Advanced Digital Signal Processing - Wavelets and Multirate
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Lecture No. # 15
Time and Frequency Joint Perspective

A warm welcome to the 15th lecture, on the subject of Wavelets and Multirate digital signal processing. We had raised a few questions in the previous lecture, which we set out to answer in this lecture. The questions pertain to the fact, that we were building more than one multiresolution analysis and specifically the question was, what are we looking for; when we go from one multiresolution analysis to the other? Why cannot we be content with Haar multi resolution analysis? We have been singing the phrases of the Haar multiresolution analysis so frequently, in some of the previous lectures.

We been saying that, it tells us most of the things that a multiresolution analysis does. Why then, should we look for others? What is inadequate in the Haar? Well, if you remember, when we began this course; we set out to do something which a basic course on signal theory or signals and systems does not. Namely look at two domains simultaneously. We have been trying to do this, for a while. So, when we took the Haar, we did at some point talk about the filtering aspect of it. We also talked about the ideal to which we strive in the filter bank and if you recall that ideal had to do with the frequency domain, not the time domain.

So somewhere, we have not been giving as much importance to the other domain so to say, as we should. We been working in the natural domain of time, but we have not quite being doing justice to the frequency domain. When we take a Haar multiresolution analysis, what does it do in the frequency domain? We have not quite understood this as yet and the first thing that we would like to do is to answer this question. What does the Haar multiresolution analysis do in the frequency domain? And then, how do these filter banks that comprise the Haar, build up to that frequency domain behavior?

So let us set out first, to answer a more basic question. What is the Fourier transform of the scaling function and the wavelet function in the Haar multiresolution analysis?

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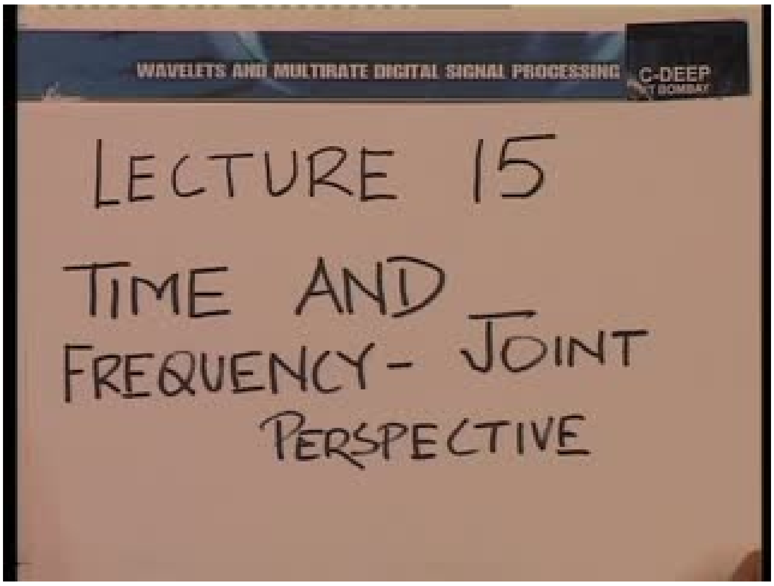

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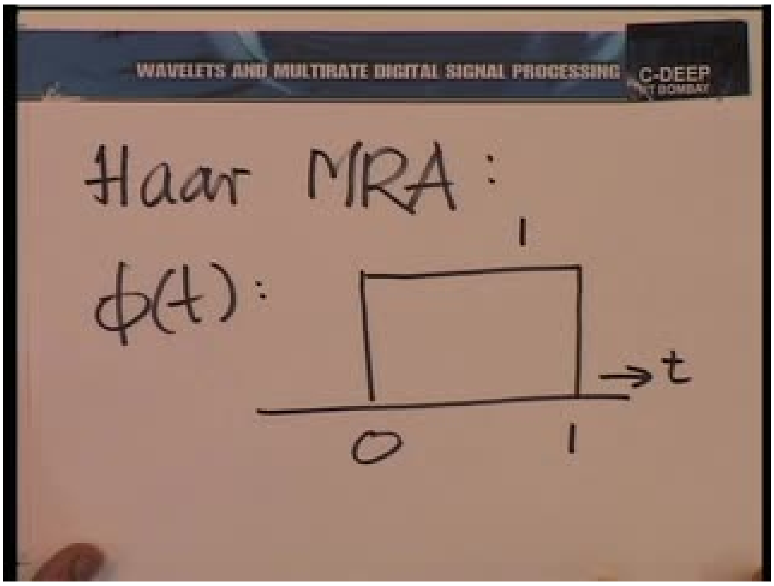

So, we shall call today's lecture a joint perspective, time and frequency and we shall set out first, to look at the Fourier transform of phi t and psi t in Haar. The Haar MRA. Let us, begin with the phi t and let us look at this Fourier transform. Remember, how phi t looked, this was phi t. Let us obtain its Fourier transform.

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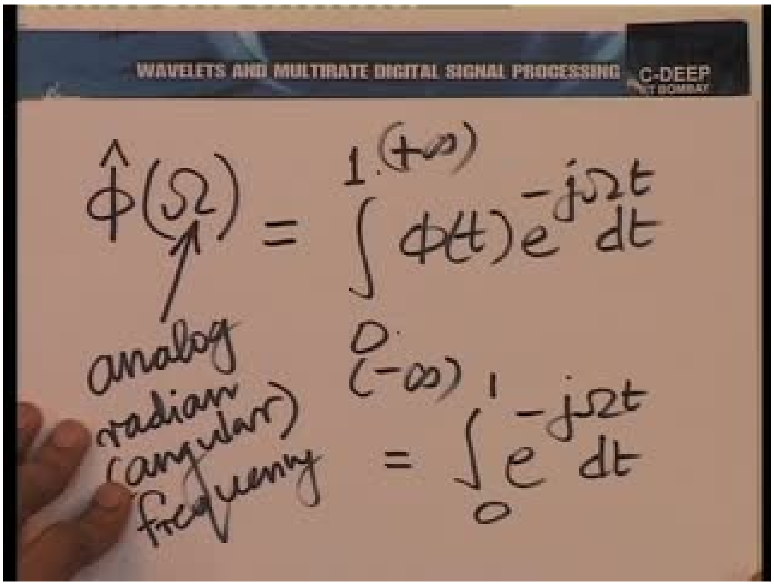

We shall use this notation for the Fourier transform, a cap on top and capital omega to denote the angular frequency, analog angular frequency, radian or angular frequency and this is easy to calculate. Well, I am saying 0 to 1; actually, I should write minus if you like, I can write minus infinity to plus infinity, but the non zero part is only between 0 and 1. So, it is also alright to write between 0 and 1 only for this specific function.

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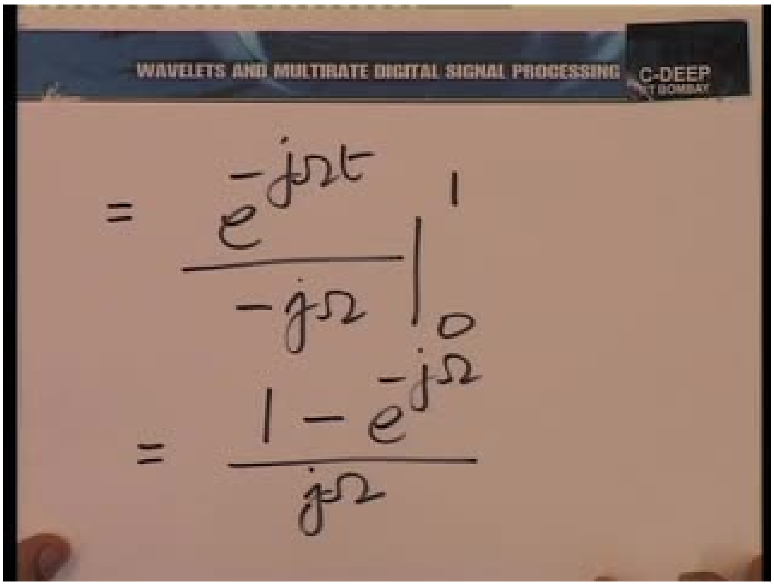

And this of course, equal to integral 0 to 1, e raise to the power minus j omega t dt, which evaluates to e raise the power minus j omega t, by minus j omega from 0 to 1 and that is 1 minus e raise the power minus j omega by j omega and we can simplify this.

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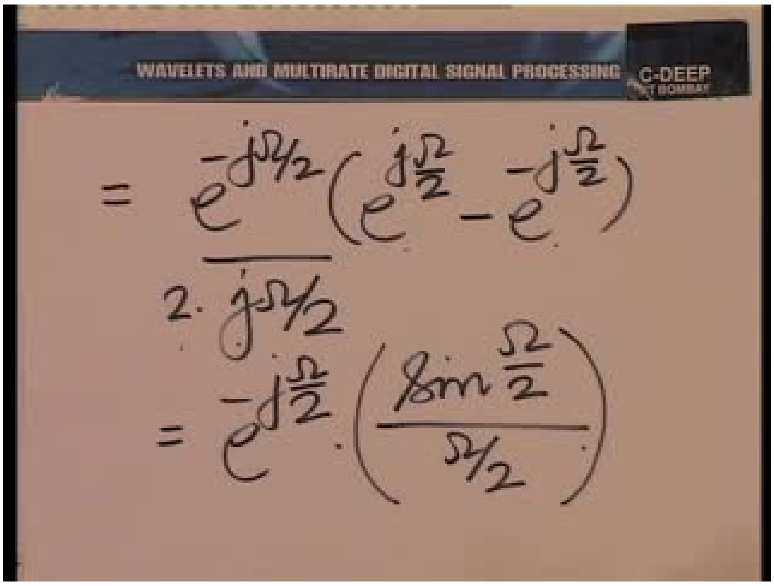

We can take an e raise the power minus j omega by 2 common, in the numerator and that is easy to interpret. In fact if you wish, you can make; even make this by j omega alright 2, and then this becomes e raise the power minus j omega by 2. If I take the 2 j and this together, I get sin omega by 2 there and omega by 2 here. So, this is the Fourier transform.

