Advanced Digital Signal Processing- Wavelets and Multirate
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Lecture No. # 12
Perfect Reconstruction: Conjugate Quadrature

A very warm welcome to the twelfth lecture; on the subject of Wavelets and Multirate Digital Signal Processing. Let us spend a minute on what we had done in the previous lecture. We had looked at the two band filter bank in the previous lecture. And, we had written down a set of conditions based on what happens when you go past a down sampler and an up sampler in the z domain. So, let us put down the conditions once again or let us in effect, put down the relation between the input and the output in the z domain; where it is valid to use the z domain, which is true in many circumstances.

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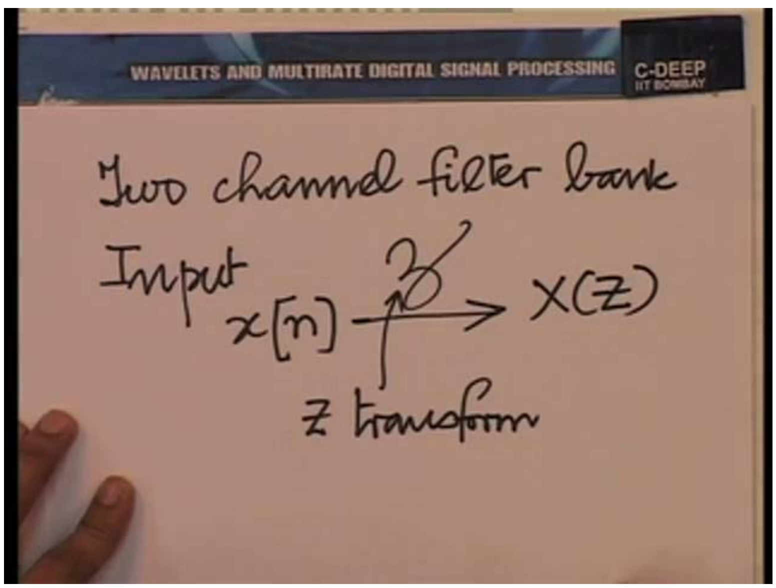

So, let us summarize what we had derived the last time. We said 2 at two channel filter bank, input x n; it is z transform.

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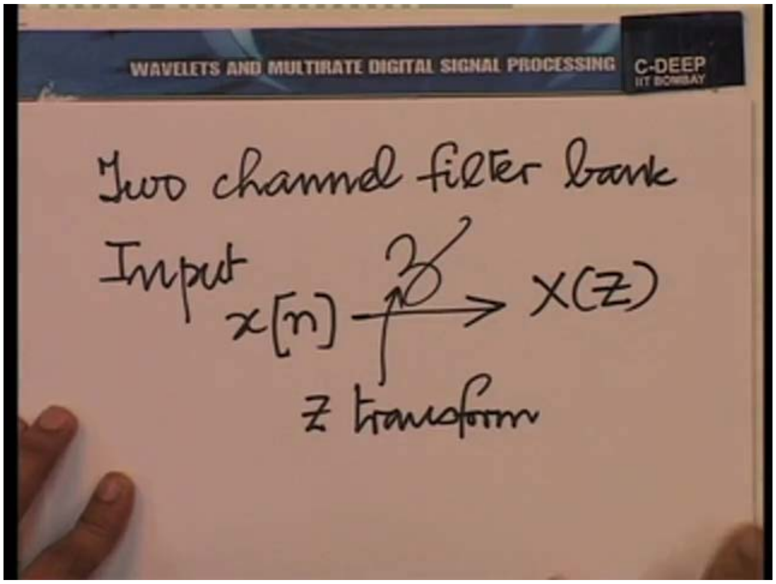

Now, use this script z to denote the z transform, capital X z.

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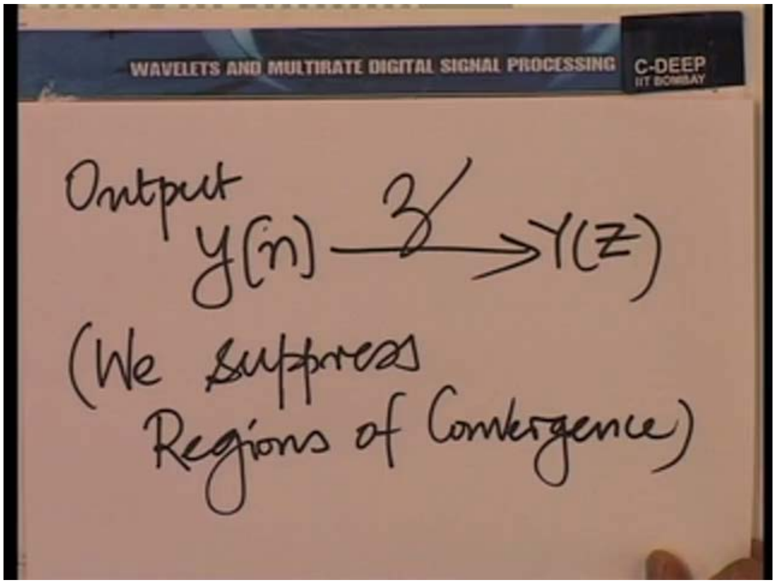

Output y n; which z transform capital Y z. Now, incidentally we suppress the regions of convergence. So, we do not explicitly mention the regions of convergence.

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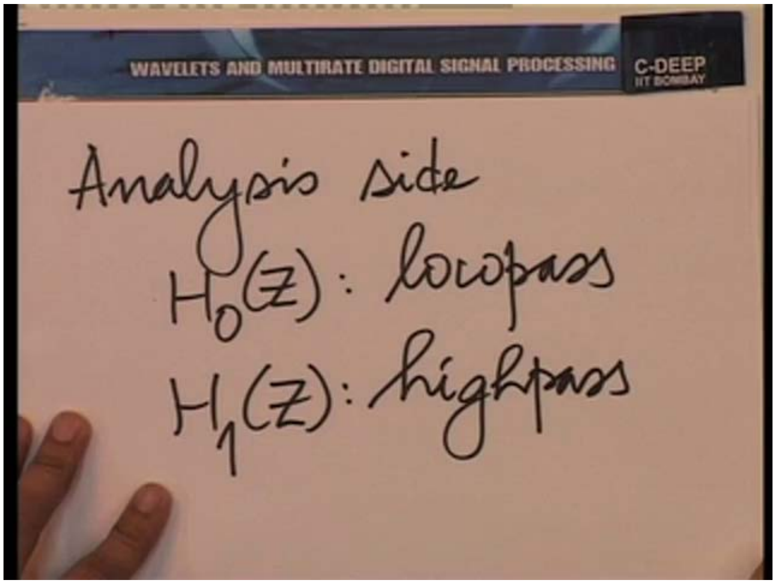

Analysis side: H 0 z, so-called low pass filter and H 1 z, the high pass filter.

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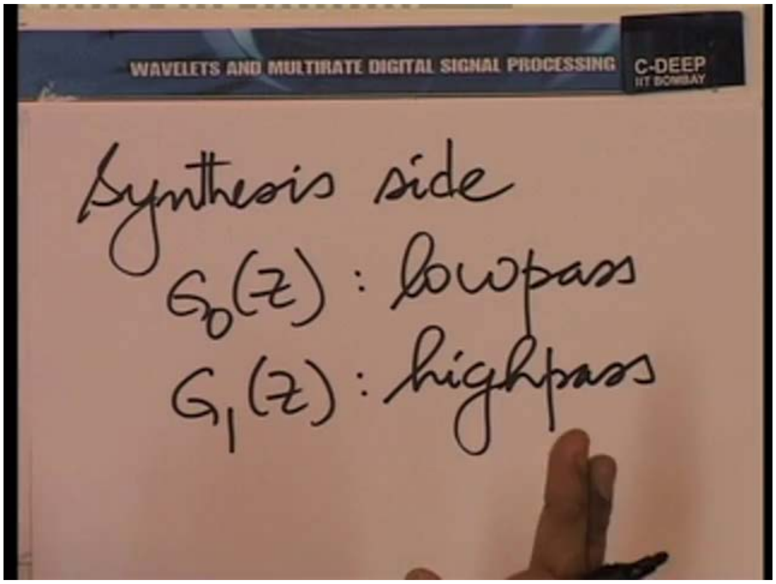

Synthesis side: G 0 z, low pass filter; G 1 z, high pass filter. So, this is the circumstance. Of course, you know where the down samplers and up samplers are. The relation between Y z and X z is as follows.

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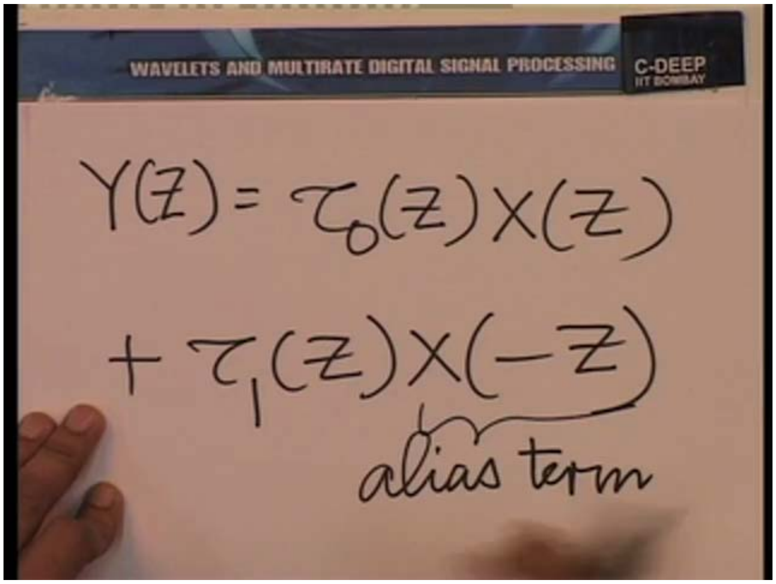

We have Y z is tau 0 z X z plus tau 1 z x of minus z. And, we call that, we call this the alias term.

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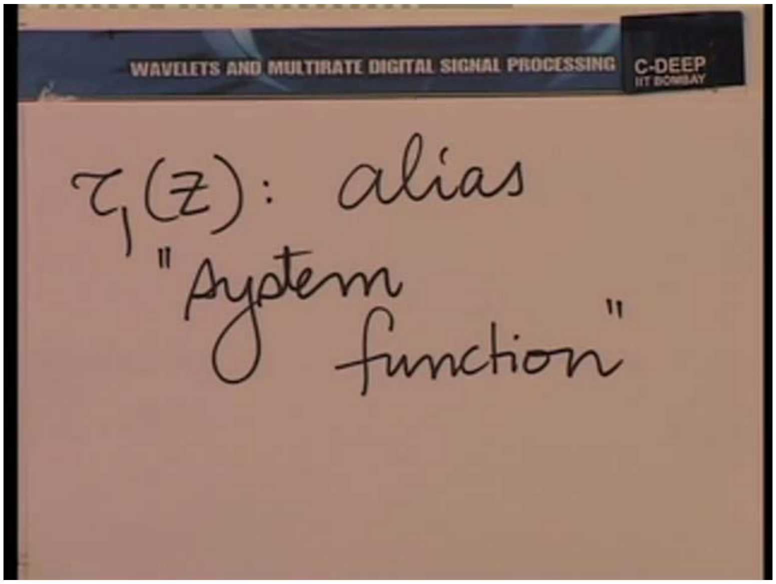

And therefore, tau 1 z was called the alias “system function”. And, put “system function” in inverted commas and it is synthesized here.

