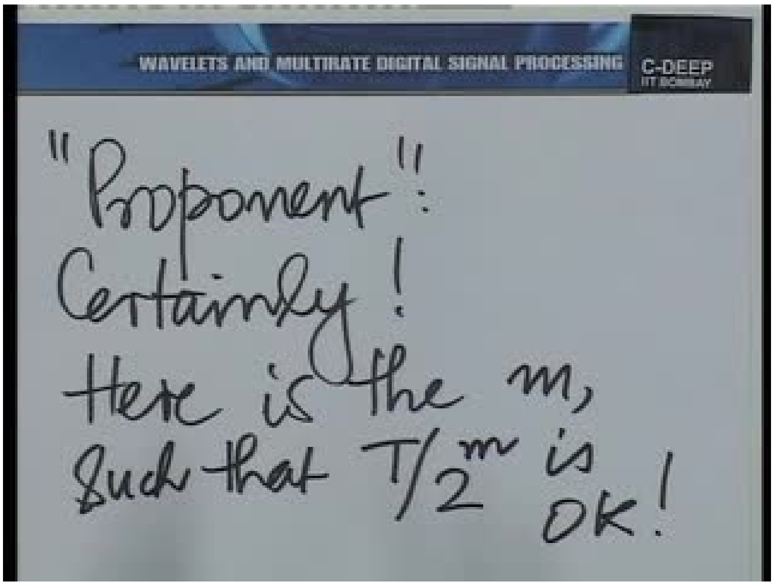

Let's say E_0 and the proponent says, certainly here is the m such that $T/2^m$ is ok! so that is the idea of proponent and opponent here

The "Adversary or the opponent" gives you a target. He says, I want this squared error to be less than this number E_0 and the proponent says, well ,here you are, if you make that interval of size $T/2^m$ low, and behold your error is going to be less than. Or equal to E_0 and what is striking, in this whole discussion is, no matter how small we make that $E_{0,}$ the proponent is always able to come out with an m. Such that $T/2^m$ I mean piecewise constant approximation on intervals of size T $^{T/2\ m}$ would give you an approximation close enough for that small $E_{0.}$ We need to spend a minute to reflect on this. This is the serious thing we are saying, in fact , let us for a moment think on how this is dual to the idea of representation of a function in terms of its Fourier series for example.

In the Fourier series representation what do we do, we say give me a periodic function or for that matter give me a function on a certain interval of time. Let us say an interval of T size T, if I simply periodically extend that function that means, I take this basic function on the interval of T I repeat it on every such interval of T, translated from the original interval.

So suppose that original interval is 0 to T then repeat whatever is between 0 and T between T and 2 T between - T and 0 between - 2 T and - T between 2 T and 3 T and go on doing this.

So you have a periodic function. Decompose that periodic function into its Fourier series representation so what am I doing in effect? I have a some of sinusoids sine waves all of whose frequencies are multiples of the fundamental frequency. What is that fundamental frequency? in angular frequency terms it is $2\pi/T$ in hertz terms it is $1/T$

So in hertz terms, you have sine waves with frequency, which are all multiples of $1/T$ and an appropriate set of amplitudes and phases, a sine to these different sinusoidal components, with frequencies of multiples of $1/T$ when addict to gather would go arbitrarily close to the original function. Of course, the original periodic function on the entire real axis or for that matter specifically on the interval from 0 to T, if you restrict yourself to the function from where you start it.

So not only does the Fourier series, allow you to represent, by using the tool of continuous functions, analytic functions remember. We talked about sine waves in the previous lecture. Sine waves are the most continuous, in some senses, the smoothest function that you can think of. The derivative of a sine wave is, a sine wave, the integral of a sine wave, is a sine wave, when you add two sine waves, in the same frequency. They give you back a sine wave of the same frequency.

So sine waves are the smoothest function that you could deal with and even if you had somewhat discontinues function on the interval from 0 to T and if you use this mechanism of Fourier series decomposition you would land up expressing a discontinues function in terms of extremely smooth analytic functions.