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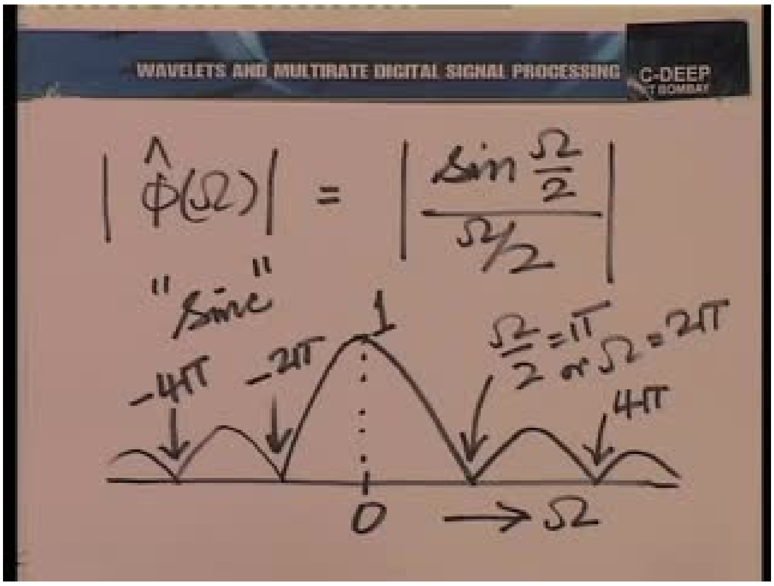

Now, let us look at the magnitude of this Fourier transform. So, in fact I could straight away sketch this. It is essentially the magnitude of sin omega by 2 by omega by 2. A sketch would look like this. This is the very familiar function to most electrical engineers. We call it, the so called Sinc function, the Sinc pattern. People have different names for it; they call it the sampling function, the Sinc function and whatever other names. Anyway this is the point where omega by 2 is equal to pi or omega is 2 pi. And this of course, is the point where omega is 4 pi and this where it is minus 2 pi, this where it is minus 4 pi and so on so forth.

At this point the magnitude takes the value 1. In fact, we call that the magnitude of the Fourier; in fact, the value of the Fourier transform at omega equal to 0 is indicative of the area and phi t. So, the integral of phi t overall t.

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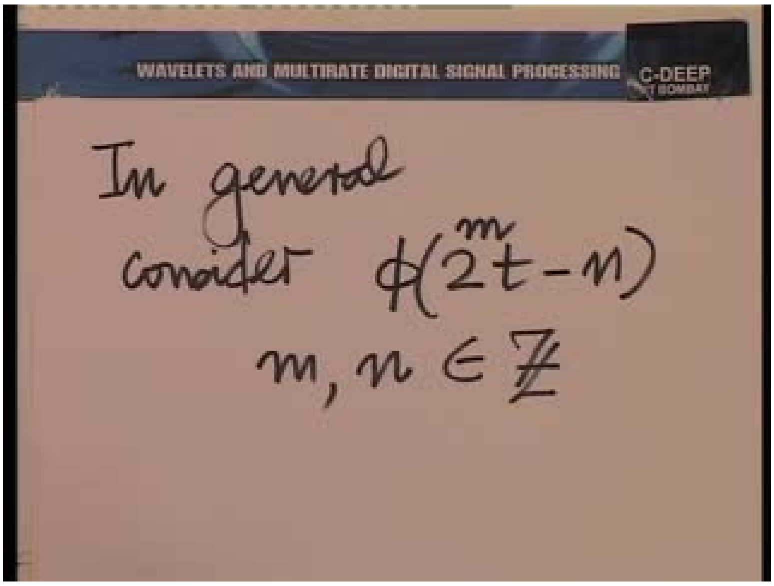

Anyway so much so for the magnitude and now, we know what to do when we compress and expand? Let us look in general, at the Fourier transform of phi 2t minus n, so we should interpret that. You know we are talking about dyadic dilates and translates. So, let us consider phi, in fact, 2 raise the power of m t minus n if you please, for m and n belonging to the set of integers.

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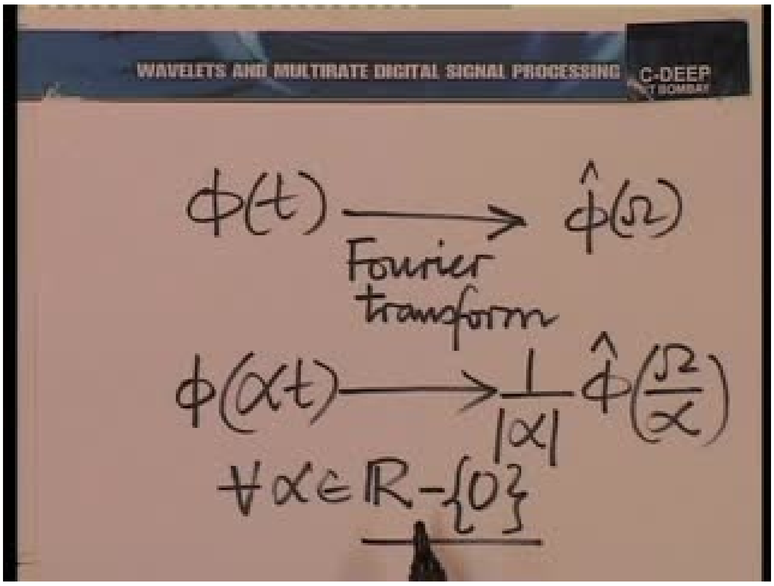

Now, you will recall that there is a very simple result in the Fourier transform which says, that if phi t has the Fourier transform phi cap omega as we do. In general, then phi alpha t has the Fourier transform, 1 by mod alpha phi cap omega by alpha, for all alpha belonging to the real numbers other than 0. This notation says all real numbers except 0. Of course, alpha cannot be 0. So for example, even if alpha is negative we can use this. In particular for example, if alpha is minus 1 we have a reflection of the Fourier transform as well.

So, using this, we now take care of the two so called distortion or modification that we made in phi; the translation and the dilation. In fact, the translation does not affect the magnitude. The translation only affects the phase or the angle of the Fourier transform. So, I can even forget about the translation, I need only look at the 2 raise the power of m term there.

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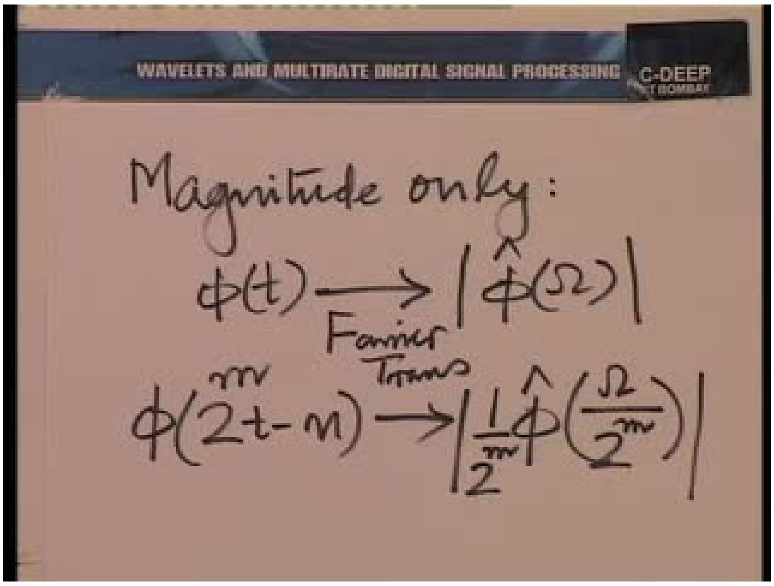

So, here, if I restrict myself to magnitude only then phi t has the Fourier transform, phi cap omega with the magnitude of mod of this. Phi 2 raise the power of m t minus n, would then have a Fourier transform mod phi cap omega divide by 2 raise the power of m. Of course, with the constant, the same 2 raise the power of m, but in the denominator here.

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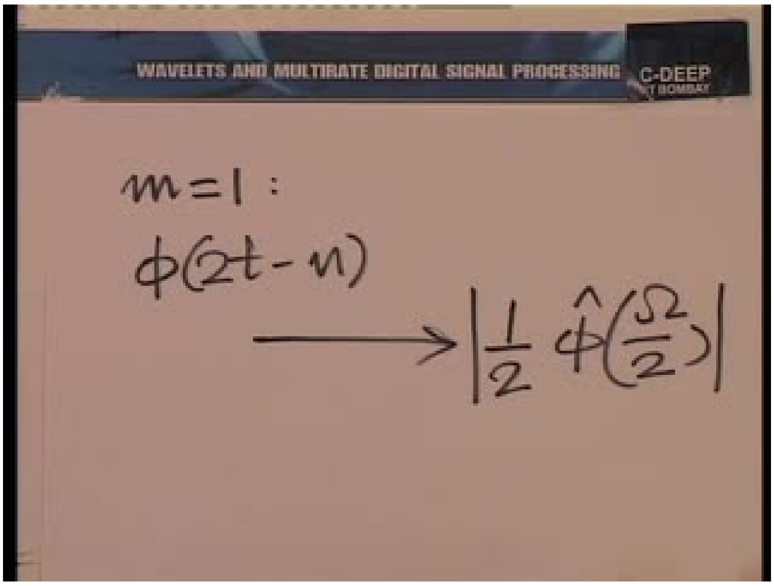

So, Of course, let us take an example, suppose m is equal to 1 and minus 1 to fix our ideas and n equal to minus 1.

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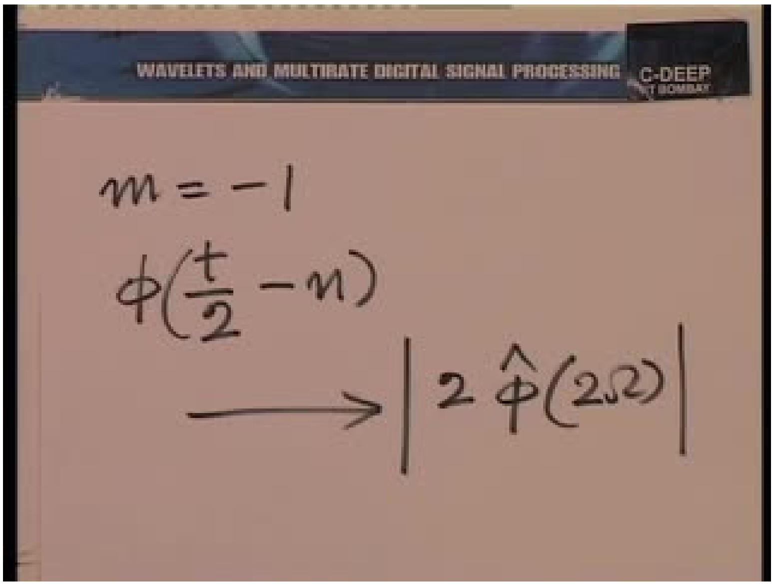

Now, notice that the n is entirely absent here.