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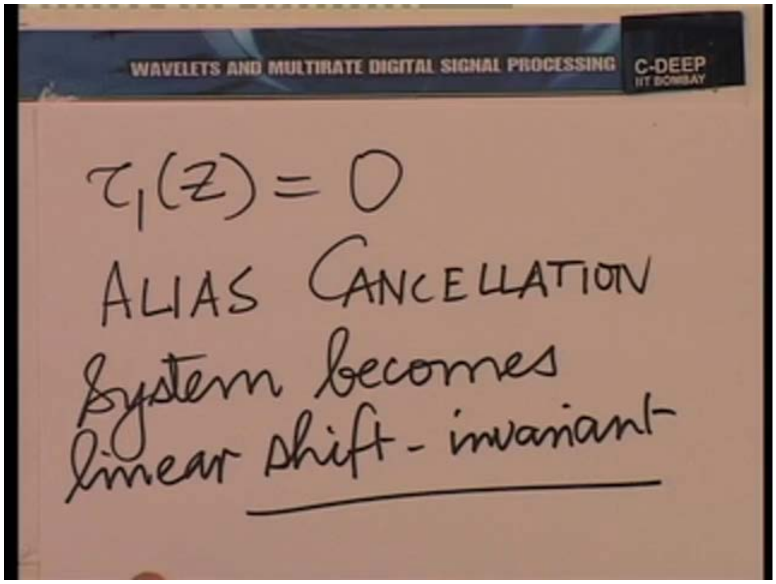

You know, when there is an alias term, the words “system function” is actually a misnomer. One should not use the term system function because the system is not linear and shift-invariant. On the other hand, if tau 1 z is 0 the system becomes linear and shift-invariant.

So, this is something that we must now take note of. Tau 1 z equal to 0, is essentially what is called the condition for alias cancellation. And, we note the system becomes linear and shift-invariant. Now, we have to look at one possibility under which tau 1 z could be zero.

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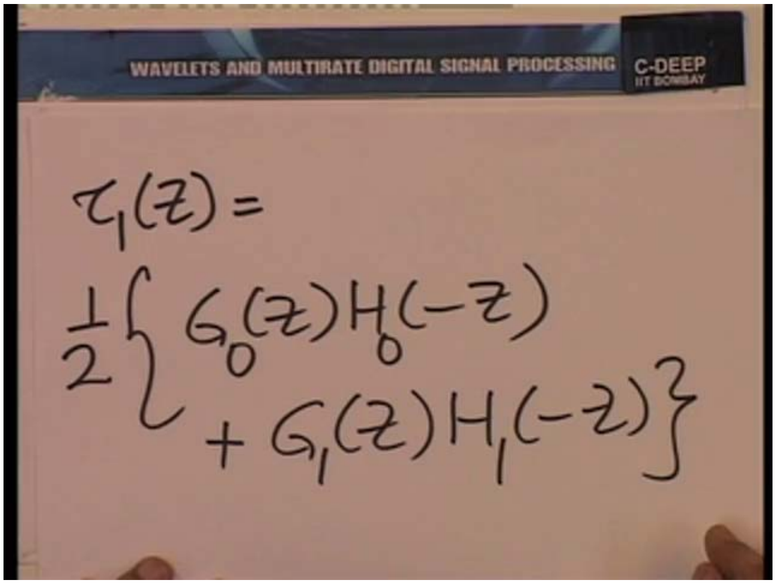

And, we had said the most general possibility can be accommodated by what is called a cancelling term R z. let us put that down again. We said that tau 1 z is essentially the function or the expression half G 0 z H 0 minus z plus G 1 z H 1 minus z.

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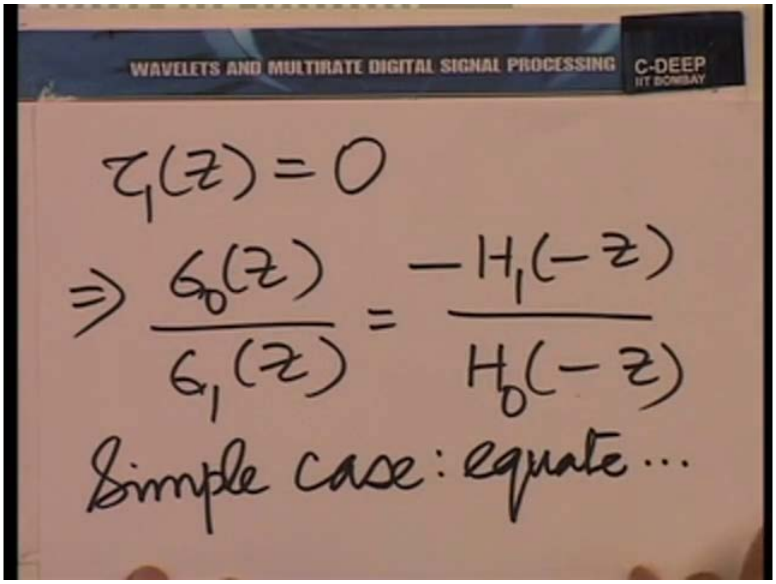

And, tau 1 z equal to 0 means essentially that G 0 z by G 1 z must be minus H 1 of minus z divided by H 0 of minus z. And, the simple case is, equate the numerator and denominator.

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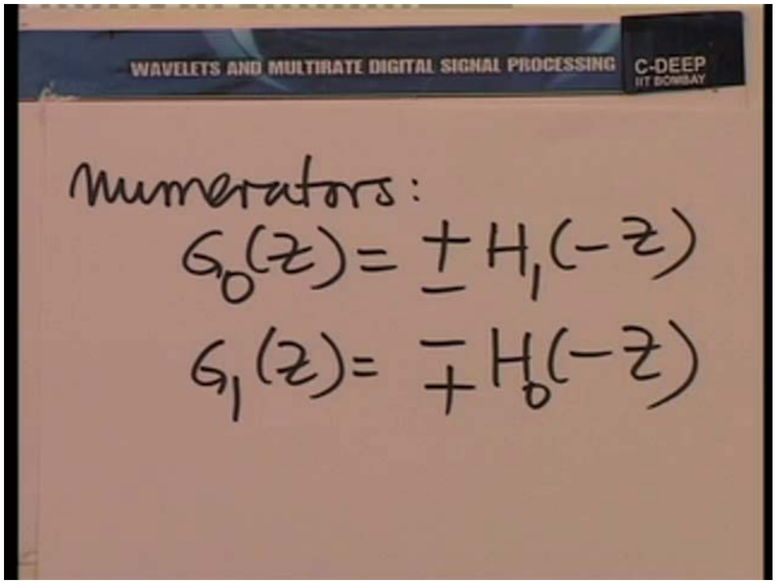

So, equate if you equate the numerator you get G 0 z is plus minus H 1 minus z and G 1 z is correspondingly minus or respectively plus H 0 minus z. We also interpreted these expressions in the ideal case. If H 0 z, for example, was the ideal low pass filter, then this would become an ideal high pass filter as we except. And, if H 1 is the ideal high pass filter, this would become ideal low pass filter. All with the cut-off of pi by 2.

So, in that sense, we have a neat interpretation for the ideal case. Even if the filter is not ideal, we have a reasonable interpretation in the sense that it could always take this to be an non-ideal low pass filter with the cut-off of pi by 2. And, this would again become a reasonably close high pass filter with cut-off pi by 2 and vice versa with this. If this is high pass filter with cut-off pi by 2, then this becomes reasonable low pass filter with cut-off pi by 2, whatever it be.

Let us now, of course consider the second condition. You see, this is alias cancellation. With alias cancellation, we are assuming that there is linear linearity and shift-invariants in the system. There is the linear shift-invariant system there. What is the system function there?

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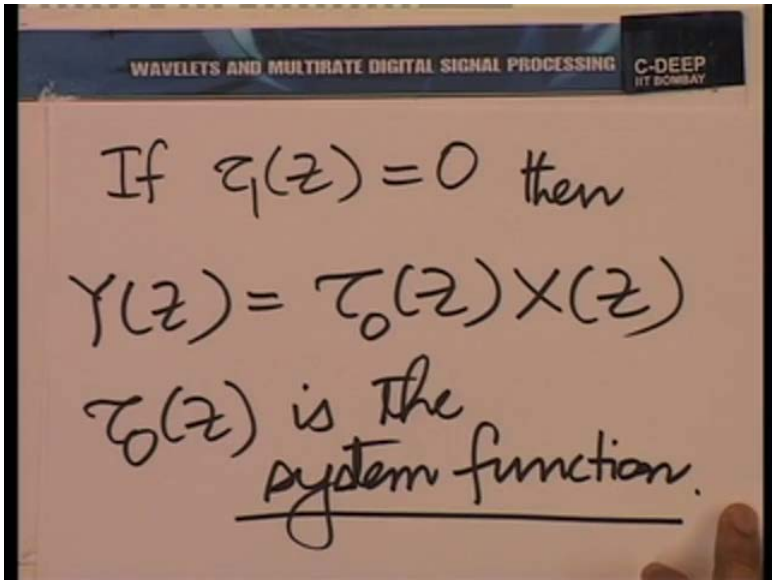

If tau 1 z is equal to 0, then you have Y z is equal to tau 0 z X z. And, we have seen effectively that tau 0 z is the system function in the true sense. So, this is an alias system with system function tau 0 z. Now, you see one of the things that one needs to worry about is what should tau 0 z be. Tau 0 z is also modifying experience for the input. And, ultimately we want decomposition and reconstruction. So, in reconstruction, we want exist to be almost same as Y z. If not quite, if not identity, at least there must be tolerable changes.

What are these tolerable changes that we can allow? Or, more appropriately what can we and what should we tolerate here. That is what we need to think about. What can we and what should we tolerate. Well, if you are talking about time systems, then we have to tolerate a delay. First, let me explain this intuitively and then let me put it down mathematically. The filter does some processing. It takes some time to process at the analysis side or the synthesis side. Now, you need finite time to process at analysis side. You need finite time to reconstruct at the synthesis side.

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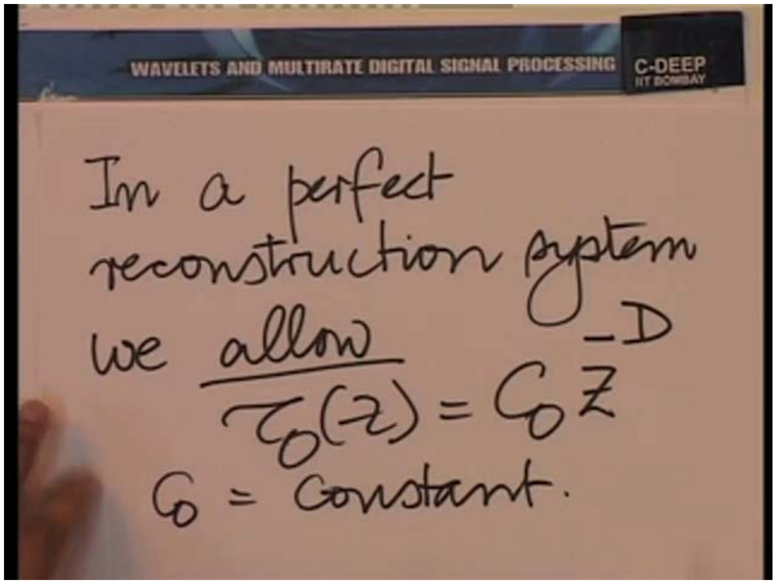

So, if you want to know other change between the input and the output, at least you have to accept the change of a delay to allow for some time to process. So, if we do not want any other change, at least we should allow for a term of the form z to the power minus D in tau naught z. The other thing that we do not mind allowing is an overall multiplicative constant. After all, we do not mind if the whole input sequences multiplied by some constant c because we can always multiply by 1 by c at the output. It is a simple operation to do; a simple amplifier or attenuator. That is not very difficult to do. So, we do not mind if the whole alias system that we had here after alias cancellation multiplies by a constant and delays by D.

And, that is exactly what we shall now put down mathematically. Assume you are saying in a perfect reconstruction system, we allow, we should say we allow tau 0 z to be the form some constant time z raised to the power minus D; is a constant.