What would you be doing in the Haar approach that we discussed a few minutes ago? Exactly the dual, even if you had this continues audio pattern you would decompose it into highly discontinues functions, which are piecewise constant on interval of size T at the resolution T, on intervals of size T/2, at the resolution T/2 and so on.

Now just as in the Fourier series representation, you have this proponent opponent kind of argument that is for a reasonably good class of functions even if they are discontinues even if they have a lot of non-analytic points and so on for a reasonably wide class of function. Remember in the Fourier series, that wide class of functions is captured by what are called the 'Dirichlet' conditions.

Now, I would not go into those details here, but there are certain kinds of conditions very mild conditions which a function needs to obey before it can be decomposed into the Fourier series, or in other words before the Fourier series can do this job, of representing that discontinues functions in terms of highly continues an analytic smooth functions.

So similar set of condition is just exist even for the Haar case, I mean if one really wishes to be finicky, one does need to restrict oneself, to a certain sub class of function, but again that restriction is not really serious, in most physical situations for the time being. in this course we may even just ignore that restriction.

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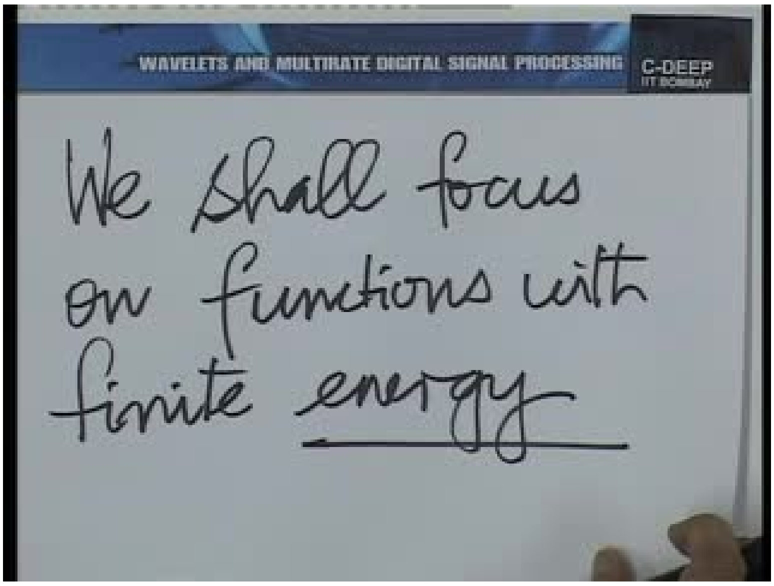

All that we asked for and that is not too unreasonable is that the function has finite energy, so let us at least put that down mathematically what we are saying is, we shall focus on functions with finite energy and what does energy mean ‘energy is essentially thus the integral of the modulus squared’.

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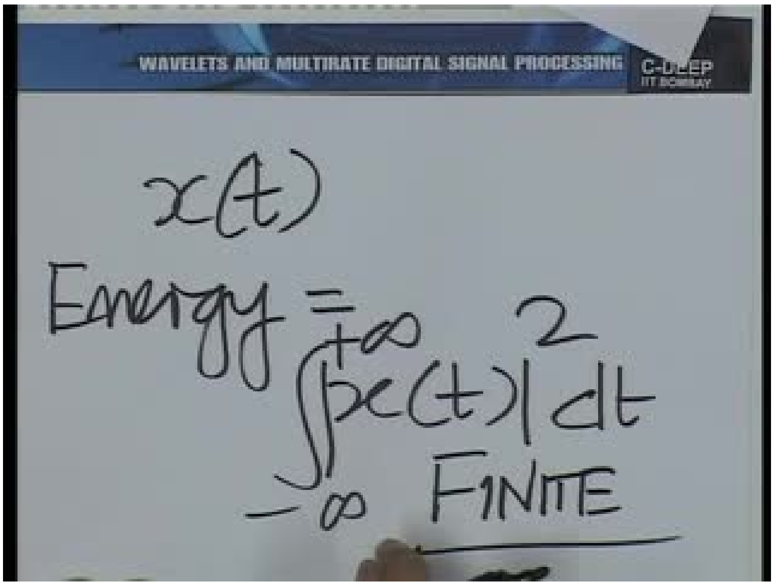

So if I have a function x(t) the energy in x(t) is the integral mod x(t) squared over all t and this needs to be finite all that we are saying is this incidentally, this quantity has a name if the mathematical literature or for that matter even in the literature on wavelets.