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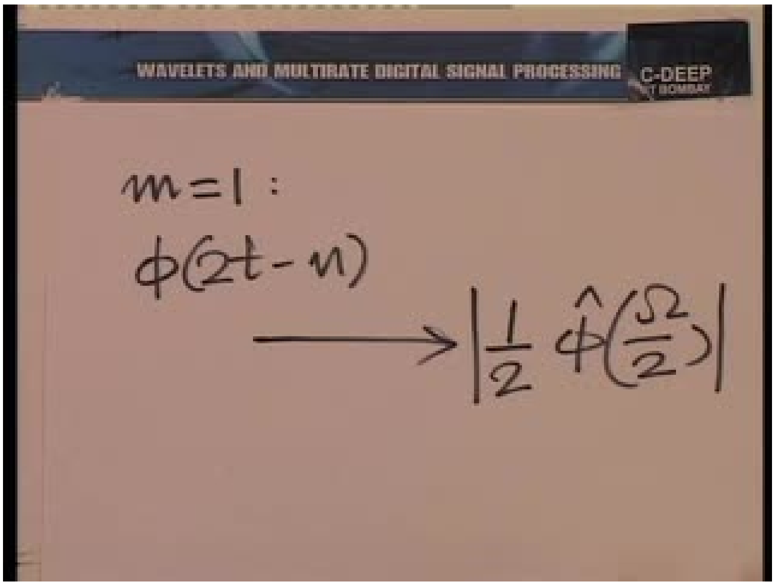

Or here.

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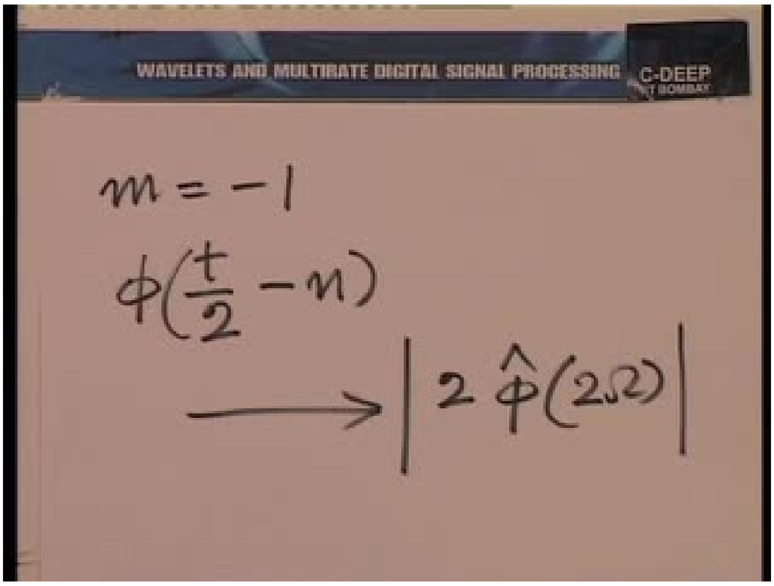

The n is irrelevant as far as the magnitude goes; the n only contributes to the phase. Let us sketch both of these, how would these look?

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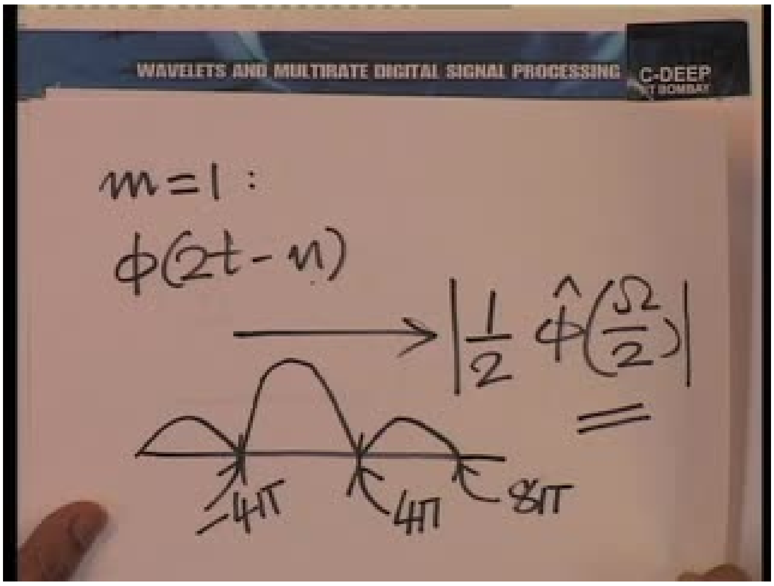

So, let me take the m equal to 1 case. Let me sketch this, so omega by 2. So, if I focus my attention on the main lobe and the principle side lobe that I have here. This was originally 2 pi, now it is become 4 pi. This was originally 4 pi, now it is become 8 pi and so to on this side. This is minus 4 pi and so on. See a stretched by a factor of 2. Let we put them together, the original Fourier transform and this Fourier transform with m equal to 1.

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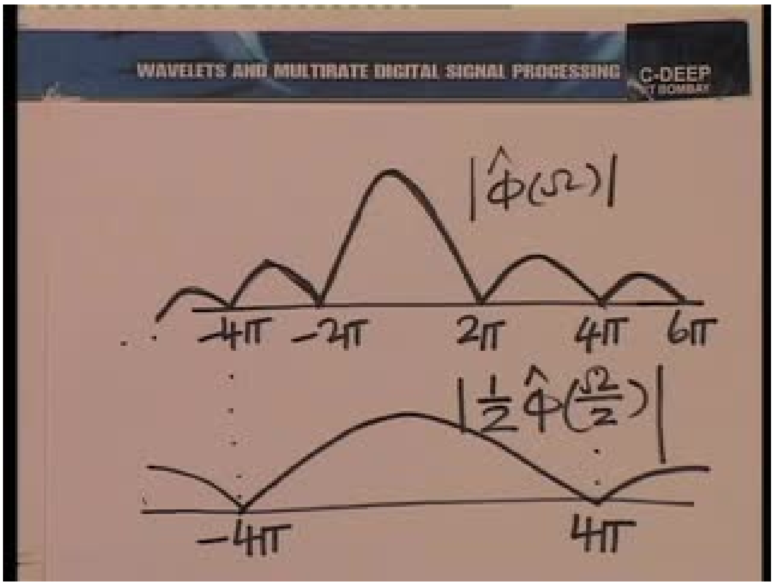

and so on. This is I am just sketching mod phi cap omega and here I sketch, mod phi cap omega by 2 multiplied by half. Of course, this should be smooth here, all these are smooth.

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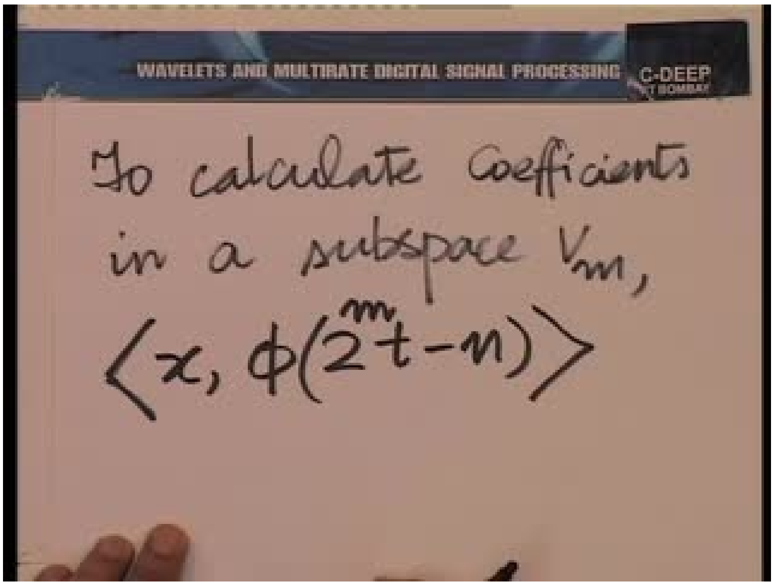

So, as expected when we squeeze in time, we have stretched in frequency and now, let us interpret what we are doing, when we take a dot product, to calculate the coefficients in a V m. To calculate coefficients in a subspace V m, what are we effectively doing? We are taking an inner product, namely, the inner product of an x with phi 2 raise the power of m t minus n, remember. Now, you know there is of course, a normalization here. So, if you want to work with an Orthonormal basis, then it should not quite be phi 2 raise the power of m t minus n. One must normalize it to make it unit norm.

So, I take an instance. Let us take any arbitrary m and let us look at the norm. So, if we consider phi 2 raise the power of m t and again, the minus n does not affect the norm. I am talking about the L 2 norm and therefore, we can as well take without loss of generality, we could take n equal to 0.

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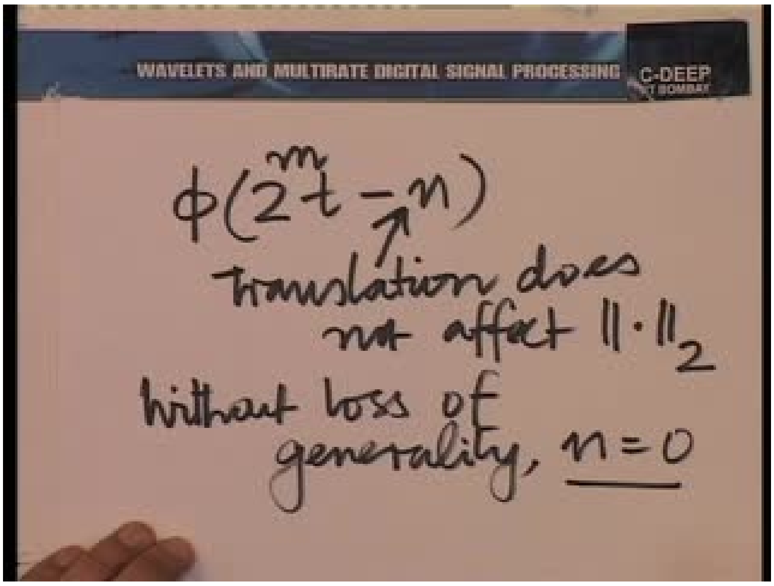

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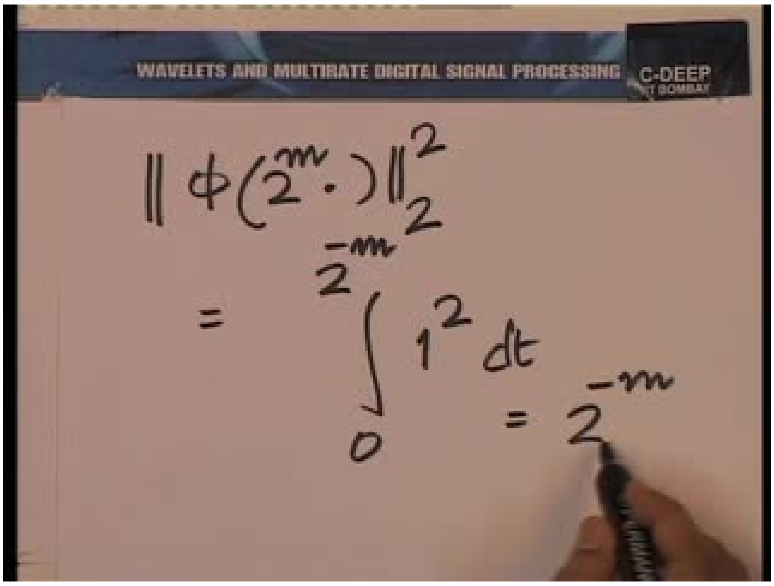

Let us find out the norm therefore. So, I am putting 2 raise the power of m and then a dot, dot denotes the argument of the function, but we are treating the function as an entity. So, I do not use the explicit argument here. The norm of this is essentially, integral. Now, you know when you go over phi 2 raise the power of m dot. You are talking about 0 to 2 raise the power minus m here, 1 square dt and that is obviously, 2 raise the power minus m. And therefore, if you want to make this unit norm, then you must divide by; this is of course, the square of the norm.