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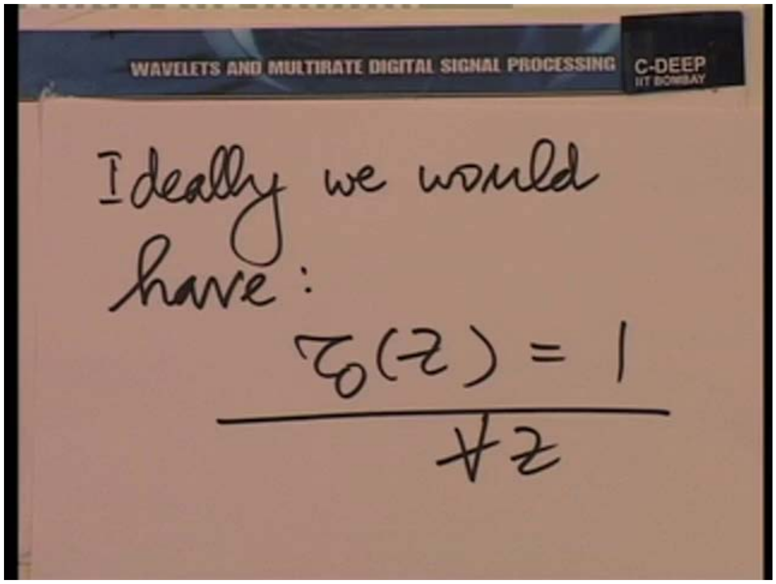

Ideally, we would have write tau 0 z is equal to 1 for all z, essentially an identity system. Now, that would, as I explained before make the system non-constant.

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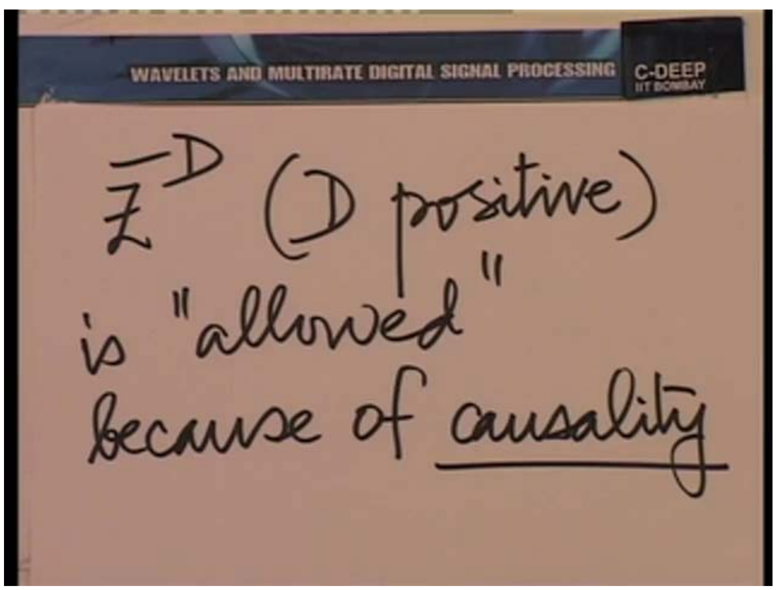

So, this allowance of the z raised to the power minus D, of course of course D is positive here; is allowed because of causality. Now, we will take the example again. as I said of the Haar MRA and the filter bank corresponding to the Haar MRA.

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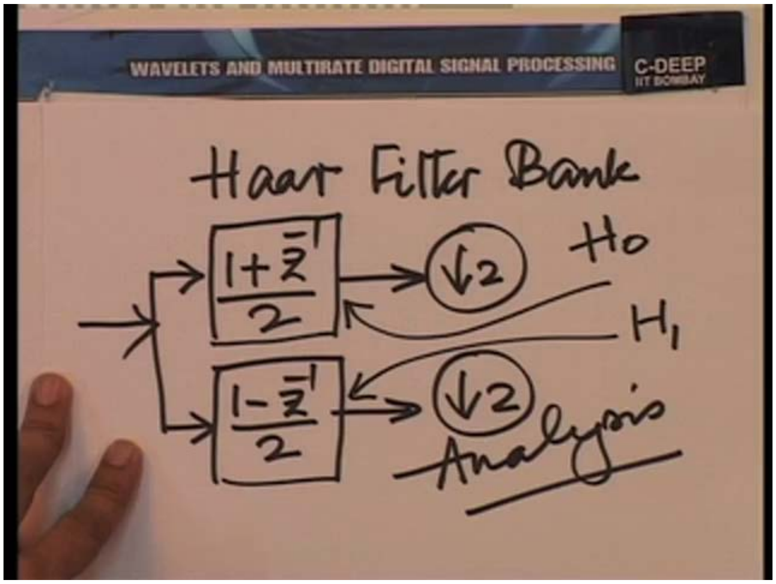

once again we will see, if we understand the haar case, we understand lot of things at once. So, next put down the filters for the Haar case. So, in the haar case, we had the following filters analysis.

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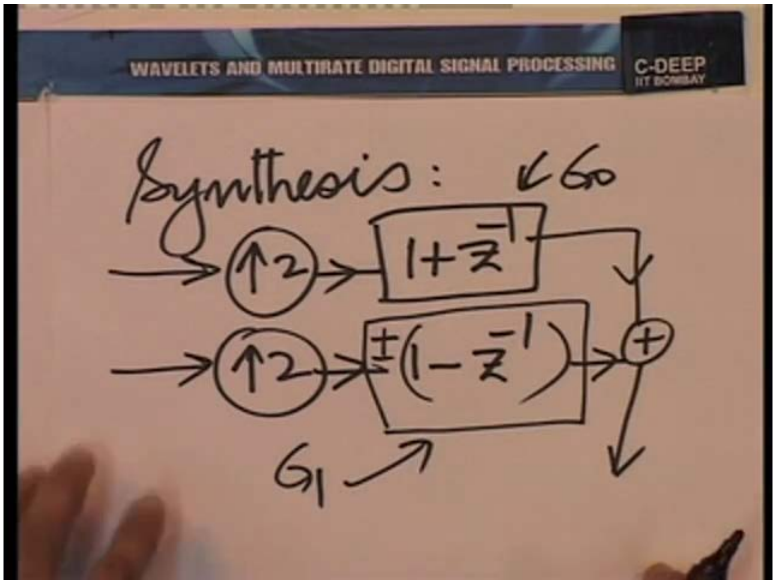

So, this is H 0 and this is H 1. Synthesis: You know, you remember, on the synthesis side, at that time if z will allow for a plus minus ambiguity here. Let us keep the ambiguity and you will see why the ambiguity is needed. This is G 0 and this is G 1.

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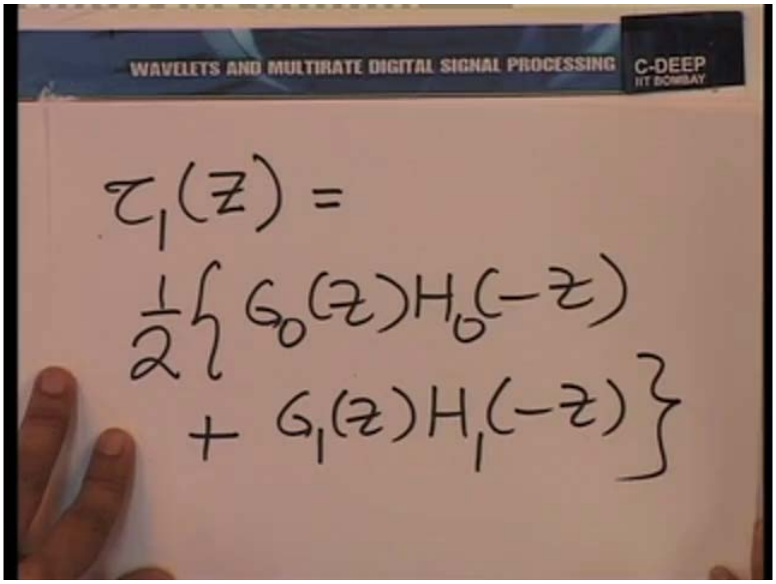

Let us try to 1 tau 1 z here, tau 1 z by definition is, of course G 0 z H 0 minus z plus G 1 z H 1 minus z.

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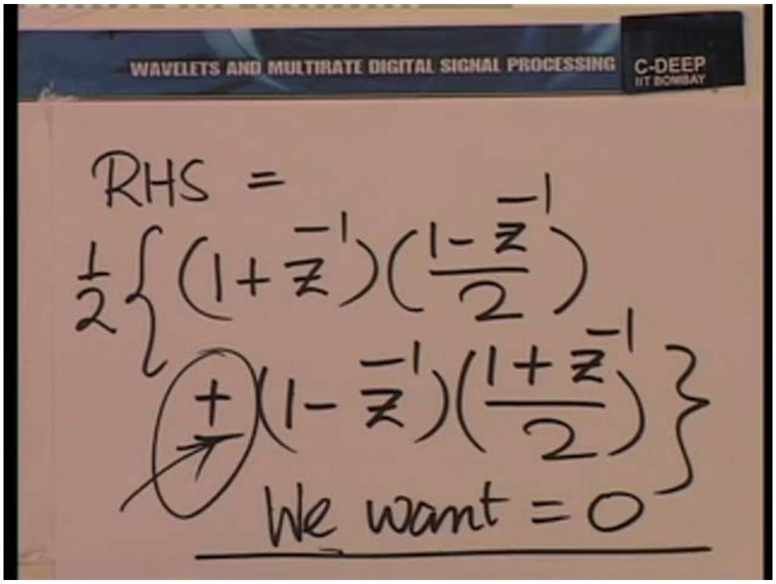

And, with that definition of G 0, H 0, G 1 and H 1,we have the right hand side becoming simply half G 0 is 1 plus z inverse. H 0 minus z is 1 minus z inverse by 2. Now, here we have a plus minus ambiguity. G 1 z is, of course 1 minus z inverse and H 1 minus z is 1plus z inverse by 2. Now, you know, when you look at this carefully, you noticed why we want this ambiguity there. We want this to become zero. And, therefore it is obvious that the minus sign should be chosen, the plus sign will not give us the zero.

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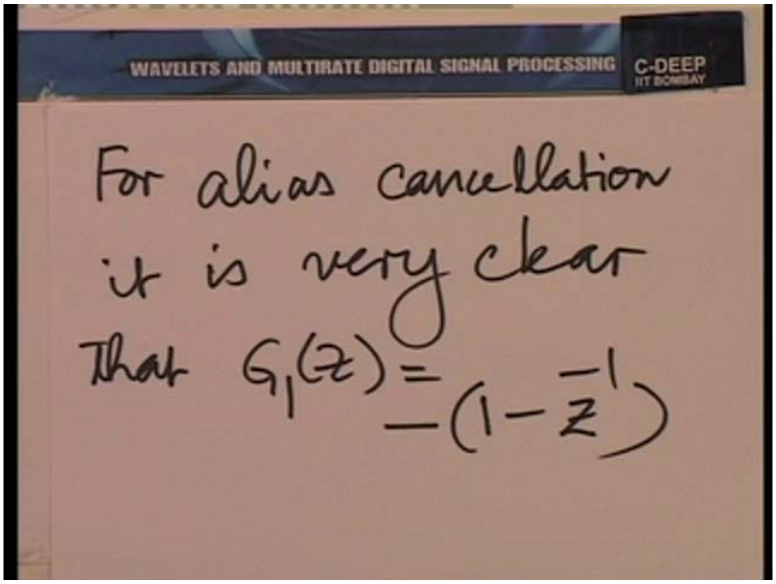

So, for alias cancellation, it is very clear that G 1 z must be equal to minus 1 minus z inverse and not plus.

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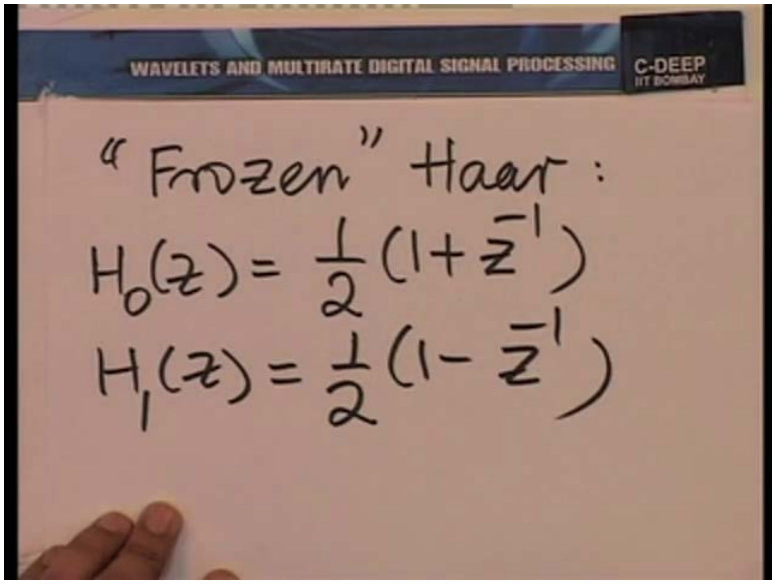

So, therefore, now let us freeze our G 0, G 1, H 0 and H 1 for the Haar case.