The energy, as we call it in signal processing, is called the L_2 norm by mathematicians and you know, it helps to introduce terminology little by little, from the beginning because if one happens to pick up literature on wavelets, these terms would be used.

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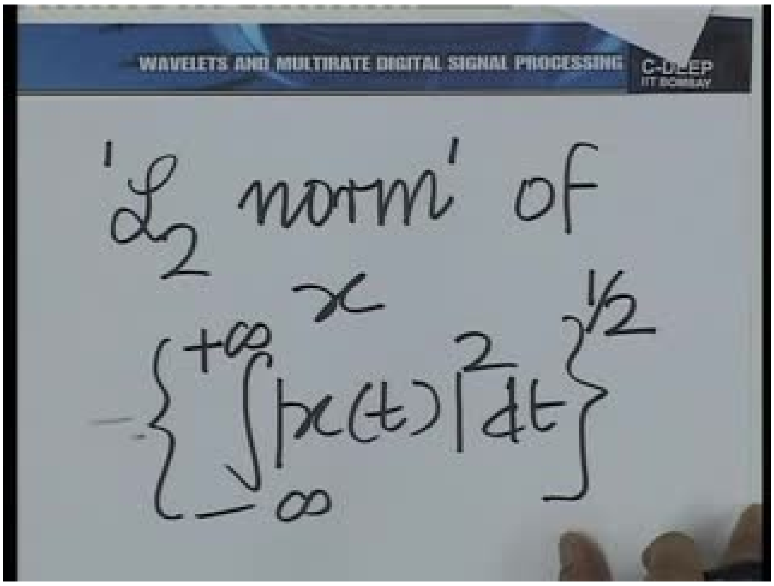

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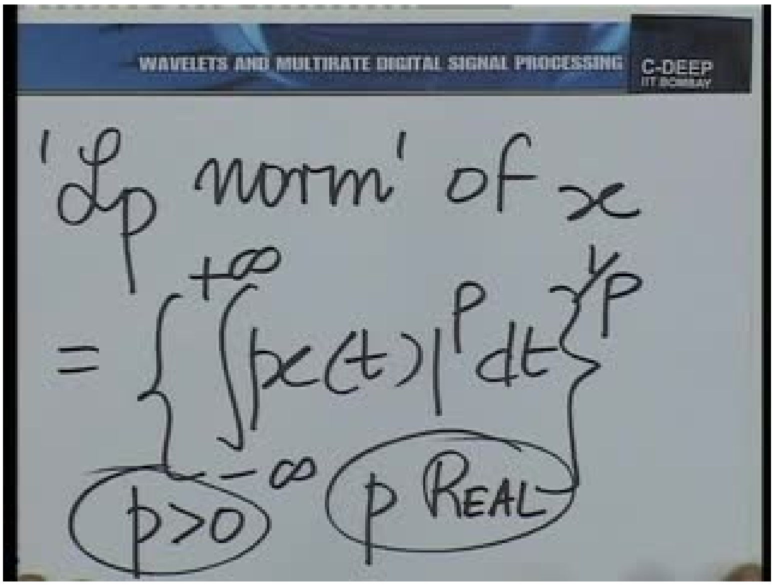

So let us introduce that notation slowly $L_2 = \{ \int_{-\infty}^{+\infty} | x(t) |^2 dt \}^{1/2}$ so we say the L_2 norm of x is essentially mod x(t) squared dt integrated over all t and to be very precise this needs to be raise to the power ½. Similarly one can talk of an $L_p = \{ \int_{-\infty}^{+\infty} | x(t) |^p dt \}^{1/p}$. L_p norm of x and that would correspondingly be mod x(t) the power P dt integrated on all time and raise the power 1/P and of course P here is a real number so for any real in fact real and positive.