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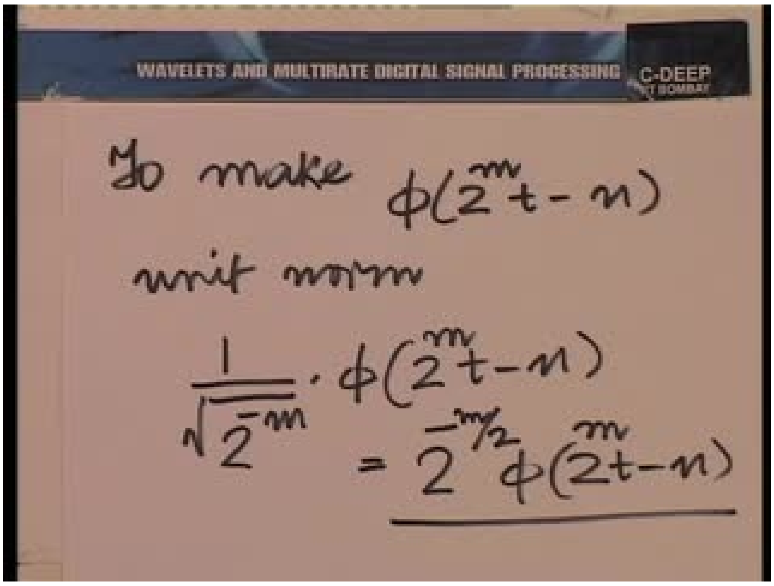

So, if you want to make this unit norm, you must divide the function by the square root of this. We must consider 1 by square root of 2 raise the power minus m times phi 2 raise the power of m t minus n and that is easily seen to be 2 raise the power minus m by 2, phi 2 raise the power m t minus n. So, this is the unit norm, this is now an orthonormal basis. So, let us make that note.

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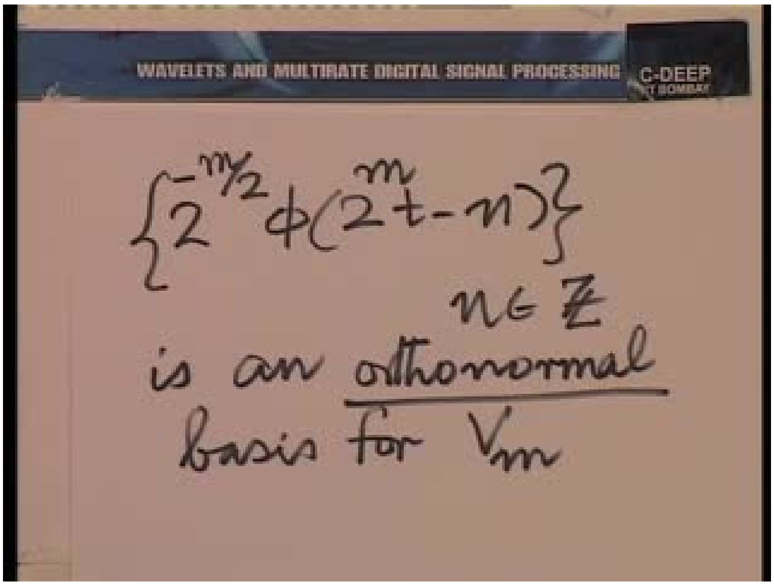

2 raise the power minus m by 2, phi 2 raise the power m t minus n, for all integer n, is an orthonormal basis for V m. You know, what V m means? V m is the mth subspace in the

ladder of subspaces that leads to L 2 R as you go right wards and to the trivial subspace with only the zero element as you go left wards. So, it is that mth subspace in the ladder, and now we have an orthonormal basis for it. Anyway, now let us interpret what happens, when we take the dot product of any function in L 2 R, with an element of this orthonormal basis.

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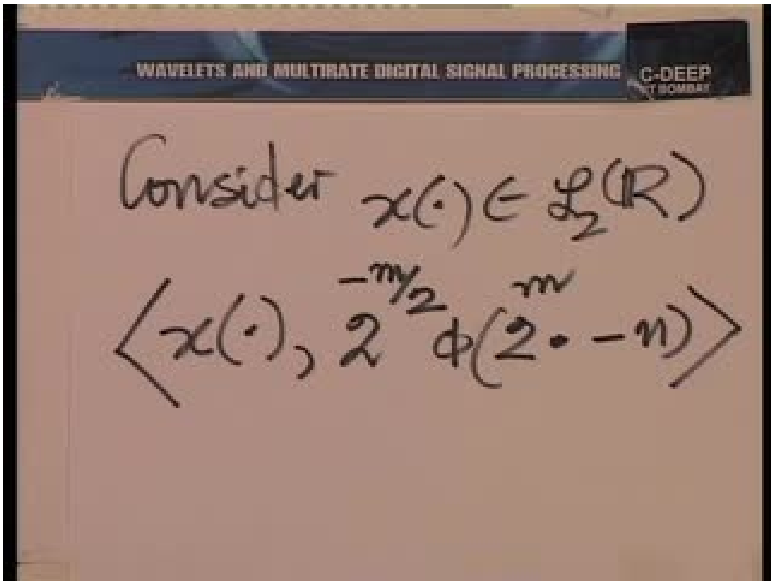

So, consider x t belonging to L 2 R or if you please x with the argument belonging to L 2 R as is the correct way to write it. And then, consider the inner product of this x, with this orthonormal basis element 2 raise the power minus m by 2 phi 2 raise the power m dot minus n. Now, we are going to invoke, the Parseval's theorem. You will remember that we had discussed the Parseval's theorem a while ago, in one of the earlier lectures. When I had talked about the relationship of functions and vectors, I had mentioned the significance of Parseval's theorem. There are different ways of stating it. Parseval's theorem on one hand says; that the inner product is preserved as we go from time to frequency.

Now, here if we use angular frequency a factor of 2 pi is needed. If we use hertz frequency, that 2 pi factor is not required. But since, we are working with angular frequency; it would be safer to retain that factor of 2 pi. But, that factor of 2 pi apart, what Parseval's theorem says is that, after all when you go from the function to its Fourier transform in effect, you are representing the same function in a different basis.

You are representing the function, with respect to the basis formed by the rotating complex exponentials or the phases for different frequencies omega. The Fourier transform is essentially, a projection of a function; in this case a function in L 2 R, on a particular element of that basis, a particular rotating phase. The inverse Fourier transform, reconstructs the original function from its components. It is worth recalling some of these points final points because, it helps for them to be firmly embedded in our consciousness in a course like this.

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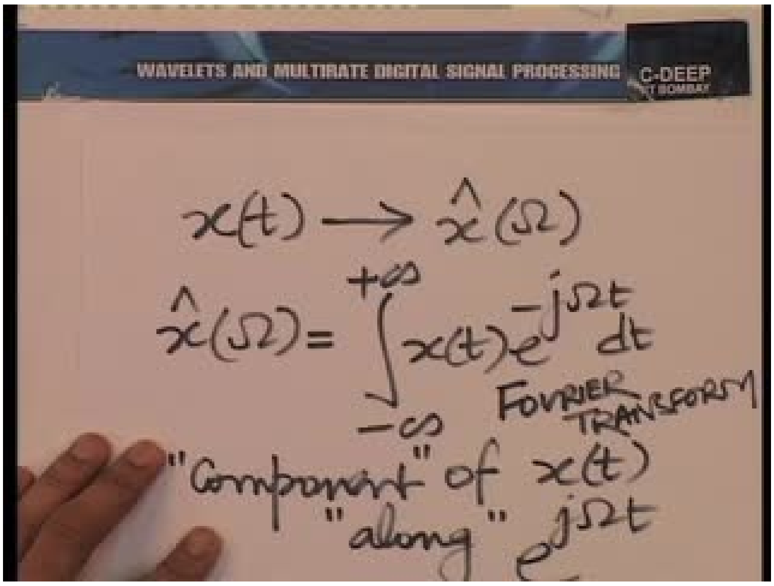

So, even if it means a little bit of repetition. Let us emphasis those points again. What we said was, if I took the Fourier transform, so if I have x(t) and its Fourier transform, x cap omega so to speak and x cap omega is essentially a projection x(t) e raise the power minus j omega t dt. A components of x(t) along e raise the power j omega t. This is the Fourier transform and the inverse Fourier transform this is so, let us write this down, this is the Fourier transform.

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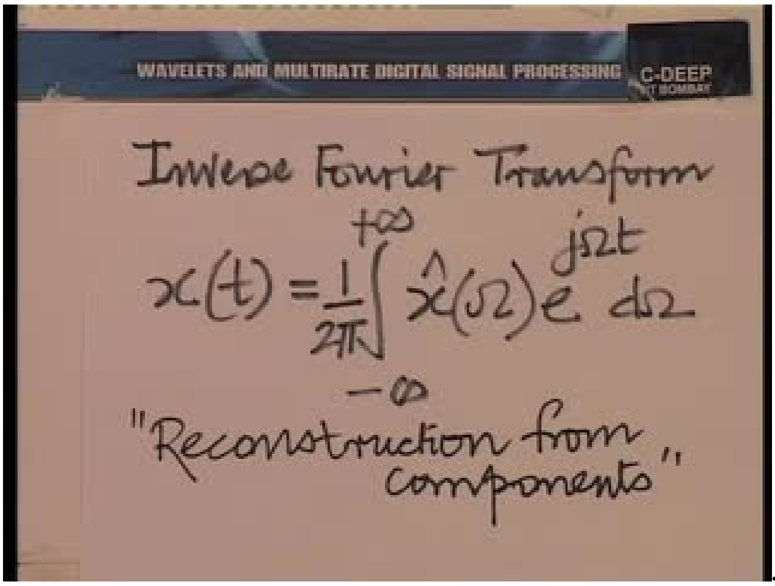

And we have the inverse Fourier transform, which reconstructs x(t) from its components with the factor of 2 pi here and you will recall the interpretation that we gave this. We had said that essentially, this is the component along a particular omega and this is the so called unit vector with the factor of 2 pi. So, if I take this and this together, it is like a unit vector here and what we are saying here is the original function is essentially, the component multiplied by the unit vector integrated over all the components, reconstruction of a vector, reconstruction from components such the interpretation.