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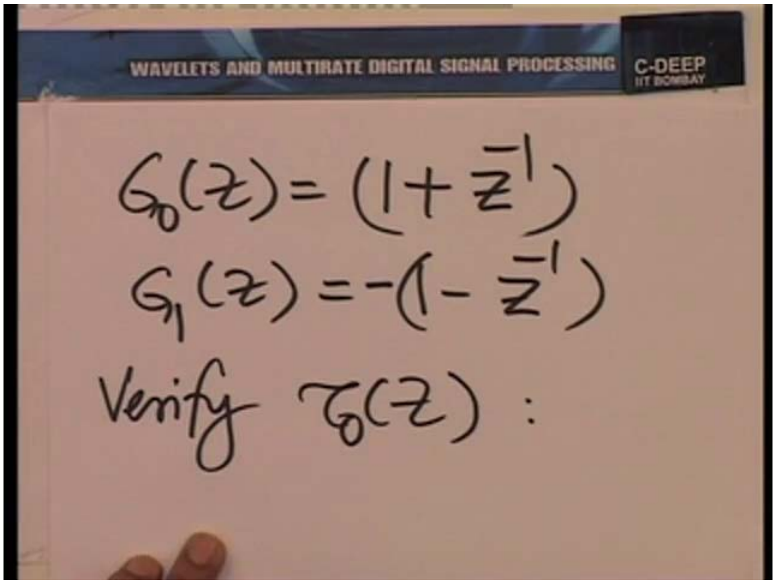

And, let us verify the perfect reconstruction condition or verify tau 0 z.

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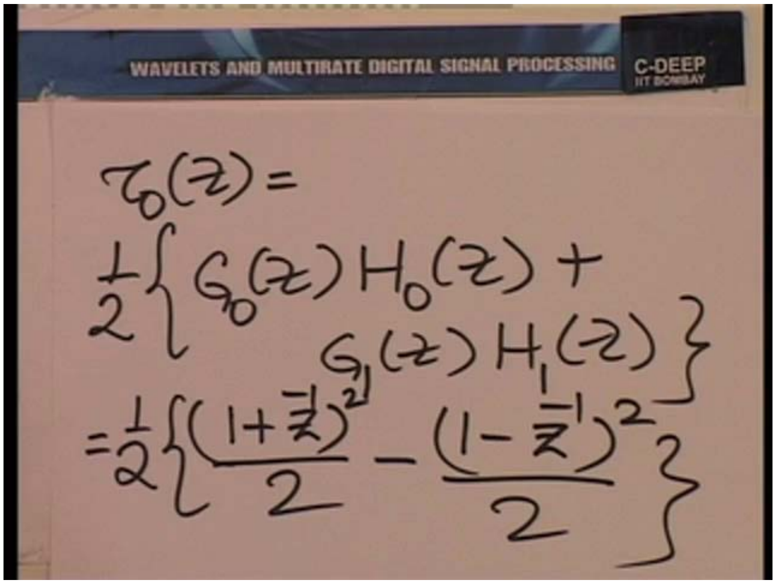

Indeed, tau 0 z is obviously, G 0 z H 0 z plus G 1 z H 1 z. And, when we expand this we get 1 plus z inverse the whole square by 2 minus 1 minus z inverse the whole square by 2. And, this is easy to evaluate.

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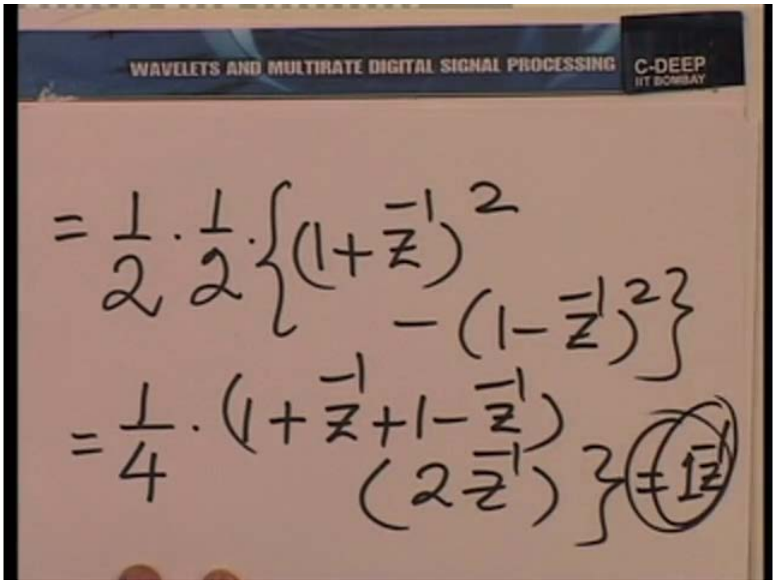

Essentially gives us half into half. And, we can use the a plus b into a minus b kind of expression and we have 1 by 4 a plus b is 1plus z inverse plus 1 minus z inverse. And, a minus b is 1 plus z inverse minus 1 plus z inverse; so, two z inverse. And here, of course z inverse cancels and here we have z inverse surviving. And, all in all, this is equal to 1. Simple and elegant. In fact it is 1, but with a factor of z inverse. So, c 0 is equal to 1 and you have a z inverse there.

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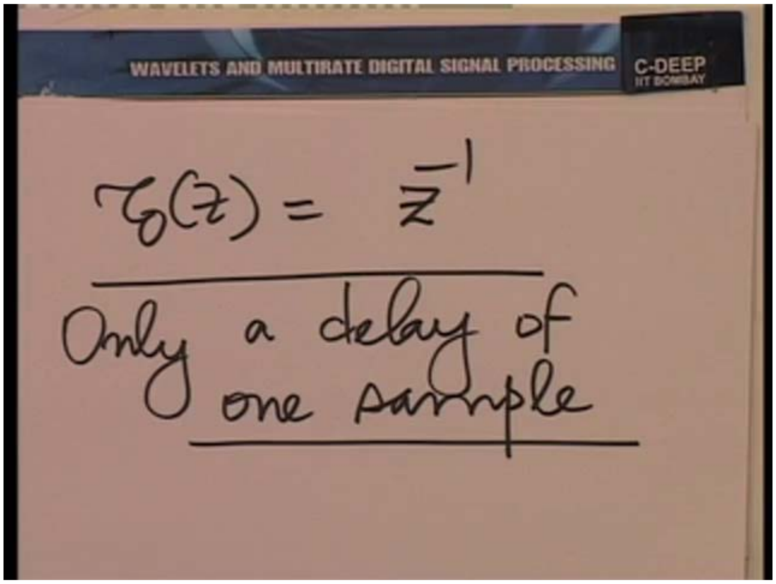

So, the tau 0 z, all in all is z inverse. What is this mean essentially? Only a delay of one sample. The constants have already been accommodated. So, . So, c 0 becomes 1. Now, why was this delay required? As i said, this delay is required on a count of causality. If we did not want this delay to be there, we would need non-causality either on the analysis or in the synthesis side.

So, for example, for example, if I do not want this z inverse term, I must multiply the output by z. In other words, I must shift the output backward by one sample. ? That means, G 0 and G 1 would now become non-causal filters; wherever causality is not an issue.

So, for example, if we are dealing with spatial data, then this is not a problem. We can get tau 0 z exactly equal to 1 without z inverse term. But, where causality is an issue, as it is when you are dealing with time data, then we cannot do this. Now, in fact in this case, let us also digest the situation and understand a little better. Let us put down the condition that we have written for alias cancellation in the specific and simplest case and see if it holds here.

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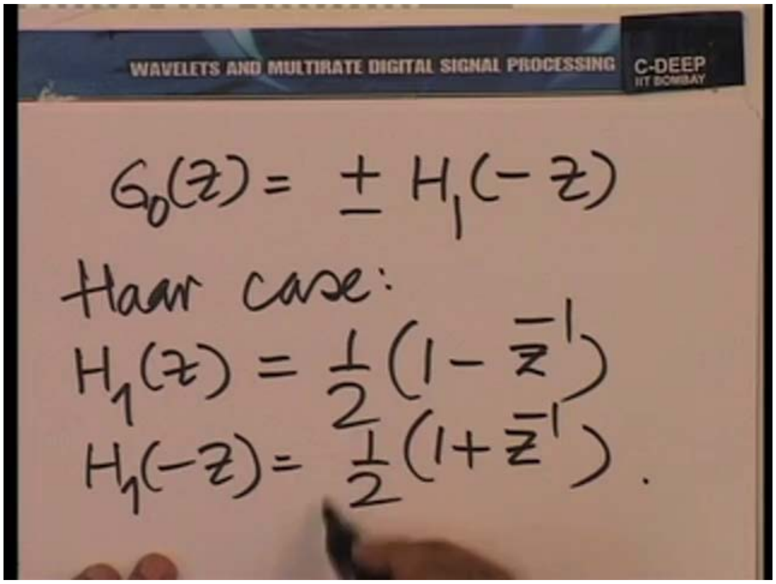

So, indeed that the simplest possibility of alias cancellation is when G 0 z is either plus or minus H 1 of minus z. And, in the Haar case, we have H 1 z is essentially half 1 plus or 1 minus z inverse rather. And therefore, H 1 of minus z would be half 1 plus z inverse. So, you notice that G 0 z is indeed plus H 1 of minus z, but without this factor of half. Now, you know that factor of half is not an issue at all.

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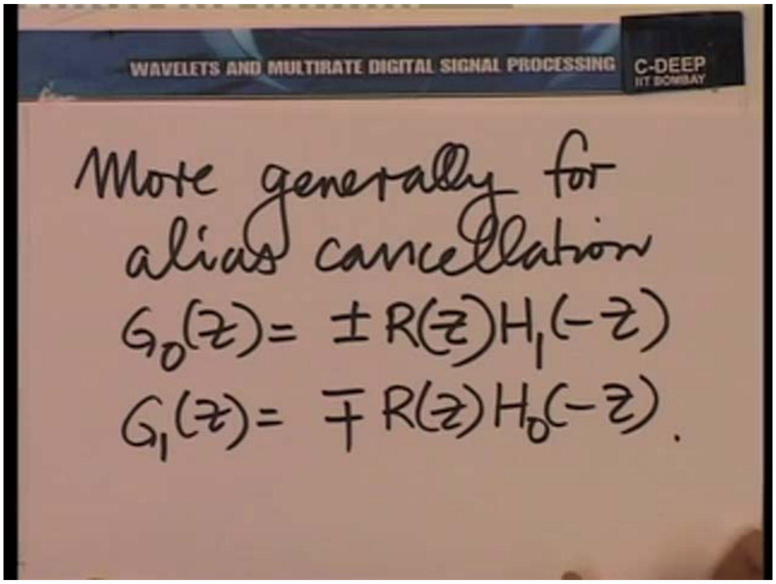

You remember, that more generally, we have written down the following requirements. We had said that more generally for alias cancellation, we need G 0 z to be plus or minus, some R z times H 1 minus z and G 1 z to be correspondingly minus or plus R z times H 0 minus z. And, in particular, you could choose R z to be a constant.

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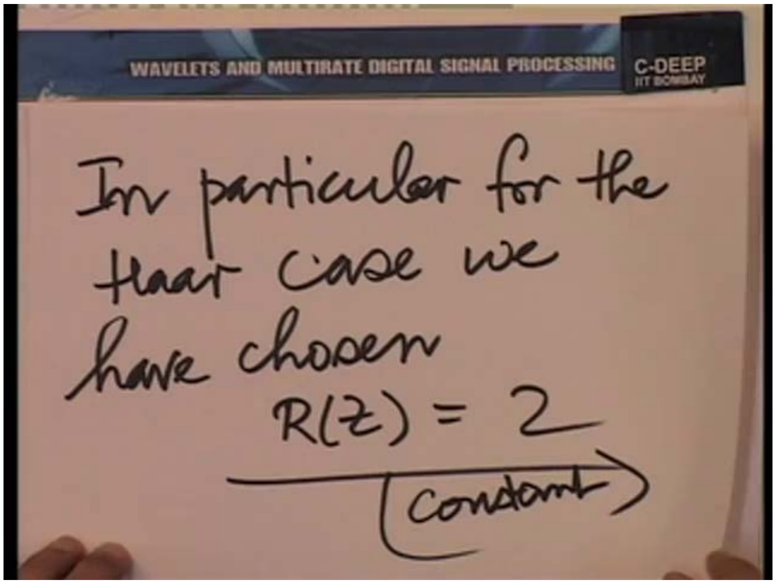

So, in particular for the haar case, we have chosen R z to be equal to 2, the constant. And, in fact I can also check for the second expression.

