Control System Design

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Lecture – 08
Laplace transforms

In the previous clip, we looked at the structure of the transfer function for a system that we have been considering so far namely this particular linear time invariant system.

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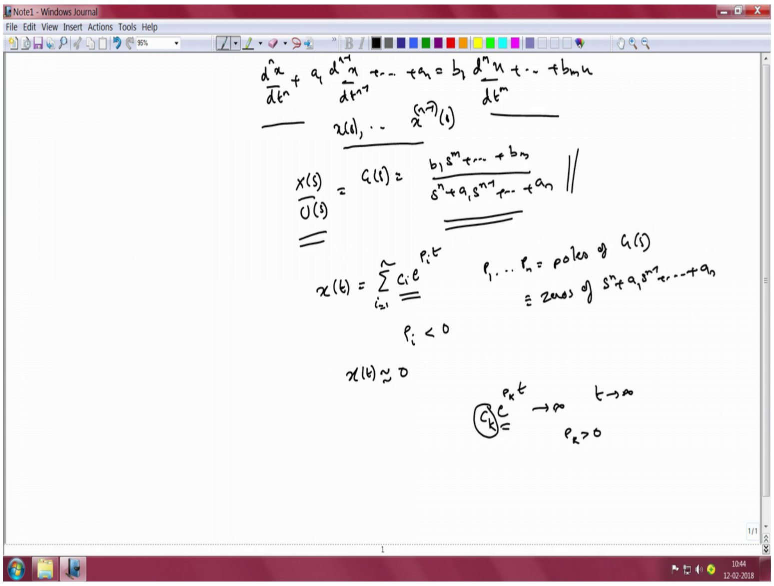

And we saw that the transfer function has this particular form namely the ratio of 2 polynomials in S. And we also saw that the initial value theorem revealed to us that the degree of the numerator polynomial has to always be less than the degree of the denominator polynomial. And under certain conditions we can approximate the degree to be of the numerator polynomial to be equal to the degree of the denominator polynomial.

Now are there any further simplifications that are possible in the structure of the transfer function? And one thing that I also want to address in this clip in addition to simplifying the structure of the transfer function; is to determine what we would do with the initial conditions that we have been carrying along all this time; let us attend to that first.

So, we know that if we have some n initial conditions; our time domain response x of t will be having a component that is related to the initial conditions which would be of the kind c i e to the power P i t, where i goes from 1 to n where P 1 to P n are the poles of the transfer function or in other words poles of G of S or in other words they correspond to the zeros of the denominator polynomial; zeros of S power n plus a 1, S power n minus 1 etcetera plus a n.

Now what we notice is that if our system is a stable system in other words if all the poles P i are less than 0, they are on the left half of the complex plane; then what we would have on the right hand side of this expression are all decaying exponentials some decay faster than others but they are all of the decaying kind. Now since they are all of the decaying kind what we can conclude is that if we wait long enough; we would have the response due to the initial conditions to have gone to 0.

And we can start to look at the response of our system to the applied input only after this particular or only after the passage of this duration of time. Therefore, we see that in case of stable systems, initial conditions do not play any significant role because their effect is erased out after a certain length of time. Likewise, if you take the case of a system that is unstable what we would have is that for any particular combination of initial conditions; we would have a non zero coefficient for that pole; let us say the kth pole which is unstable and that causes the solution to explore. So, tend to infinity as time t tends to infinity.

So, this will tend to infinity as t tends to infinity when P k is greater than 0. Now you may argue that there may be a certain combination of initial conditions that would get the coefficient c k to be exactly equal to 0. Well mathematically that may be possible, but physically it is impossible to ensure that c k will always be 0, there will always be some perturbation, some term, some noise input to your system which will get the coefficient c k associated with this exponentially divergent term to be non zero and that will cause solution to explode.

Therefore, even in case of systems which are unstable we would notice that the initial conditions are not very useful because regardless of what they are after a certain length of time the solution of the system would have exploded. Therefore, in either case whether you are dealing with a stable system or an unstable system; we see that initial conditions do not play a significant role. So, henceforth we shall not talk too much about the response of the system to initial conditions. And we shall focus exclusively on the response of the system to inputs.

In other we will focus exclusively on X of S by U of S which is given by the transfer function G of S. So, G of S as it appears in this particular slide is still rather complicated in its appearance; it is it is a ratio of 2 polynomials one of mth degree, the other of nth degree and clearly it is difficult for us as engineers to build our intuition to deal with systems that are of such high orders.

So, are there further simplifications possible in the structure of G of S?