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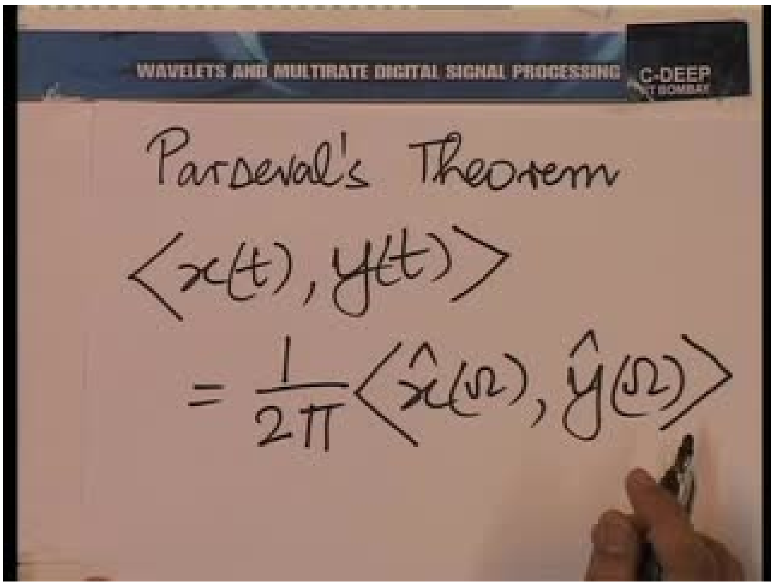

Now, with this spec round, what we have said in Parseval's theorem is the following. The inner product of x(t) and y(t); so, you know I am talking about two different domains is equal to the inner product of x cap omega and y cap omega, but with the factor of 1 by 2 pi. In the language of components, what is the interpretation? The interpretation is that, it in calculating the inner product, it does not matter whether one is using one orthonormal basis or another. The result is the same. The inner product has nothing to do with the choice of basis. The inner product between two vectors remains the same, whatever basis we choose to express the vectors. That is the statement being made in Parseval's theorem here.

We are saying, represent the functions in the natural basis of impulses or represent them in its Fourier basis, the inner product is the same. Of course, to within a factor of constant, this constant appears, because of the angular frequency radians, radians per second I mean. Otherwise, if you want to take hertz frequency this factor would be absent as well. Anyway, this was an important result that we had seen, when we looked at functions from a perspective of vectors and now we shall use this Parseval's theorem to interpret this idea of projection on to V m.

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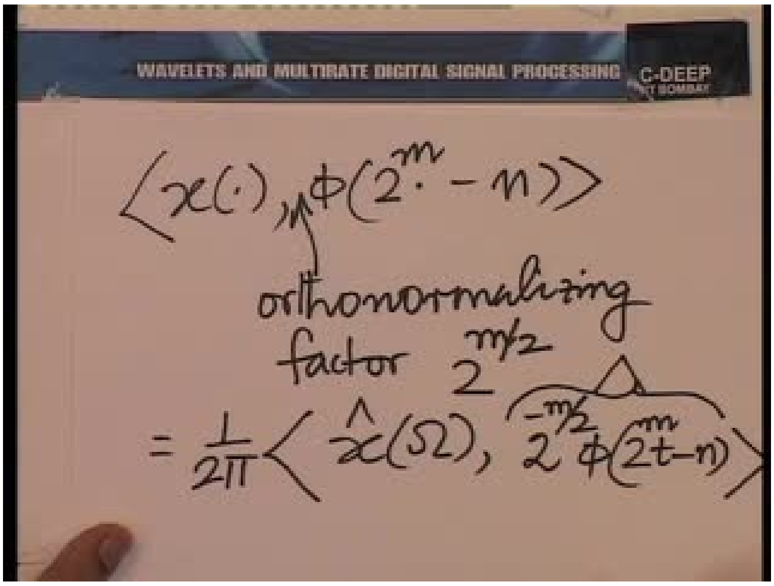

So, when we consider, the inner product that we were doing of few minutes ago. The inner product of x, with phi 2 raise the power m t minus n and all if you wish to make it orthonormal, then introduce the orthonormalizing factor, 2 raise the power m by 2. This is going to be equal to 1 by 2 pi times the Fourier transform of each of them. So, the Fourier transform of x, of course, with its argument. The argument, remember is going to be omega here and I shall write here, the Fourier transform of this whole things. So, please do not misunderstand what I am writing to be the expression itself, but understand it to be the Fourier transform of this.

So, you have 2 raise the power of m t minus n, but take the Fourier transform. So, I am saying Fourier transform of this whole thing here. Now, I would like to interpret this graphically first. What are we going to do when we take the inner product in the Fourier domain; so, what is this going to look like?

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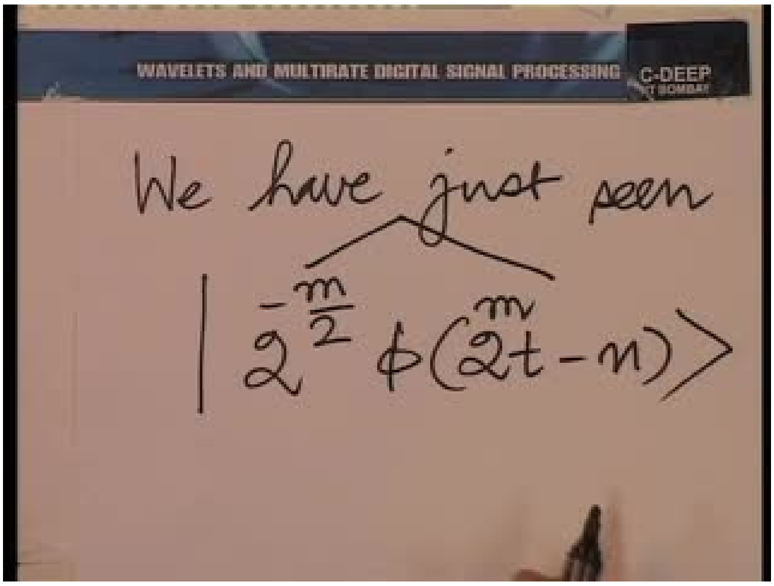

We have just seen the magnitude of this quantity. And that looks like this. This is the place where, you have 2 pi multiplied by 2 raise the power of m. Some effect you have expanded that main lobe and all the side lobes by a factor of 2 raise the power of m. In particular with m equal to plus 1, you have expanded by 2. If m is equal to minus 1, you would have contracted it by a factor of 2 and so on.

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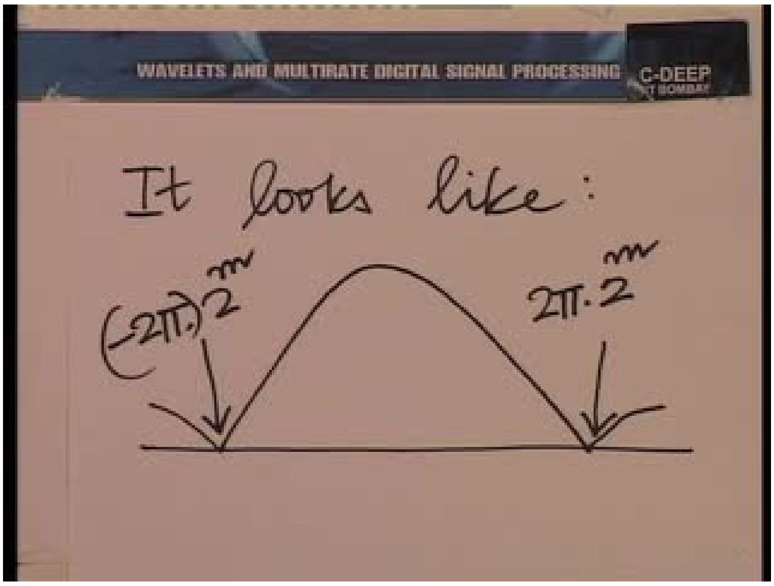

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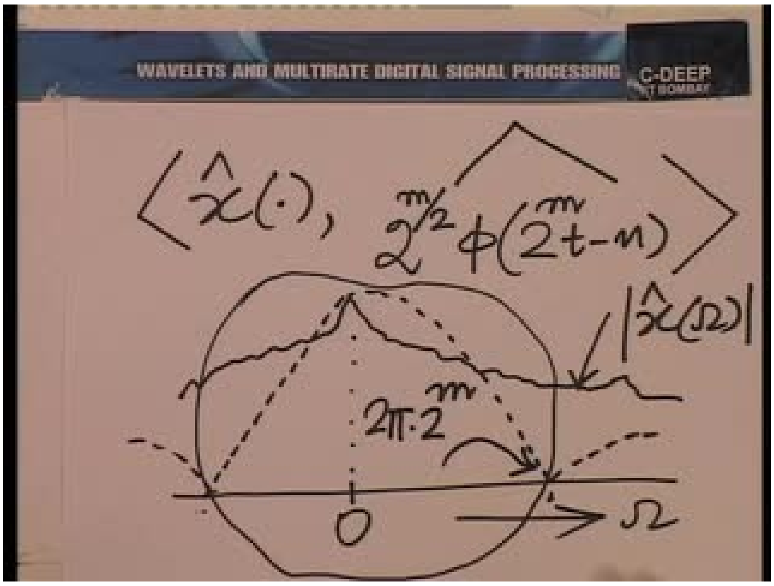

So, when we take the inner product, in the inner product x cap and the cap of this. What are we going to do? We have the Fourier transform, so let us understand it graphically as I said. Let us say, this is the Fourier transform of x whatever it be I mean, let us understand in the magnitude sense first. So, let this be the magnitude of the Fourier transform of x. This is 0, let us say here and this is the magnitude of the Fourier transform of the other argument, I am showing only the main lobe and a part of the other side lobes. When you multiply them, their magnitudes are going to get multiplied and therefore essentially, you are going to extract this band so to speak.