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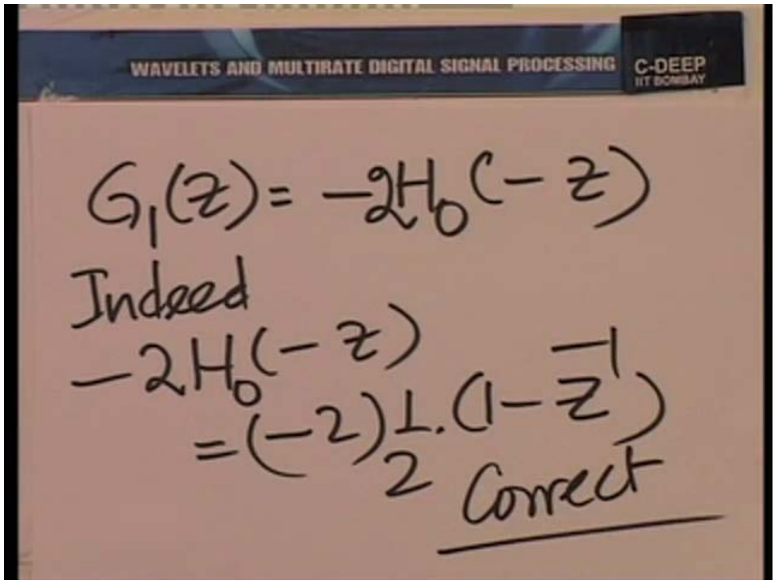

G 1 z in the haar case should then be minus H 0 of minus z or rather with a factor of 2; so, two times. And, indeed minus 2 times H 0 minus z is minus 2 times half into 1 minus z inverse, which is correct. So, things have all fallen into place. It is convenient.

I once again, point out how beautifully one can understand several concepts at once, when one takes the specific example of the Haar. The Haar MRA embeds in it. Several concepts explained in a simple way. But, of course we cannot be content with the haar and we shall slowly understand why. The first step in understanding this is to understand where the haar is the baby and where we need to grow further. Why is the haar, just the beginning of family of multi resolution analysis. In what sense is it in the best case. Towards that objective, let us look at that low pass filter and that high pass filter from a slightly different perspective. What does it do to a certain class of sequences, let us see that.

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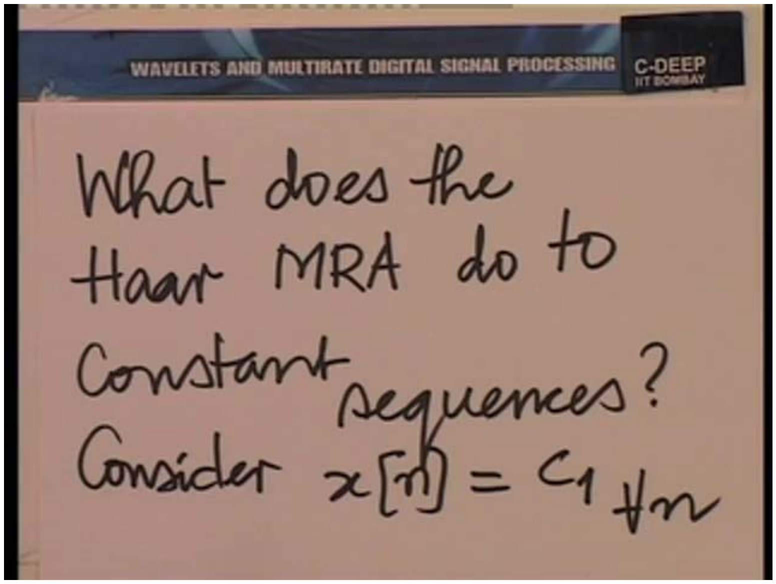

So, let us put the following question. What does the Haar do to constant sequences? In other words, consider x n equal to some constant say c 1, for all n; extreme case. How would the outputs of the various points in the Haar filter bank look?

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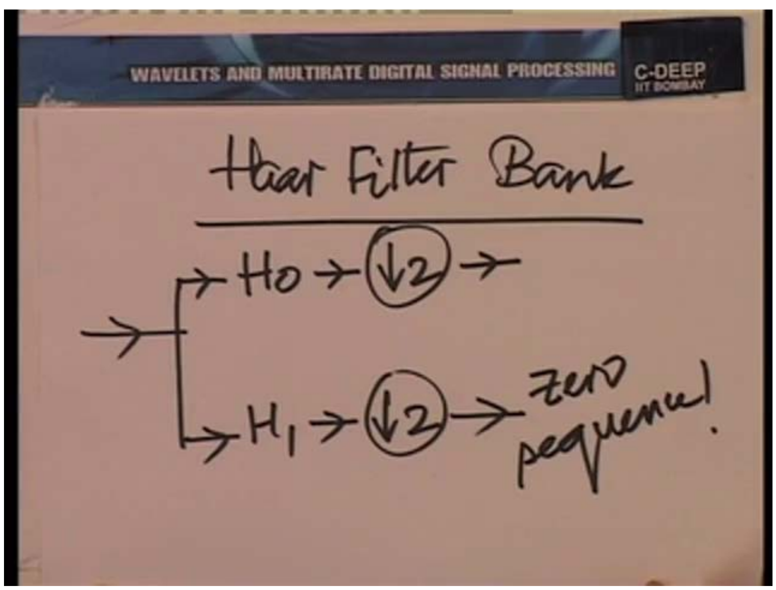

So, it is very easy to see that if you take the haar MRA, I will not keep writing the filters again. I will just show them symbolically. I have H 0 here; I have H 1 there. And, if I take just the analysis side, it is very easy to verify that the output here is going to be the zero sequence.

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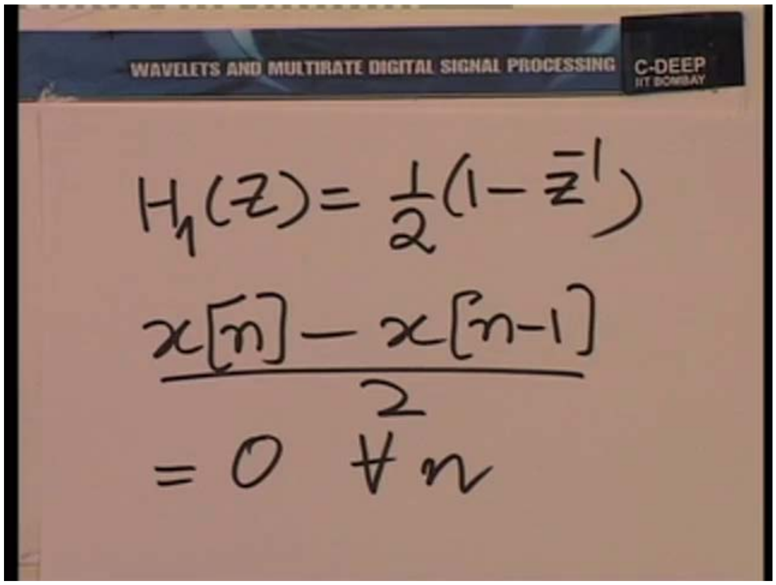

In fact, I will take just a minute and verify it. Essentially, H 1 z operates 1 minus z inverse. And, this essentially means your operation x n minus x n minus 1 by 2, which is identically zero for all n. So, this is a very significant observation we are making. We saying on the haar filter bank if there is any constant component in the input, it is destroyed on the high pass branch. This is a slightly different way of working with the haar filter bank. In fact, now we will go one step further. In the haar filter bank, I had one term of the form one minus z inverse. Suppose, I had two such terms what would happen? So, let us put that down.

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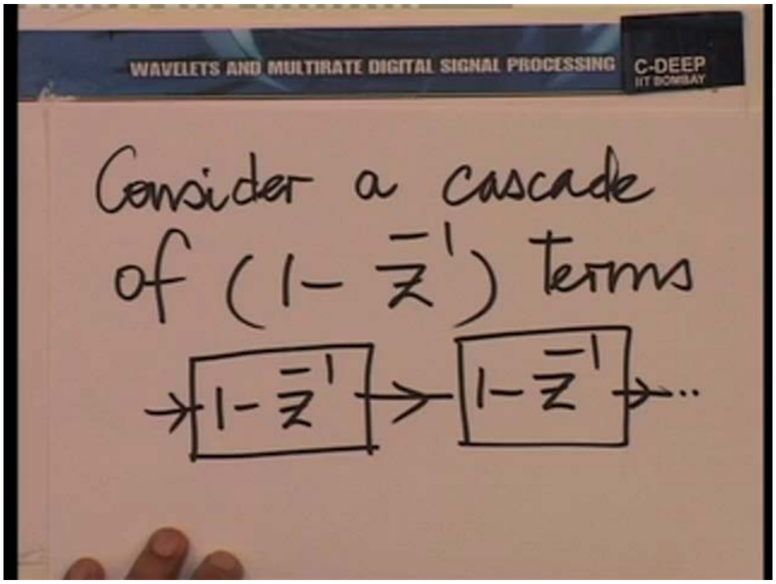

So, we consider a cascade of 1 minus z inverse terms. So, you know you have a system like this; 1 minus z inverse fit into 1 minus z inverse and so on. We shall now proof a very simple and a very elegant result.

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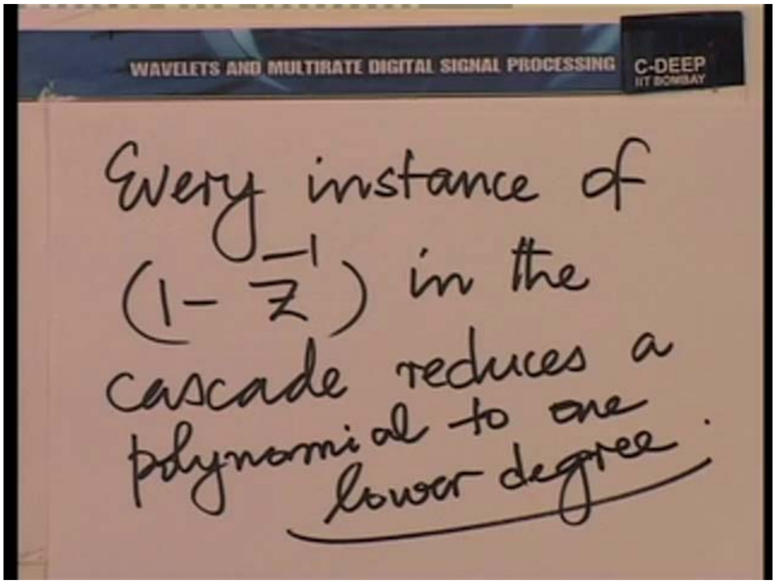

We shall show that every cascade, every instance of 1 minus z inverse in the cascade reduces a polynomial sequence to one degree lower. So, you know I am looking at the situation from a slightly different perspective. Now, I am not talking about frequencies or sinusoids anymore. I am saying, suppose you think often input sequence as having polynomial components, now, where on earth to an encounter polynomial kind of expansion valued. We know about the Taylor series. After all, the Taylor series is essentially a polynomial expansion of an input. And, when we make a polynomial expansion of the input and we subject a few terms in this polynomial expansion to the action of 1 minus z inverse, we have an interpretation. That we are talking about here. So, you know visualize a region in which you are talking about the sequence… and you know.