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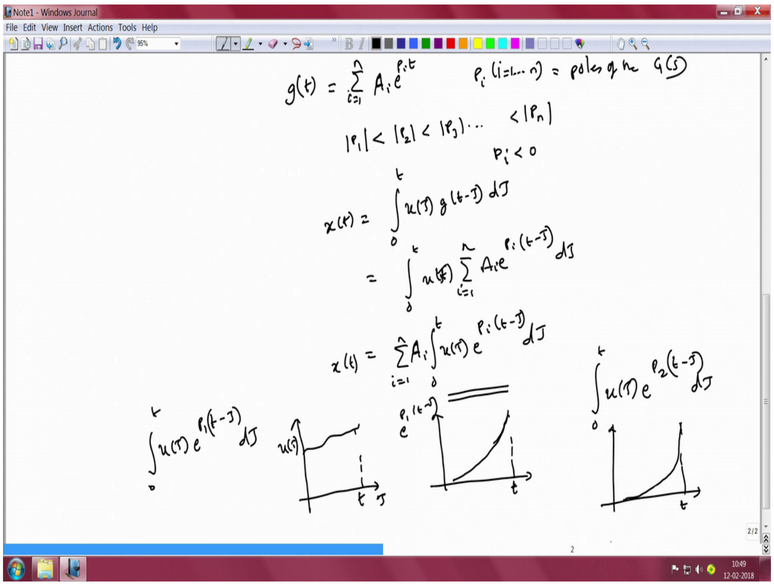

To address this question let us first go back to the time domain where we know that g of t which is the Laplace inverse of G of S would be of the form sigma A i e power P i t, where P i i goes from 1 to n are the n poles of the transfer function.

So, i goes from 1 to n; now let us undertake the exercise of arranging these Pi's in an ascending order of their magnitude. So, I shall assume that P 1 is the smallest of the lot. So, the magnitude of P 1 is less than the magnitude of P 2; is less than the magnitude of P 3 and so on and so forth and the largest value is P n. I shall also assume for the moment that we are dealing with stable systems in which case all of these P i are less than 0. This is going to be the case especially when we are dealing with closed loop control systems because we are always going to be targeting stable closed loop systems.

Now if we can arrange this in this manner then we know that our response x of t in the time domain is given by integral 0 to t; u of tau g of t minus tau d tau. And that is going to be equal to from the expression here 0 to t, u of tau sigma A i e to the power P i t minus tau d tau; where I goes from 1 to n. Now I can write this in turn as sigma i goes from 1 to n; A i integral 0 to t; u of tau, e to the power P i; t minus tau d tau.

So, it is a summation the response x of t therefore, is a summation of n integrals which are given by these specific terms. Now let us graph these particular terms let us take for instance the case of P being equal P i; i being let us for example, take the case of i being equal to 1 in which case integral 0 to t, u of tau, e to the power P 1 tau P 1 t minus tau; d tau would be the integral how would this graphically appear? u of tau could be some input which varies in a certain manner up to tau equal to t.

So, this may be u of tau and e to the power P 1; P minus tau would be an exponential function which would decay starting from the time tau equal to t in this particular manner, this is decaying because we know that P 1 is less than 0. So, u of tau times e to the power P 1 t minus tau is essentially the product of these 2 functions, this is e to the power P 1 t minus tau. And integral of u of tau times P 1 of t minus tau is the area under the product of these 2 functions.

Now if you look at the next term which is integral 0 to t, u of tau e to the power P 2 t minus tau d tau what you would have is therefore, that this integral is equal to the area under the curve which is a product of u of tau. And e to the power P 2 times t minus tau how does e to the power P 2 times t minus tau look?

Since we have arranged P 1 P 2 and so on in this particular manner; P 2 times t minus tau would be an exponential function that is decaying much faster than e to the power P 1 times t minus tau. Therefore, you agree with me that if u of tau looks something like this then in general u of tau times P 2 of t minus tau will have much lesser area under that curve compared to u of tau times e to the power P 1 times t minus tau.

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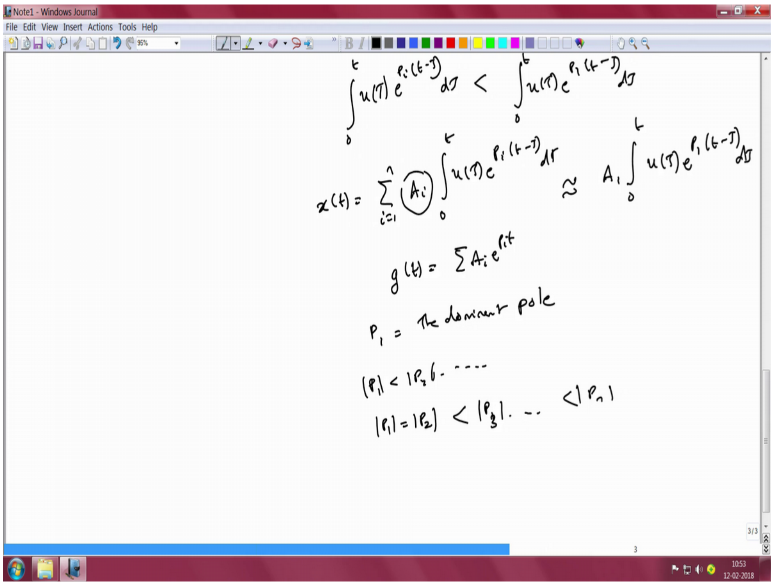