In a notional sense, we are going to emphasis most the area of the Fourier transform around the main lobe of phi. You see after all, it is the magnitude which plays the significant role here. When we take the dot product, we are going to multiply the two Fourier transforms and integrate overall frequencies. Where the magnitude is larger, the contribution will be larger. Where the magnitude is smaller, the contribution will be smaller. So, the side lobes would kind of suppress, that part of the Fourier transform and the main lobe would emphasis, the corresponding part of the Fourier transform contained under the main lobe. What we are saying is essentially, that part of the Fourier transform of the original function x, which is contained in the main lobe is emphasized as against all the rest, in calculating the area.

Now, this also gives us an interpretation of what happens when we increase or decrease m, in frequency. In fact, let us look at this drawing once again.

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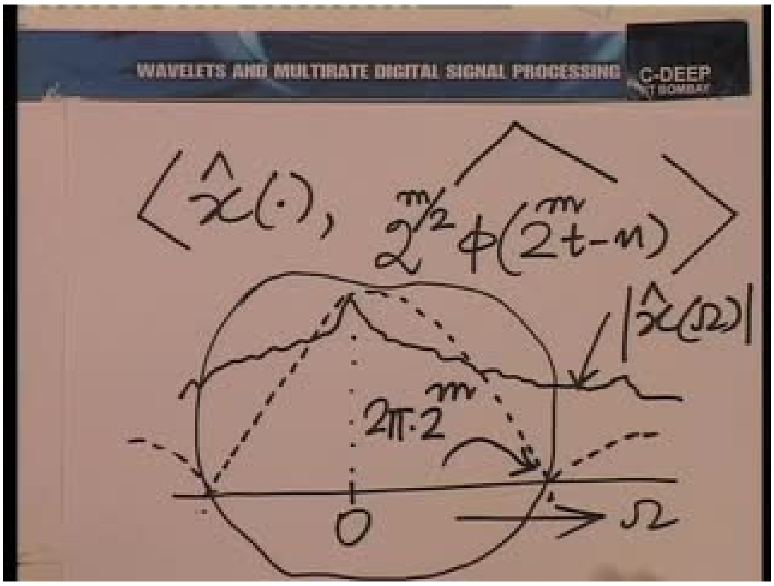

This point is 2 pi multiplied by 2 raise the power of m. So, take for example, m equal to 0. You are essentially this dotted line here, is the magnitude of the Fourier transform of the appropriate dilate of phi and of course, we do not care so much about the translate. The effect of the translation is only to change the angle and it does not reflect in the magnitude. So, for m equal to 0 we are essentially emphasizing, a region of the frequency axis broadly speaking, between minus 2 pi and plus 2 pi. When we take m equal to 1, we are emphasizing a region between minus 4 pi and plus 4 pi. When we take m equal to minus 1, we are emphasizing a region between minus pi and plus pi and so on so forth, add and fill it up.

What are we saying? When we increase m; m 1, 2, 3 and so on; we are effectively keeping more and more information around the zero frequency, we are broadening it. Of course, we are narrowing in time but, we are broadening in frequency. So, we are keeping a larger band of frequencies, but all around the zero frequency. Now, what happens, when we consider psi t that is unequally interesting interpretation? Let us do that. In other words, here we have analyzed the implications of taking the projection on one of those subspaces in the ladder. What happens when we project on one of the incremental subspaces? That is also an interesting question.

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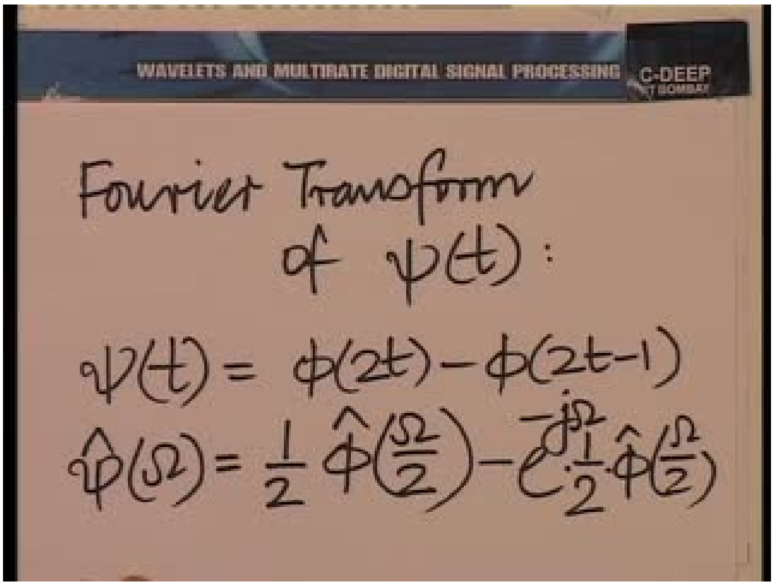

So, for that you must first consider the Fourier transform of psi t and that is easy to do. Psi t as you know is phi of 2 t minus phi of 2 t minus 1 and you can easily find its Fourier transform. Psi cap omega is therefore, going to be equal to; well, when you multiply by 2 here, you would be dividing by 2 in the other domain and of course, here we need to take care of the minus 1. So, e raise the power minus j omega times the same expression. We can evaluate this. So, you know it is I think maybe we should it would be better to come to this a little more systematically.

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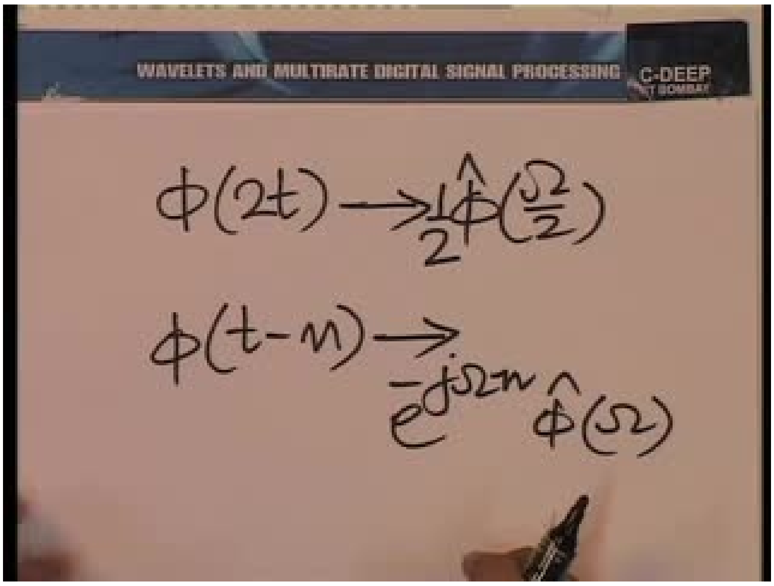

So, phi 2 t would have the Fourier transform, phi cap omega by 2 multiplied by half. Now, phi of t minus n would in general have the Fourier transform, e raise the power minus j omega n times phi cap omega and when we consider phi of 2 t minus n in general.

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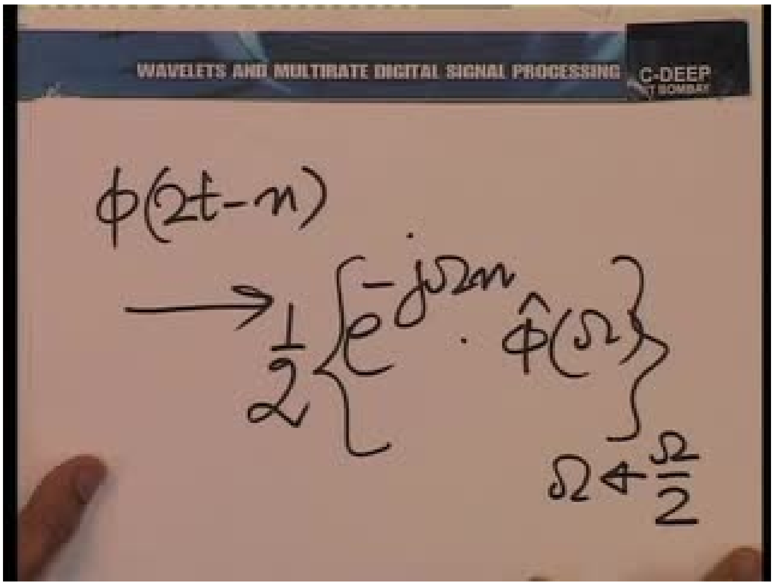

So, here we are going to replace t by 2 t. We should take this replace omega by omega by 2 and then multiply by half. So, this replacement must be in the whole thing.

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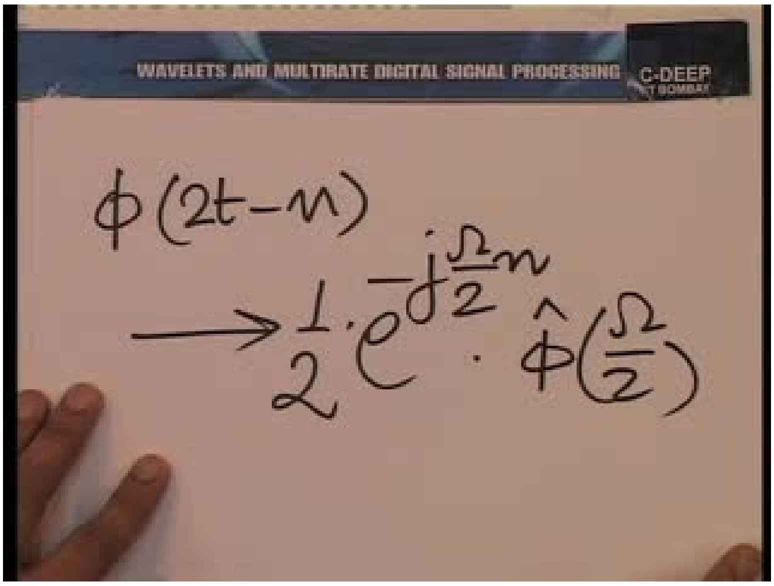

And therefore, phi of 2 t minus n would have the Fourier transform, half e raise the power minus j omega by 2 times n, phi cap omega by 2 and therefore, we need to make a little correction here.