So, let the sequence, for example, come from an analytic continuous function and let, then that function be expanded in a Taylor series around a certain point, which means you have polynomial terms. Now, let those polynomial terms be subjected to the action of this cascade of 1 minus z inverse. That is the situation in which we should visualize aspects. It is a different way of expanding an input. Anyway, so putting that context in perspective coming back to the polynomial.

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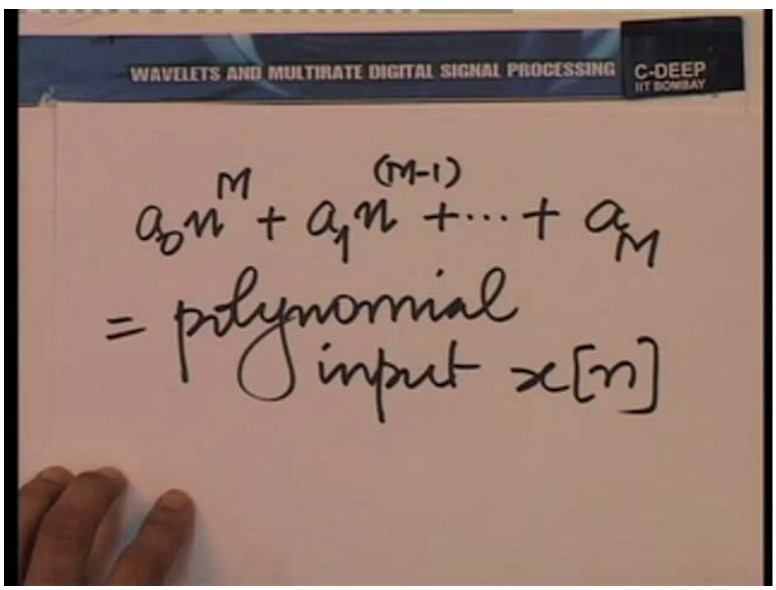

So, we show that if I feed any polynomial of the form say a 0 n to the power of capital M plus a 1 n to the power M minus 1 and so on, up to a M; which is the polynomial input sequence.

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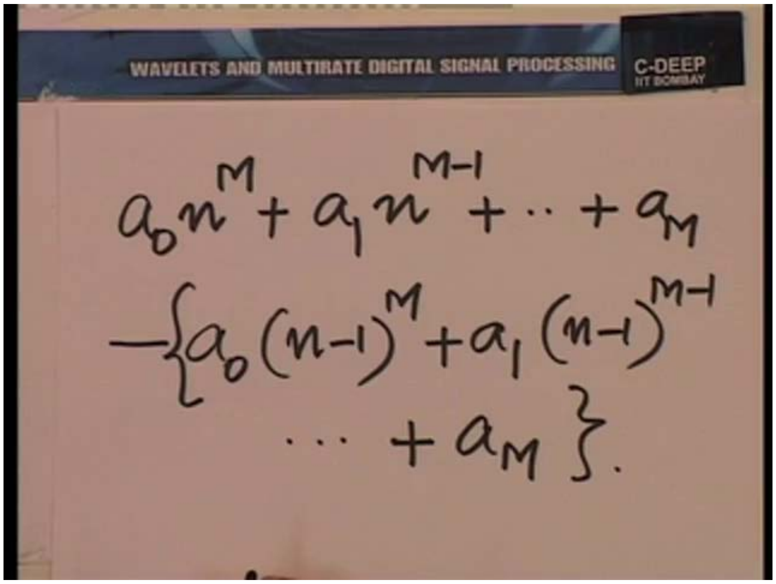

Every time, we subject this polynomial to 1minus z inverse, what is going to happen? So, let us subject it to 1. The first time we subject it, we are doing this. Now, what is happening in this process?

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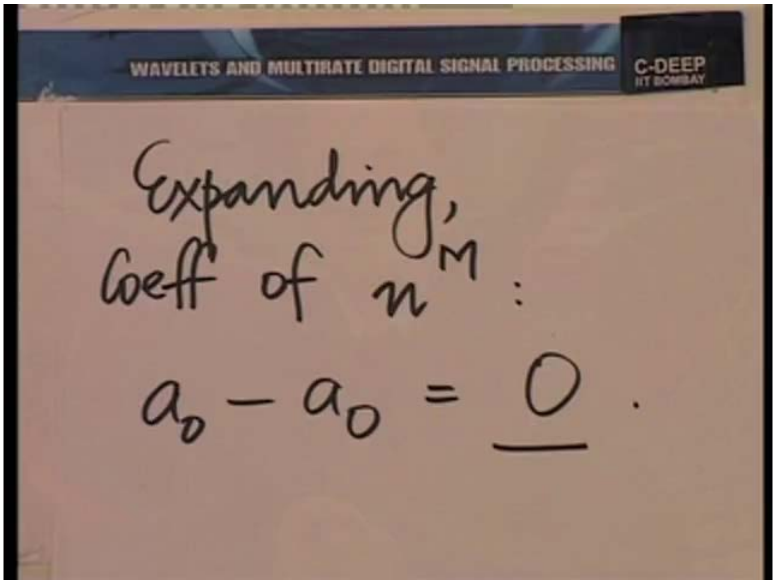

It is very clear that when we expand this, the coefficient of n to the power of M is easy to evaluate. It is essentially a 0, I mean, you know you have the term a 0 n raised to the power of M coming from here and the term a 0 n minus 1to the power of M, which contributes the coefficient of n raised to the power of M here. And, that coefficient is again a 0; so, a 0 minus a 0, which is 0. That is interesting. Each time, we subject this polynomial sequence to the action of 1 minus z inverse, we are reducing the degree of the polynomial by one. In fact, let me illustrate this. If I am taking a sequence which is polynomial of degree 1 and let us subject it to the action of this filter. So, you have essentially something like set...

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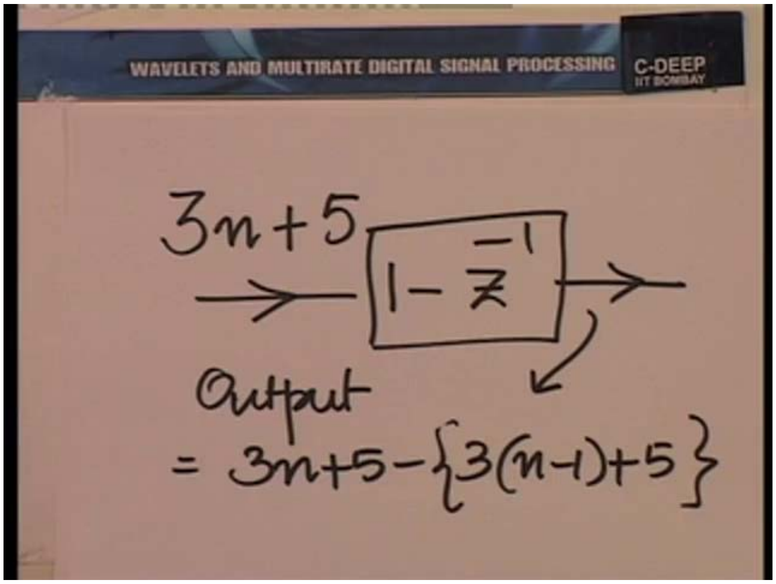

So, let us take a concrete example. So, let us take 3 times n plus 5 and let us subject it to action of 1 minus z inverse to fix our ideals.

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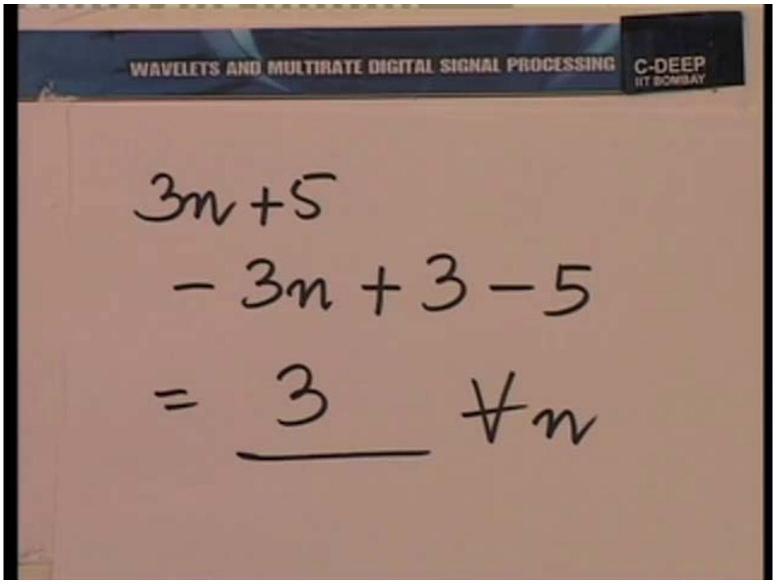

What emerges here is essentially 3 n plus 5 minus 3 n minus 1 plus 5; that is easy to evaluate. It is essentially, 3 n plus 5 minus 3 n plus 3 minus 5 and that is just 3, for all n. See, you have brought the degree of the polynomial down by one. You have a degree 1 polynomial; now you have a degree 0 polynomial. That is exactly, what happens for any degree polynomial. So, what we just showed a minute ago, and I will put back that discussion is that the coefficient of the highest power n to the power M vanishes. And therefore, when this sequence goes into z raise to the power minus 1 or 1 minus z inverse, so to speak, you only have n raised to the power M minus 1 and lower degree terms left. So, now we just prove to simple lemma. Each instance of 1 minus z inverse brings the polynomial degree down by one.

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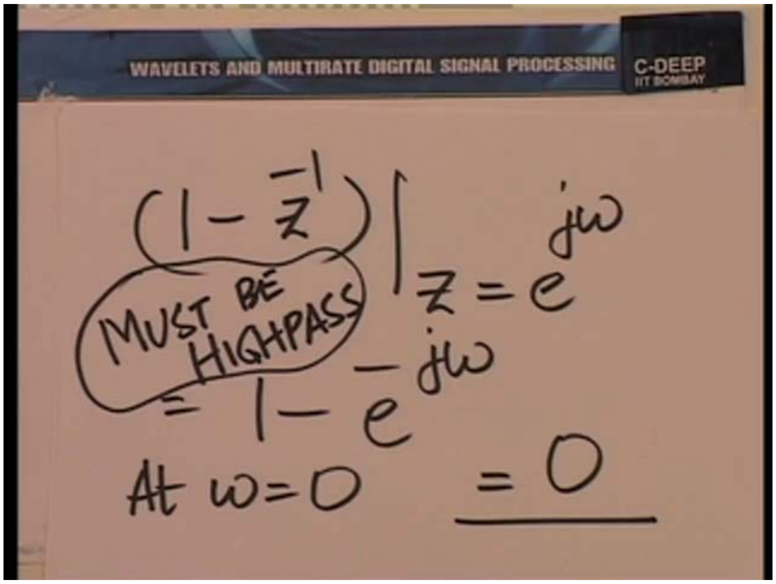

So, in other words, in a certain sense, the more 1 minus z inverse terms you have and by the way, we soon see these terms are going to be on the high pass branch; is not on the low pass branch. You know 1 minus z inverse, when we substitute z equal to e raised to the power of j omega, vanishes at omega equal to 0. It is verified.

So, when we take 1 minus z inverse and substitute z equal to e raise to the power of j omega, we get 1 minus e raised to the power of j omega, and then we put omega equal to 0. We get this is equal to 0. So, in other words, this is 0 d c. So, to speak 0 at 0 frequency, 0 response at 0 frequency, this cannot possibly be low pass in its behaviour. So, it must be high pass. In other words, if we do want terms of the 1 minus z inverse, they must only be present in the high pass filter. They cannot be the terms present in the low pass filter.