Now you can go on like this and you can show that if you have arranged the poles in an ascending order in terms of their magnitude then all the higher integrals integral 0 to t, u of tau, e to the power P i t minus tau d tau is generally less than the first integral 0 to t; u of tau e to the power t 1 t minus tau d tau. And what is our response equal to? Our response is equal to x of t equal to sigma i going from 1 to n; a i integral 0 to t, u of tau e to the power P i t minus tau d tau.

Now, if we are lucky with the problem in that these terms A i are also comparable to one another. So, A 1 is comparable to A 2 is comparable to A 3 and so on and so forth; then x of t can be approximated by noticing the fact that the other integrals for P P 2 P 3 P 4 and so on. The area under the curve for those integrals is much less than that for P 1. This can be approximated as a one times integral 0 to t; u of tau e to the power P 1 t minus tau d tau.

In other words, despite the high degree of the numerator polynomial and the denominator polynomial of my transfer function; if it so happens that the impulse response g of t which we have wrote in written it down as A i; e to the power P i t is such that all these coefficients A i are comparable to one another. And we can arrange the poles P 1 to P n in an ascending order in the manner that we did, then the slowest pole is the one that contributes the most to the response of the system; that dominates the response of the system hence P 1 is called the dominant pole.

Now, this arrangement that we did of poles in ascending order magnitude of P 1 less than magnitude of P 2 and so on and so forth; it is possible provided the pole P 1 is a real pole. So, the simplest dynamic system that approximates the response of our high degree transfer function G of S is a first order system, if its slowest pole; if its dominant pole is a real pole. However, it is also possible that the pole P 1 is at complex pole in which case the upper they appear in complex conjugate pairs.

So, we would have them magnitude of P 1 to be equal to magnitude of P 2. So, we would have 2 poles which appear as complex conjugate pairs and therefore, they have to be considered together and their magnitudes would be identical. So, when we arrange the other poles also then we would have P 1 is equal to P 2 which would be less than or equal to which would be less than P 3 and so on and so forth; up to P n.

So, in this case the simplest system that approximates a response of our more complicated transfer function G of S is going to be a second order system. So, in other words if the dominant pole is a real pole then the simplest approximation to the response of G of S would be the response of a first order system. If on the other hand, the slowest poles in our G of S are complex poles are in that case; the simplest system that approximates the response of our G of S would be a second order system. Hence first order systems and second order systems occupy a special place in our study of the linear dynamic systems.

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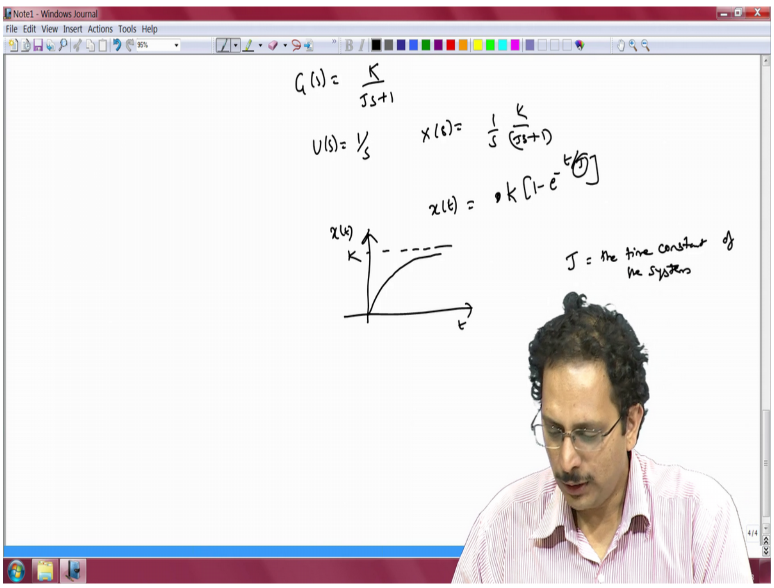

So, let us briefly visit first order and second order system responses especially 2 step inputs before we conclude this clip. So, I shall consider a first order system to be 1 that has a transfer function of the kind K by tau S plus 1; of course you can also have a 0.