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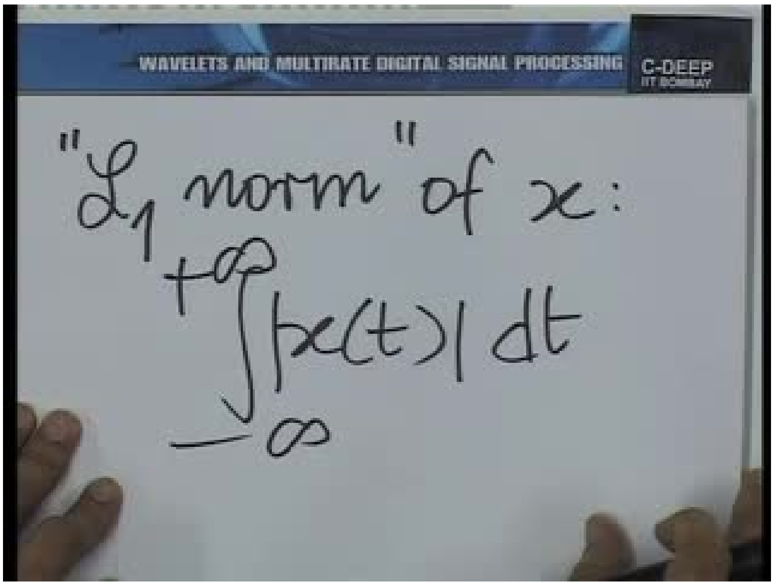

So you could talk about an L_1 norm, you could talk about an L_2 norm , you could talk about an L infinity norm .What would an L infinity norm? be let us take some examples what would an L_1 norm it would essentially be integral mod x(t) . L_1 norm of x: $\int_{-\infty}^{+\infty} |x(t)| dt$

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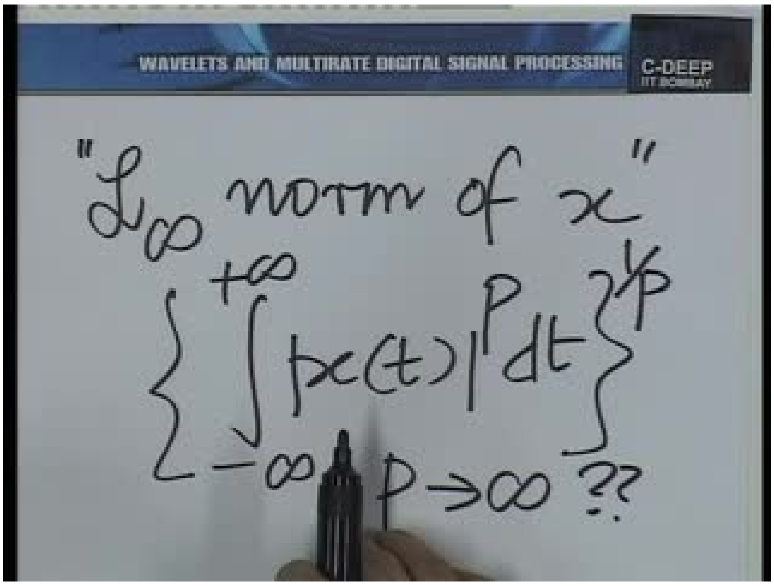

The L_2 norm we already know. What would the L_∞ norm be, that is interesting so you see in principle it would be something like this, but what an earth is this? What do we mean by this? You see as P becomes larger and larger what are we doing. We are emphasizing those values of x (t) which are larger.

So for a larger value of P, we are emphasizing those values of mod x(t) which are larger and as P tends to a larger and larger where your P tends to infinity, we are in some sense highlighting that part of x(t) which is a largest.

So in other words, the L_∞ norm of x essentially would correspond to the maximum or the supremum. You know the very largest value that x (t) can attain all over the real axis. So it has a meaning, even as P tends to ∞. Anyway, this was just to introduce some notation, which we are going to find useful.