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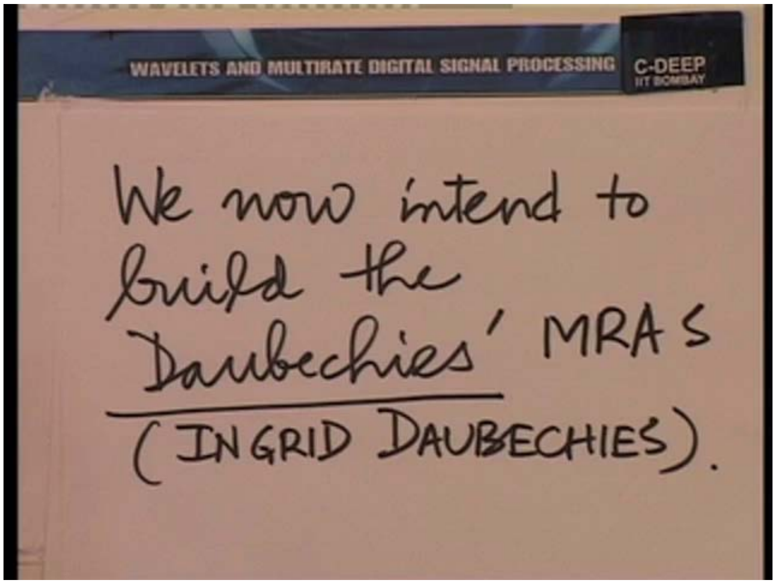

Otherwise, you know you would have a 0 response at zero frequency… for a low pass filter. In fact, what we are now going to build up is a whole family of multi resolution analysis in which you have more and more 1 minus z inverse terms in the high pass branch. And, that is, in fact very well known as what is called the Daubechies' family in multi resolution analysis. Let us write that down.

We now intend to build what is called the Daubechies' family. You know, this name is actually the name of a mathematician, scientist, whatever you might want to call her and the full name is this. I believe this is correctly pronounced as Daubechies, but I could be wrong. I think, we could just published Daubecheis and the content. So, anyway we now intend to build the Daubechies filter bank. So, Daubechies' MRA s.

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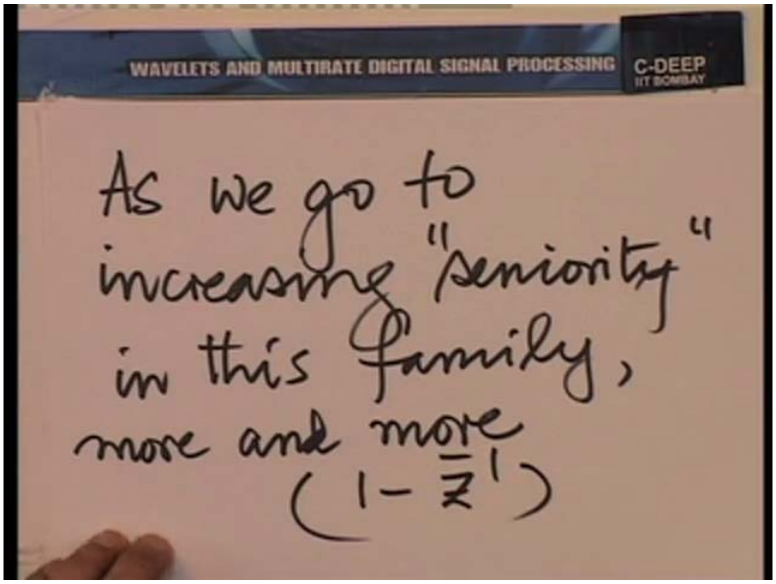

And, the features of these Daubechies MRA s is that as we go to seniors members of the family, as we go to increasing "seniority", there are more and more 1 minus z inverse terms. So, on the high pass branch, we are effectively cancelling or killing higher and higher order polynomials. So, the other way of looking it is, if you are cancelling or killing them on the high pass branch, they must go to the low pass branch. So, we are retaining more smoothness on the low pass branch. That is the other way… And addition to doing this, we also want the same kind of analysis and synthesis filters. So, all these together leads us to a specific class of filter banks, which we shall now put down explicitly. And, these filter banks are called conjugate quadrature filter bank.

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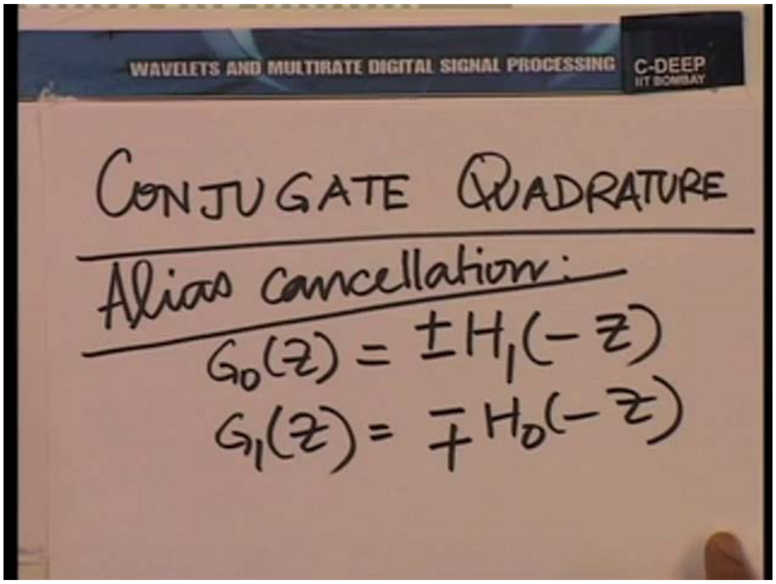

So, we are looking at one class of what are called conjugate quadrature filter banks. Now, we describe the equations of these conjugate quadrature filter banks are very simple. We will start from the alias cancellation condition. The alias cancellation condition says G 0 z needs to be essentially H 1 of minus z. And, we have the freedom to put plus or minus here. Similarly, G 1 z needs to be correspondingly minus or plus H 0 minus z.

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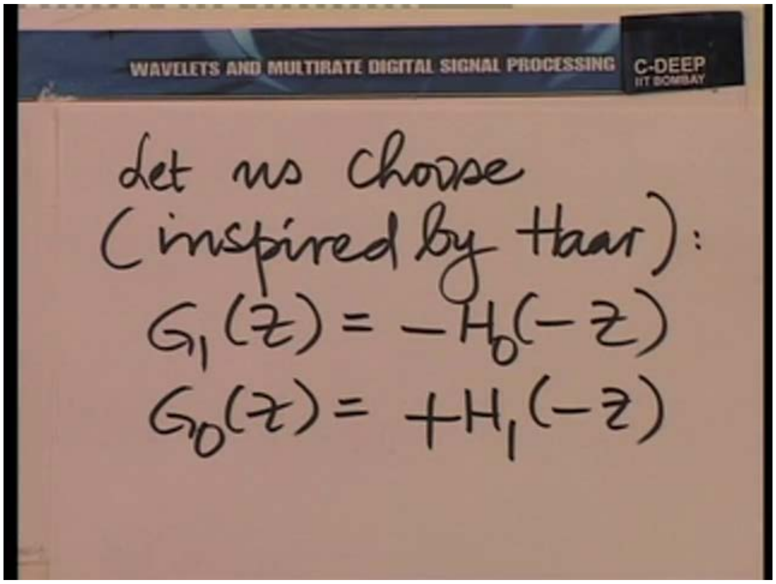

Now, taking inspiration from the haar, let us take the following choice. G 1 z is minus H 0 minus z. And therefore, G 0 z is then, plus H 1 of minus z. Now, you know, we will keep away the factor of two for the moment because after all, that factor can be observed in the G 0 that we allowed.

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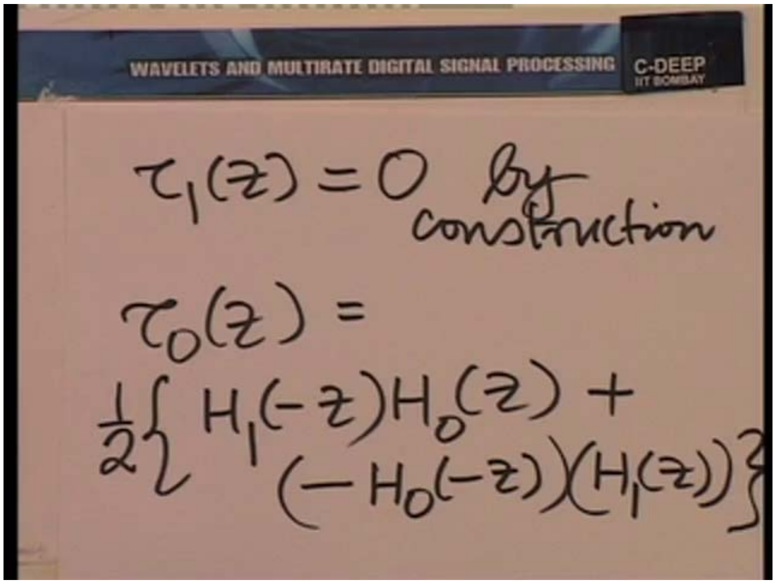

So, with this substitution, what do we get? The tau 0 z, then of course tau 1 z is identically 0 by construction. But tau 0 z takes the following form then. Tau 0 z takes the form half G 0 z, which is essentially H 1 minus z times H 0 z plus minus H 0 minus z time H 1 z. So, we have an interesting situation here. We have this H 0 z H 1 minus z product. You know, H 1 z is essentially a high pass filter; H 1 minus z therefore essentially becomes or aspires to become low pass filter with a cut-off of pi by 2. So, here you essentially have a cascade of two low pass filters with cut-off pi by 2. And, correspondingly this becomes a cascade of high pass filters with cut-off pi by 2. And, you are effectively saying that the overall system functions with this cascade of low pass filters of cut-off pi by 2 and this pair of high pass filters with cut-off pi by 2 must go towards a perfect frequency fraction situation. That is the interpretation of the equation here. So, we now focus on H 1 z.

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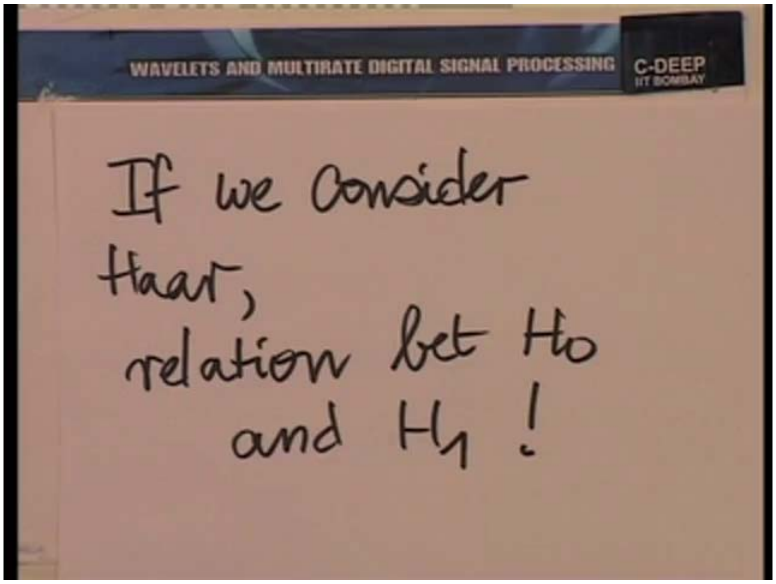

You see, if we look at the haar, once again there is the relation between H 0 and H 1. In fact, what we trying to say is, we of course rate the synthesis to the analysis for the purpose of alias cancellation. But, now we have a perfect frequency structure requirement. So, we want this tau 0 z to essentially go to a delay and multiplying constant. Now, that means you need a relation between H 1 and H 0. And, one simple thing to do is to make H 0 z related to H 1 minus z. That is what we have actually done in the haar.

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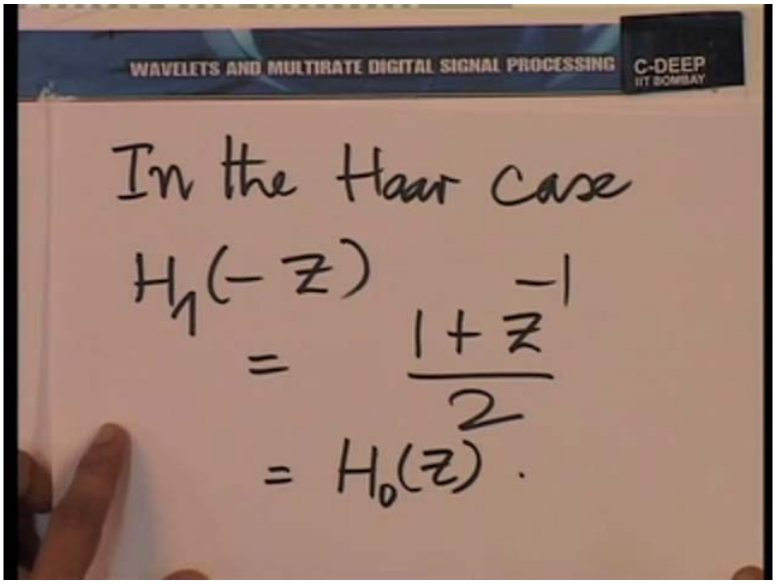

If you look at it carefully in the haar case, H 1 minus z is essentially 1 plus z inverse by 2. So, you know H 1 minus z and H 0 z are very closely related. Now, let us generalize this. In fact, the only catch is you know we will later on need to make a little adjustment here. So, at this moment, in which simply aspect H 1 minus z to be equal to, see in this the haar case, this is equal to H 0 z, but we may need to make a little adjustment here.