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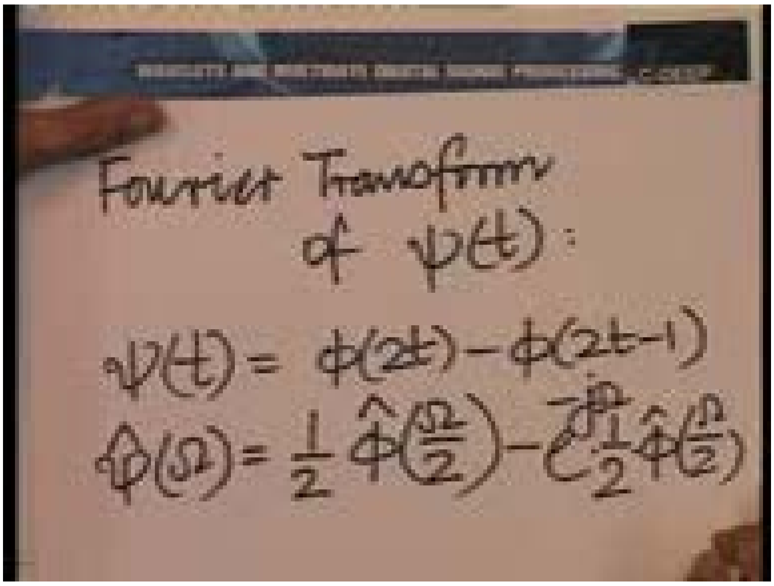

The correction is we need to replace omega by omega by 2 here. So, that we can rewrite that part of the expression. It is more convenient.

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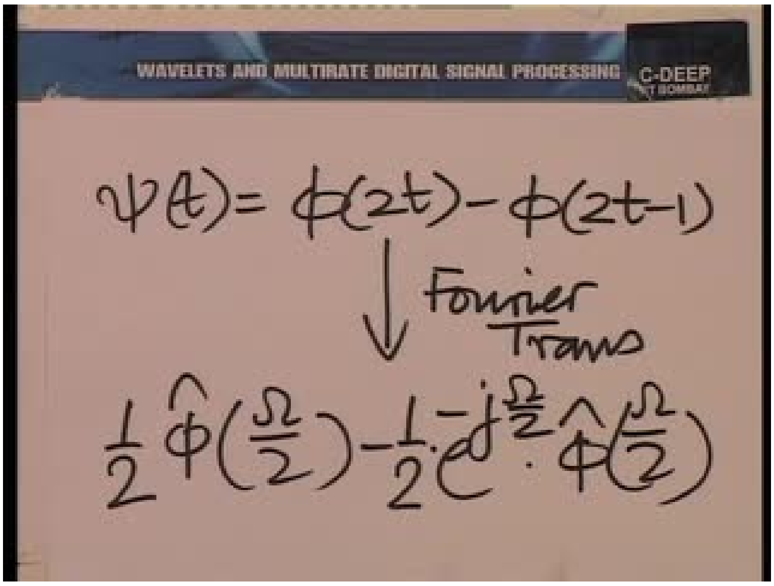

Psi t which is phi 2 t minus phi 2 t minus 1 has the Fourier transform, phi cap omega by 2 minus e raise the power minus j omega by 2 times phi cap omega by 2.

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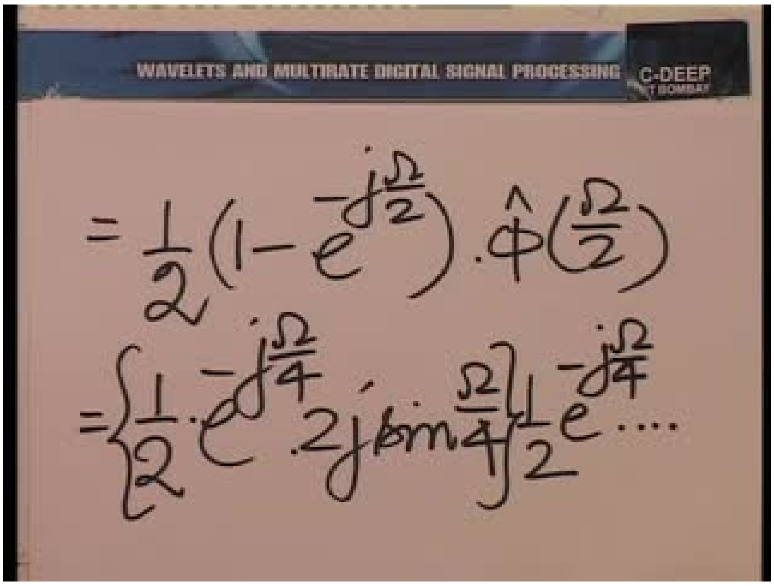

And if we aggregate terms, we have phi cap omega by 2 and this is easy to write. This is half; well we can play the same trick. We can extract an e raise the power minus j omega by 4 common from here and get 2 j. I am skipping the couple of steps; sin omega by 4 and you know the expression for this. This is essentially, 1 by 2 e raise the power minus j omega by 4 times; well, I continue on the next page. It is really complicated, but not too difficult.

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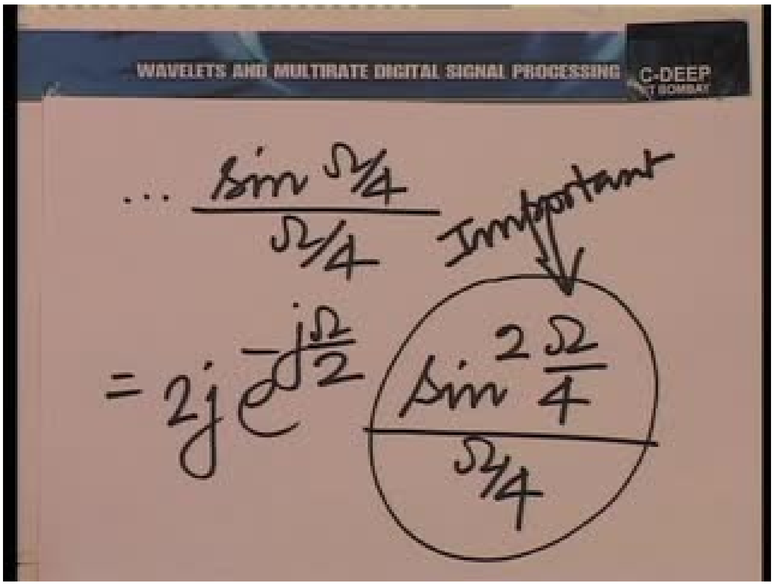

Sin omega by 4 divided by omega by 4. In fact, let us multiply all this and put it together. So, I have e raise the power minus j omega by 2 coming together j there, sin squared, omega by 4 divided by omega by 4 and a factor of 2 remains outside. So, what we need to focus on here is essentially this part. This is important, the rest of it is not because, it is this that really affects the magnitude seriously, rest of it is essentially a constant magnitude.

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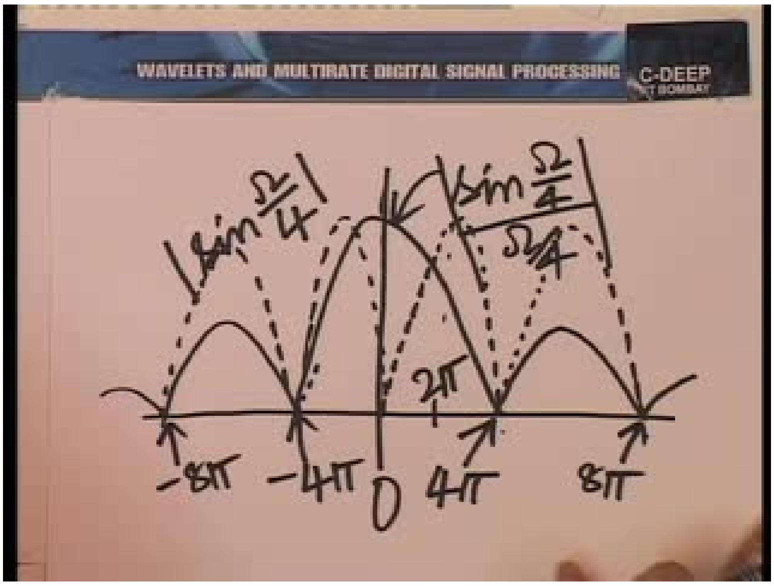

So, let us look at the magnitude of sin square omega by 4 by omega by 4. First, let us look at sin omega by 4 by omega by 4. That of course, would look like this, so this is the point 4 pi, minus 4 pi and so on and then, on the same graph I will draw this is mod sin omega by 4 by omega by 4. And on the same graph, I will show in dotted mod sin omega by 4 once again. So, that is going to look like this. It is going to have a maximum at pi by 2 and then it is going to see omega by 4. So, one cycle will be completed when omega by 4 equal to pi that is omega equal to 4 pi.

So, one cycle been completed here, mod sin is going to look like this. So, in each span of 4 pi you have one half cycle being completed. So, this is one half cycle. Now, in the next span of 4 pi again, you have one more half cycle being completed, like this. This is the situation. Now, look at the situation, this was the zero frequency here. The solid line has a maximum at zero and it tapers of up to 4 pi on both sides. The dotted line reaches a maximum in between at 2 pi, when omega by 4 is equal to pi by 2 alright.

So, omega is 2 pi, this point, the maximum here occurs at 2 pi and therefore, when you multiply this decreasing function by increasing function here. There is going to be some point of maximum in magnitude somewhere in between 0 and 4 pi and it is going to taper off to 0 at 4 pi again. So, let me focus, using the other side lobes. It is easier to understand. For example, between 4 pi and 8 pi, it is very easy to understand. You are multiplying essentially, two similar looking functions and so, you can see this is going to be a maximum, somewhere it is going to taper off. And it is also clear that the maxima in the other side lobes are going to be weaker in the than the maxima in the main lobe here. All this, when you multiply the dotted function and the solid function, you are going to get a pattern something like this which I now sketch.

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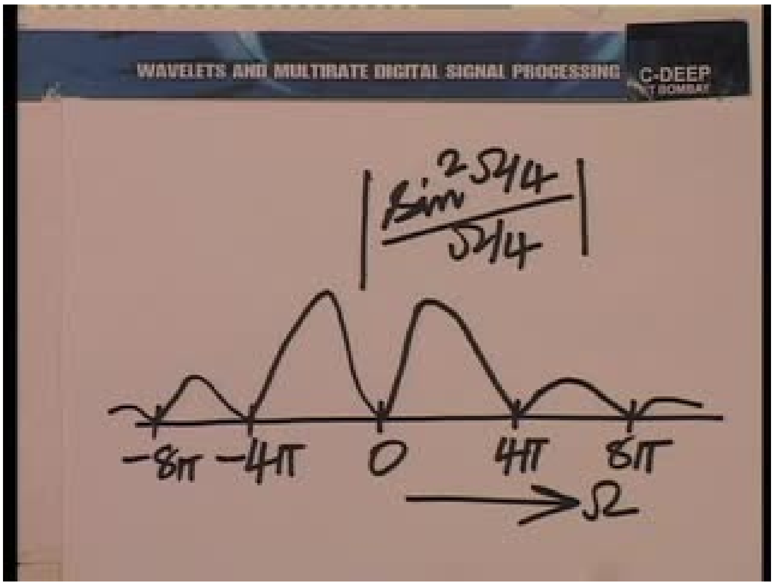

Something like this this is going to of course, be mirror in this side and then, this way, this way and so on. So, this is mod sin squared omega by 4 by omega by 4, as a function of capital omega. Let us take a minute to look at this.