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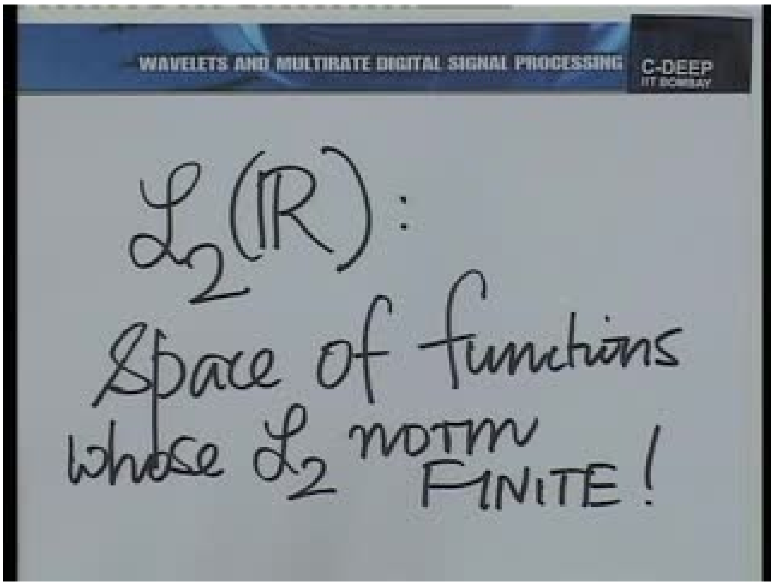

And what we are saying in this language is, that we are going to focus on functions, which belong now. Here we are going to start talking about functions that belong to a space. We say the space L_2 , what is this space L_2 . L_2 (IR) it is a space of functions and it is a space of functions whose L_2 norm is Finite simple!

Similarly, you could have this space, L_P the space L_P is the set of all functions whose L_P norm is ∞. Now the word space, is used with an intent. You see space really means, if I take a linear combination of functions in that set, it gets back to a function in that set.

So if I take any finite linear combination of functions in a space L_P the resultant is also in that space in that set L_P and that is why we call it as space.

So L_P, all the L_Ps for any particular P are spaces. Linear spaces, they are closed under the operation of linear combination.

So in other words we are saying, let us focus our attention on the space L_2 . Now what we have said, in the Haar analysis that we talked about a few minutes ago is that, if you take any function in the space L_2, I mean if you are adversary picks up any function in the space $L_2$and puts before you a value e_0

Saying, please give me an m, so that, when I make a piecewise constant approximations on intervals of size $T/2^m$, my error squared error is less than e_0 . The proponent is able to do, so the proponent is able to come up with an m, which gives this answer and this could be done, no matter how small the e_0 is. The proponent will always come out with a suitable m that is the idea of what is called ‘closure’.

So what we are saying is, when we do an analysis using the Haar wavelet, in other words, when we start from a certain piecewise constant approximation, on intervals of size, let us say 1 for example, and then bring it to intervals of size ½ , ¼ , ⅛ , 1/16 ,as small as you desire, you can in principle, go as close, in the sense of L_2 norm. That means, if I look at the L_2 norm of the error between the function and its approximation that L_2 norm of the error, can be brought down as much as you desire.

And in that sense, whatever the Fourier series was doing, after all what does the Fourier series do, it allows you to bring the L_2 norm of the error between the function and its Fourier series as small as you desire for a reasonable class of functions. For a wide class of functions, give me the epsilon give me the E_0 and I will give you a certain number of terms that you must include in the Fourier series.

So the adversary says, well here is an E_0 for you, the proponent says ok, include so many terms in the Fourier series and you can bring your error down as low as you desire. The same kind of thing is happening here - the proponent adversary principle. Now this is a deep issue, that one function $\psi(t)$ is able to take you, as close as you desire to the functions, that you want to approximate and by the way, this is only one $\psi(t)$ which can do it. The whole subject of wavelets allows you to build up many such $\psi(t)$s.

Here we had a good physical, a very simple physical explanation. We started from piecewise constant approximation, we said well, when you want to refine your piecewise constant approximation, you could do it by using the Haar wavelet and this you could do to go from any resolution to the next resolution.