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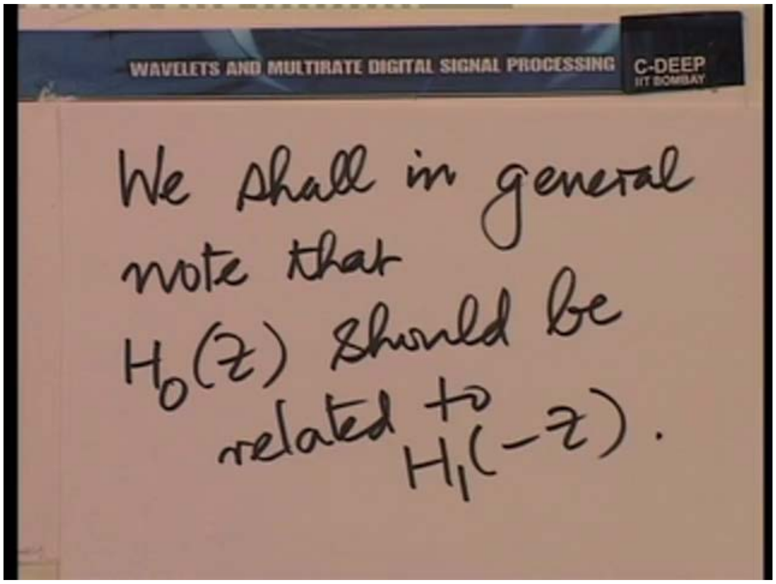

So, what we will do is, we shall in general, note that H 1 or rather H 0 should be related to H 1 minus z . So, in the haar case here equal, but in general we ask for a relation, a very close relationship. The other way of saying it is that if you replace z by minus z in H 0, we should get the H 1.

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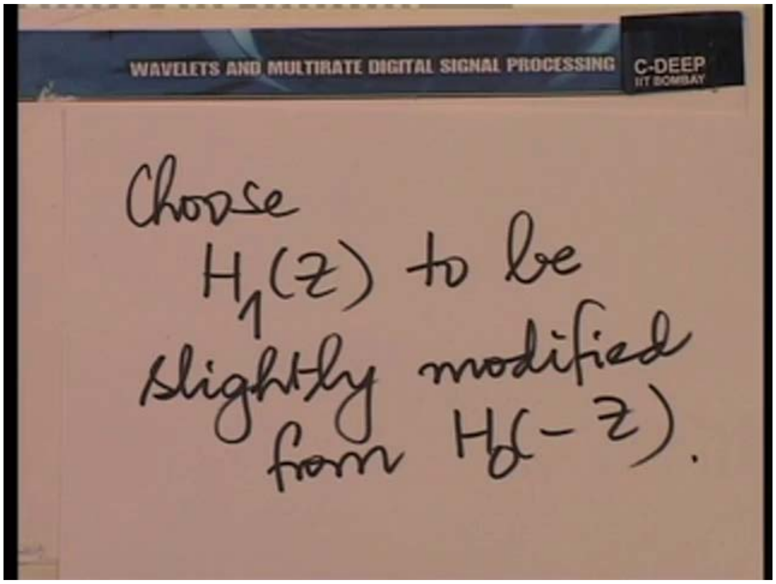

So, what we are saying is choose H 1 to be derived form or to be slightly modified form H 0 of minus z. Modified in what way? So, in fact you know here again there is a little bit of an issue. So, what I am now going to do is to put down a choice for H 1 by knowledge or by an exposure to the filter banks that I have and justify it later. So, you will have to bear with me for a little while, not too long, may be just about a lecture. I shall put down the choice here; I shall partially justify the choice in this lecture and completely justify the choice in the next lecture, where we once again look at the whole system in total.

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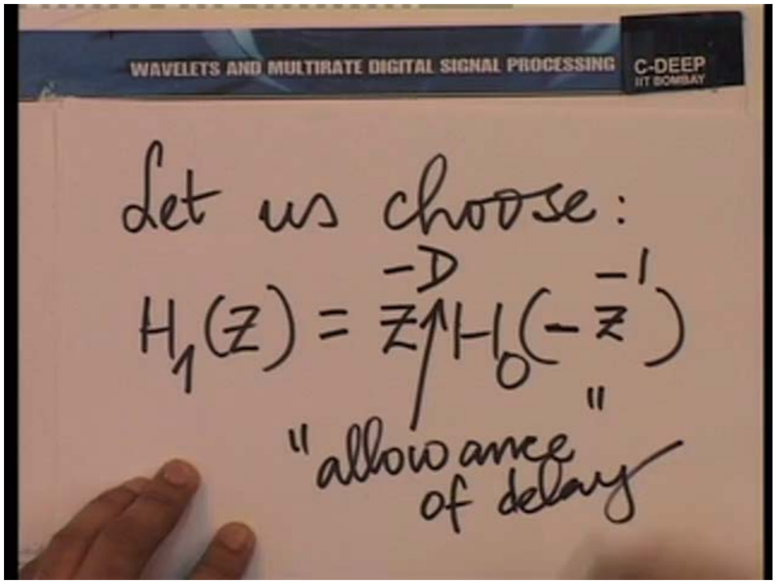

So, let us choose H 1 z, not to be H 0 of minus z, but H 0 of minus z inverse. And, we will also allow for the possibility of z raised to the power D here. So, we will say minus D. We will allow for this possibility.

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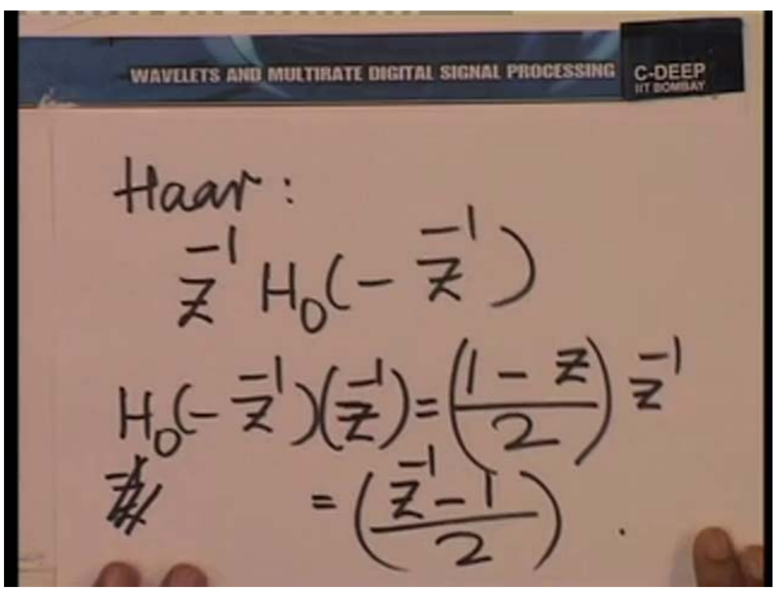

If we do that, in fact we can verify for the case of Haar, it is very easy. For the haar case, let us take z inverse times H 0 of minus z inverse and indeed H 0 of minus z inverse for the Haar case is essentially 1 minus z by 2. And then, if I take z inverse times H 0, so if I multiply this by z inverse, multiply both sides of z inverse.

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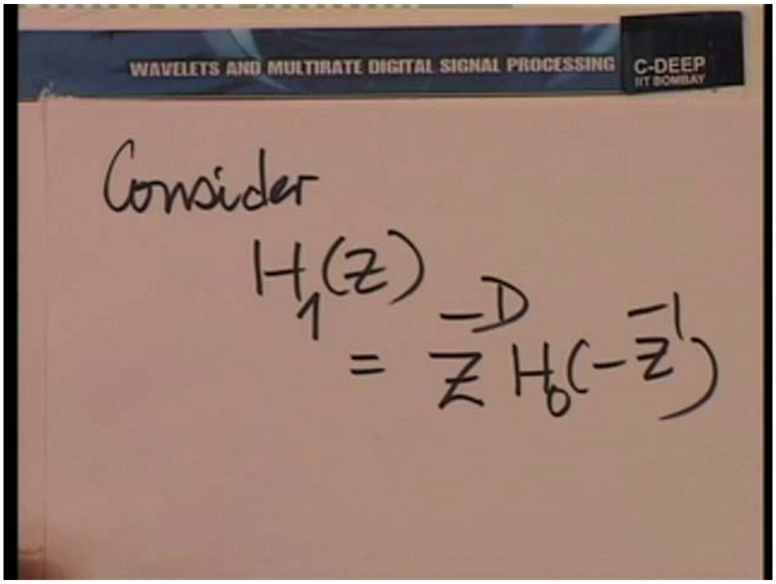

I have essentially z inverse minus 1 by 2. So, we are doing well. Now, let us make this the most general case. So, we will consider H 1 z to be of this form; z to the power minus D H 0 minus z inverse. And, write down the tau 0 z for this.

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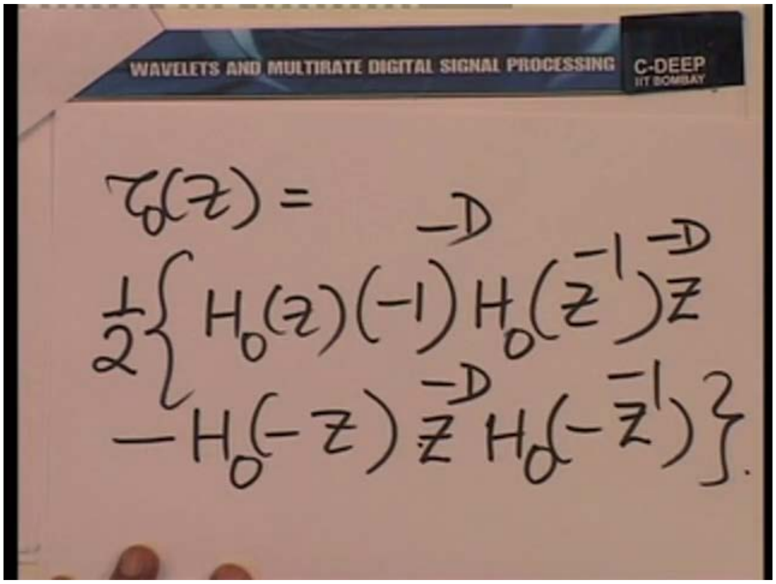

Tau 0 z would become half, then H 0 z H 1 of minus z. So, H 1 of minus z becomes minus 1 raised to the power minus D times H 0 z inverse. Now minus, you see you have G 0 z. Again, now we have G 0 z and let me put back the expression for you just for convenience.

We want tau 0 z to be this; H 1 minus z times H 0 z, for which we have this term H 0 z into H 1 minus z. All right. We have z raised to the power of D and minus H 0 minus z times H 1 z. Now H 1 z, it is accepted to be z raised to the power minus D times H 0 minus z inverse. So, we have z raised to the power minus D here.

Now, what we intend to do in the next lecture is essentially to look at this expression. So, we have H 0 z, H 0 z inverse here, H 0 minus z, H 0 minus z inverse there. The z raised to the power minus D terms there. And, of course you have the minus 1 raised to the power minus D here and minus 1 here. And, we need to choose these strategically. It is clear. And then, we also need to put some conditions on H 0. So, this indeed becomes a perfectly reconstructions situation. We shall complete this in the next lecture and take it further from there. Thank you.