But that would make this is that will make the system a proper transfer function, but not a strictly proper transfer function and hence we shall not discuss it at this point. Now if I were to apply a step input to this system in other words if I were to apply U of S is equal to 1 by S; then the response would be given by X of S is equal to 1 by S times K by tau S plus 1.

If I take the Laplace inverse of this I would get x of t and t hat would be equal to k times 1 minus e power minus t by tau. So, the response of a first order system therefore, would look something like this its steady state value namely the valued assumes when t tends to infinity is equal to k. So, this is k assuming that its initial conditions are 0 we have discussed in the past that we do not need to consider initial conditions in our analysis of systems and their control.

So, we shall assume that all initial conditions are 0 henceforth; then the response would look something like this; the speed with which the time scale in which it raises to its steady state value of k is characterized by this term tau and therefore, tau is called the time constant of the system. Now let us take the case of a second order system.

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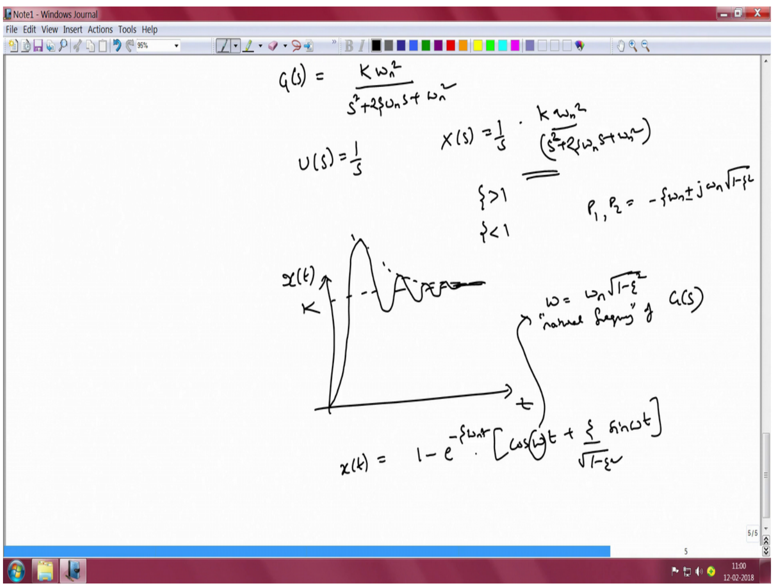

So, a second order system I shall in this example I shall consider a second order system that has a transfer function that looks something like this K times omega n square by S square plus 2 zeta omega n S plus omega n square.

Now this typically represents a mass spring damper system or an LCR circuit and if I were to apply a step input to this system, the response of the system prove of course, be 1 over S times k times omega n square by S square plus 2 zeta omega n S plus omega n square. Now, if zeta is greater than 1; I would have 2 distinct real roots in which case I can think of my second order system as a cascade of 2 first order systems. And therefore, the output of the first order system would be the input to the second first order first order system; it is therefore, not a very interesting example to consider.

So, it is only when zeta is less than 1 that we have a pair of complex conjugate poles for this system; namely the locations the poles P 1 comma P 2 would be minus zeta omega n plus minus j times omega n square root of 1 minus zeta square; these would be the 2 poles of our transfer function. So, for this case the step response which we obtain by taking the inverse Laplace transform of X of S would look something like this.The steady state value would be obtained by applying the final value theorem and we find that it is equal to k.

Now, when we apply a step input then the response overshoots and oscillates back and forth and dices down gradually with time before reaching its final steady state value of k. The frequency of this oscillation is called the natural frequency of oscillation of the system and it is given by omega is equal to omega n square root of 1 minus zeta square; this is the natural frequency of G of S.

More specifically the mathematical expression for x of t would be x of t is equal to 1 minus e power minus zeta omega n t, which represents this exponential decay in the amplitude of this oscillations as the system settles down to its steady state value times cos omega t plus zeta by square root of 1 minus zeta square sine omega t; where omega is a natural frequency given by this term here.

So, we took special look at first order and second order systems because despite the complexity of G of S; some of its poles will be slower than others. And generally one will expect G of S to have either one slowest pole or two slowest poles; one slowest pole if that pole is real and two slowest poles if that pole is complex; in which case in most circumstances we can approximate G of S by either a first order system in one case or a second order system in another case. And hence the centrality of first order and second order systems responses to standard inputs as far as our analysis of control systems are concerned with this; we conclude this clip.

Thank you.