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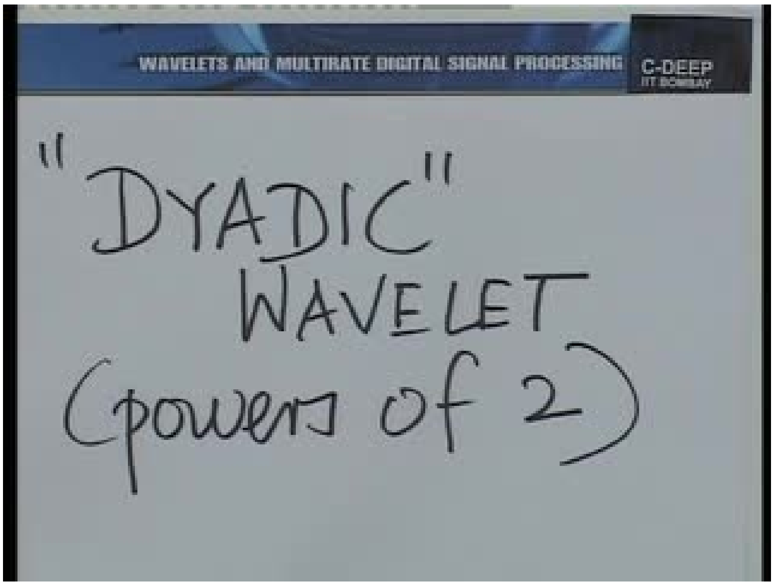

Please remember here, we are increasing the resolution or improving the amount of information contained by factors of 2 each time and that is why, we use the term "Dyadic". Let me write down that term Dyadic, so what we have introduced in this lecture, is a notion of a dyadic wavelet and dyadic refers to, powers of 2 (steps of 2) every time the Haar wavelet is an example of a dyadic wavelet. And in fact, for quite some time in this course, we are going to focus on dyadic wavelets.

Dyadic wavelets are the best studied. They are the best and most easily designed. They are the best and most easily implemented and I dare say, the best understood. So, for quite some time in this course, we shall be focusing on the dyadic wavelet, the Haar is the beginning, I mentioned in the previous lecture, that if one understands the Haar wavelet and if one understands, the way in which the Haar multi-resolution analysis is constructed, many concepts of multi-resolution analysis would become clear.

What we intent do now after this in subsequent lectures is, to bring this out explicitly, so let me give you a brief exposition of what we intent to do in subsequent lectures and then, we shall go down to doing it mathematically step by step.

You see we brought out the idea of the Haar wavelet explicitly here, what is the Haar wavelet we know, what function it is and we know that dilates and translates of this function can capture information, in going from one resolution to the next level of resolution in steps of 2 each time.

Now, how is this expressed? In the language of spaces after all we talked about the space L 2 R L 2 R is the space of square integrable functions

So how can we express this? In terms of approximation of that whole space so can we express this, in terms of going from one subspace of L 2 R, to the next subspace and in that case, can we express this Haar wavelet, or the functions constructed by the Haar wavelet and its translates and perhaps also, dilates in term of adding more and more to the subspaces to go from a courser subspace, all the way up to L 2 R on one side, and all the way down to a trivial subspace on the other.

So we are going to introduce this idea of formalizing the notion of multi-resolution analysis. We need to think of what is called, a ladder of subspaces, in going from a course subspace to finite and finite subspaces until you reach, L 2 R at one end and courser and courser subspace until you reach the trivial subspace at the other end.

Further we are going to see that, the Haar wavelet and its translates at a particular resolution, at a particular power of 2. So to speak actually relates to the basis of these subspaces.

So, we are going to bring out the idea of basis of these subspaces, and how the Haar wavelet captures what is called the different subspace. In fact, the orthogonal complement to be more formal and precise, simple but beautiful and what we do for the Haar will apply to many other such kinds of wavelets.

Let us then carryout this discussion in more detail, in the next lecture, where we shall formalize whatever we have studied today for the Haar wavelet, by putting down the subspaces that lead us towards L 2 R at one end, and towards the trivial subspace at the other.

Thank you.