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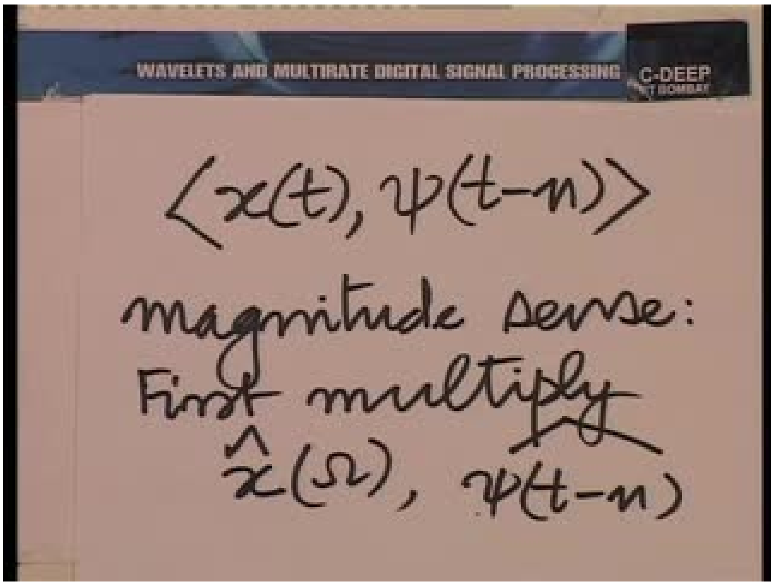

When we take an inner product, an inner product of a function x(t). Now, here I am writing t, to explain that inner product is in time, with of which is psi of t minus n. In the magnitude sense, what are we doing? We are first multiplying the Fourier transform of x(t), with the Fourier transform of psi t minus n. This should be understood to mean the Fourier transform of psi t minus n, a cap on the whole function. So, when you multiply them and then, integrate that part of the Fourier transform of psi t minus n, which is significant in magnitude is going to be extracted out of the Fourier transform of the original function x and what is that significant part?

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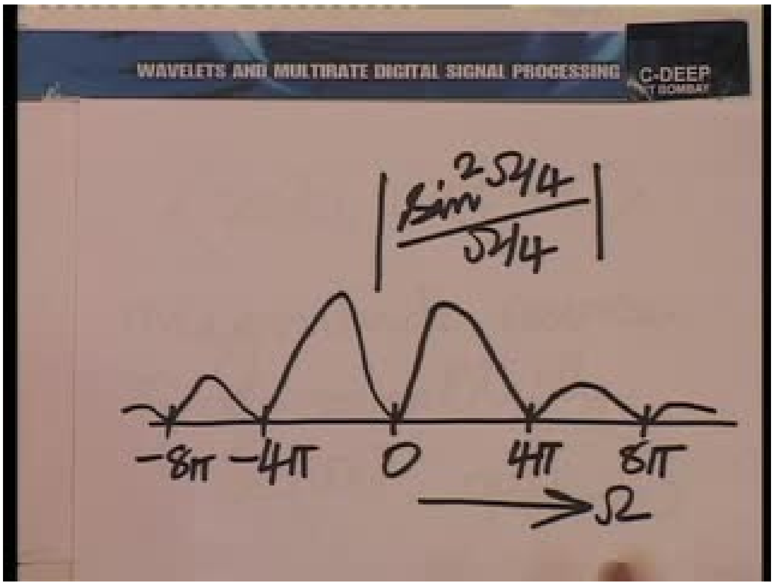

Let me put it back here for you; it is therefore, going to extract give emphasis to a band now. Not a band around zero, a band around some other frequency here. Of course, just symmetry, is always symmetry in frequencies. So, you could focus on the positive side of the frequency axis, but what is going to be done is to emphasis a band here and instead of psi of t, if you take psi of 2 t, for example.

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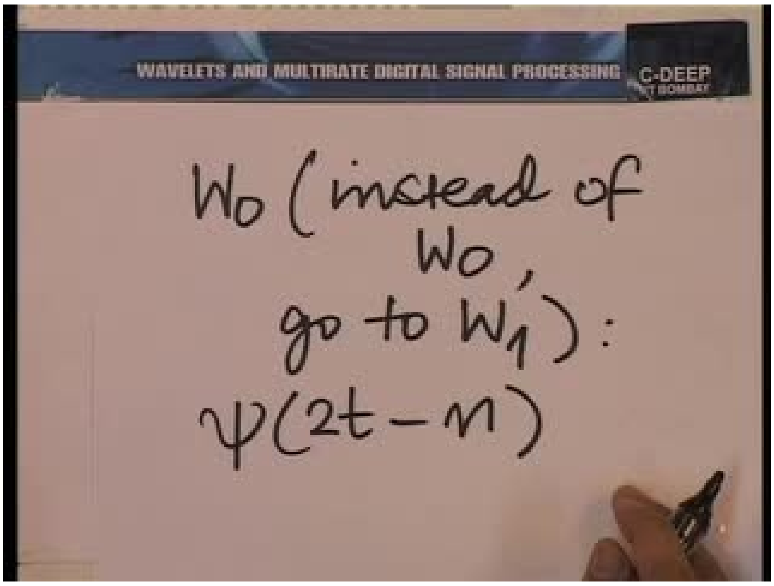

So, if you take, say instead of W 0. If you go to instead of W 0, go to W 1, in which case you would have psi of 2 t minus n and how would the Fourier transform of psi of 2 t minus n look?

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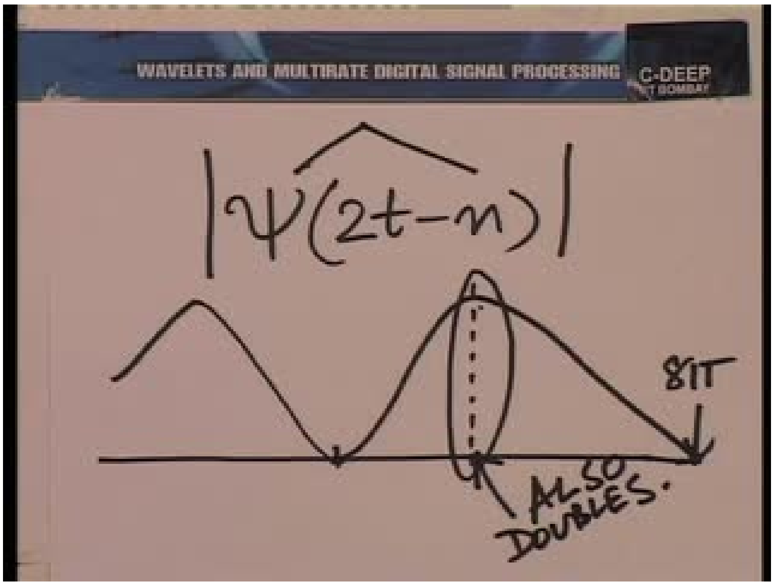

So, if we take the magnitude, if we take the Fourier transform of psi of 2 t minus n and plot its magnitude, its appearance would be something like this, it will be stretched. So, instead of going from 0 to 4 pi now, you would have a band between 0 and 8 pi, of course, symmetrically on the negative side. Now, there are two things. You see you must keep in mind that, unlike the case of phi, in psi we have two changes taking places. One is that the band expands, the main lobe expands and each of the side lobes expand, but the second is that the center frequency, the point where, this was a maximum here.

You see this center frequency also doubles both the band and the center frequency double. That means now, you are emphasizing the different band, the center frequency is different, each time you go to a different W, you take W 0 you are emphasizing one band. When you take W 1, you are doubling the center frequency. So, you are emphasizing the different band and of course, the band itself has doubled also. When you take W 2, you are again doubling the center frequency, so you are emphasizing a different band and again the band is doubled. So, each time you are doubling the band and you are doubling the center frequency.

So, in a certain sense; that idea of complementarity. You know each time you put one increment layer. You are putting one more band and the band size is doubled here. Now, where in all this is our discontent with the Haar, why are we discontent? The discontent is because, even if we say we are emphasizing a band, it is only true to a certain extent.

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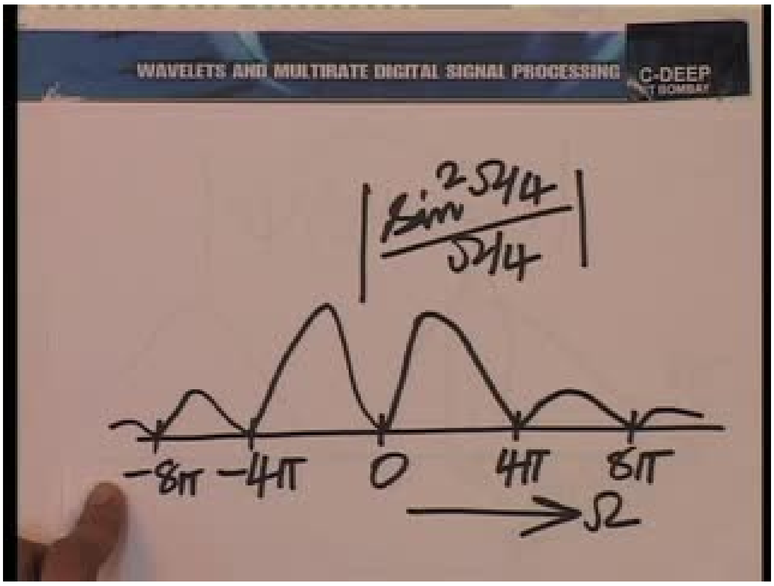

Let me put back this Fourier transform, and illustrate. We might say to an approximation that we are emphasizing this band, but that is only approximate. We are also, to some extent keeping this band and this and there of course, the negative corresponding pieces and that is where we are not content with the Haar. We want to keep a certain band, focus our attention on it. We do not want interferences from the other side lobes that are there and in going to other multiresolution analysis; we are essentially trying to reduce that unwanted presence of the side lobes as much as we can. We have given a feel of, what our discontent is like. We shall build on this further in subsequent lectures. Thank you.