Control System Design

Prof. G. R. Jayanth
Department of Instrumentation and Applied Physics
Indian Institute of Science, Bangalore

Lecture – 07
Laplace transforms (Part 1)

Hello in the previous clip we looked at Fourier transforms and how it helps us to obtain the impulse response of a system without applying an impulse input. But then in the process we discovered that the mathematics of Fourier transforms allows us to transform a convolution integral which is what we need to solve in the time domain to obtain the response of a system to any particular input to a product.

So, the Fourier transform of the response is equal to the product of the Fourier transform of the input and the Fourier transform of the impulse response. Now since a product operation is significantly more intuitive compared to a convolution integration operation; we saw some benefits in not working analyzing systems in the time domain.

But rather by sticking to the frequency domain itself for our design and analysis purposes. However, as control engineers we discovered that despite all the benefits of Fourier transforms in terms of increase of intuitiveness and so on it still has drawbacks in that there are some signals which are of great relevance to us as control engineers which do not possess Fourier transforms. So, we gave the examples of a ramp function an increasing exponential all of these do not possess Fourier transforms and other functions; such as the step function or sinusoidal input which you know how delta functions in their Fourier transforms are rather difficult to deal with.

And these once again are signals that are very commonly encountered in the course of control system design. And that led us to investigate if weighing down such functions with decreasing exponentials would allow for such functions to possess Fourier transforms and while we facilitate down that path, we came upon the definition of the Laplace transform.

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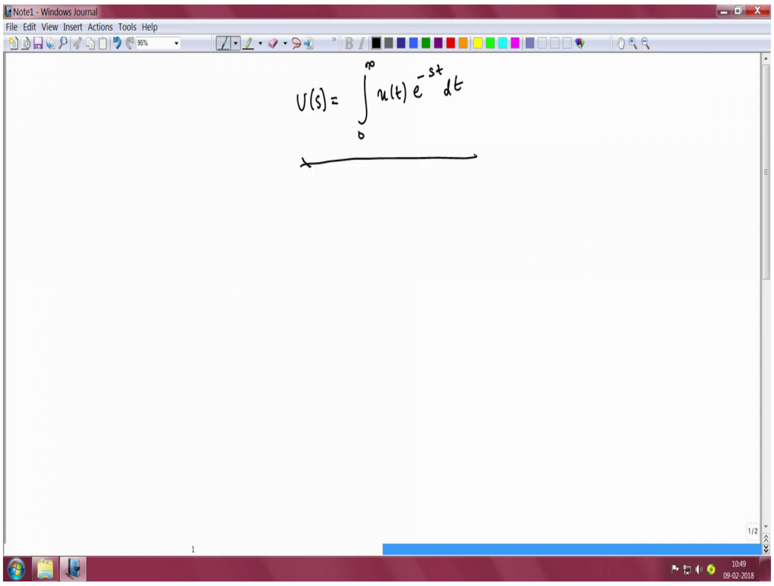

So, what I have shown here in this slide is the expression for the Laplace transform of a signal u of t it is given by capital U of S is equal to integral from 0 to infinity u of t e power minus s t d t; it is assumed that the signal u of t starts at the time t equal to 0.

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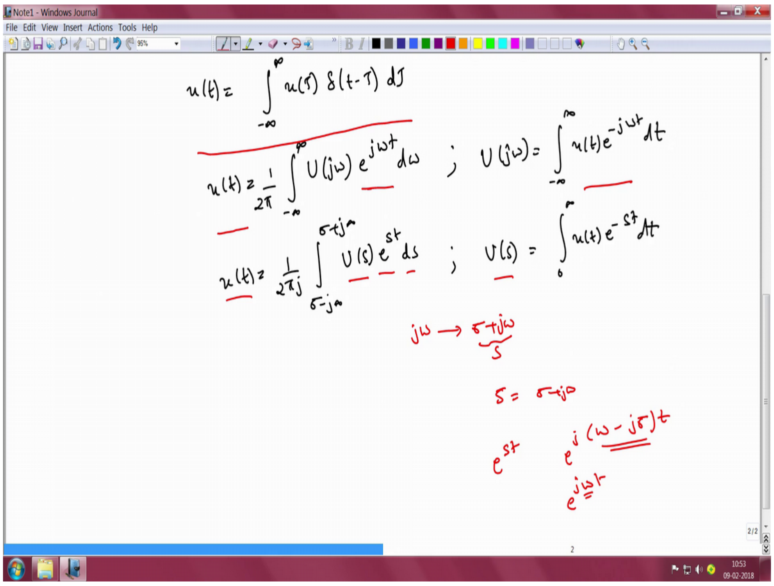

In order to highlight the unity in our quest for representing signals using different elementary functions; I have shown in this slide our attempt so far. We started off by representing our signal in terms of impulses; so, that led us to this particular convolution integral where we represented u of t as a train of impulses delta of t minus tau.

But this we discovered had problems because delta was a spiky function which could potentially damage our system. So, we tried to represent u of t in terms of other more benign functions and this was what was accomplished by using Fourier transforms, we were able to represent u of t once again this time not in terms of delta functions; but in terms of complex exponentials e power j omega t.

So, just as in the previous case delta of t minus tau had to be taken in the proportion u of tau times delta tau and added up to get u of t. Here we had to take any specific complex exponential e power j omega t in the quantity u of j omega times d omega and add it all up to get u of t. And that quantity or the frequency content of the signal u of t we called as the frequency spectrum and u of j omega which was also called to Fourier transform of u of t and that was given by this particular expression here. Now what we have managed to do is to represent u of t; not in terms of complex exponentials e power j omega t, but rather in terms of complex exponentials e power s t.

The reason we had to do this was because there are a few signals of great interest to us which could not be represented by using the mathematics of Fourier transforms. So, by replacing j omega by the term sigma plus j omega which we defined as s; we were able to represent a bigger set of signals including those of that of importance to us by using a complex exponential e power s t.

And the quantity once again of each of these complex exponentials e power s t that we need is characterized by the terms U of S times d s. So, therefore, you see that in each case we have attempted to represent u of t in terms of elementary functions delta function in the first case complex exponential e power j omega t in the second case and e power s t in the third case. And what you also see is this interesting similarity between Fourier transforms and Laplace transforms in that in Laplace transforms, we have s to be equal to sigma plus j omega.

So, in the case of Fourier transforms we have e to the power j omega t where omega is a real number, a real frequency whereas, in the case of Laplace transforms I can write e power s t as e power j times omega minus j times sigma times t. So, what we notice is that if we replace a real frequency namely e power j omega t a real frequency omega by a complex frequency omega minus j sigma; we will essentially transition from Fourier transforms to the Laplace transforms.

So, having justified the usefulness of Laplace transforms for control engineering purposes let us now examine some of the important properties of Laplace transforms.

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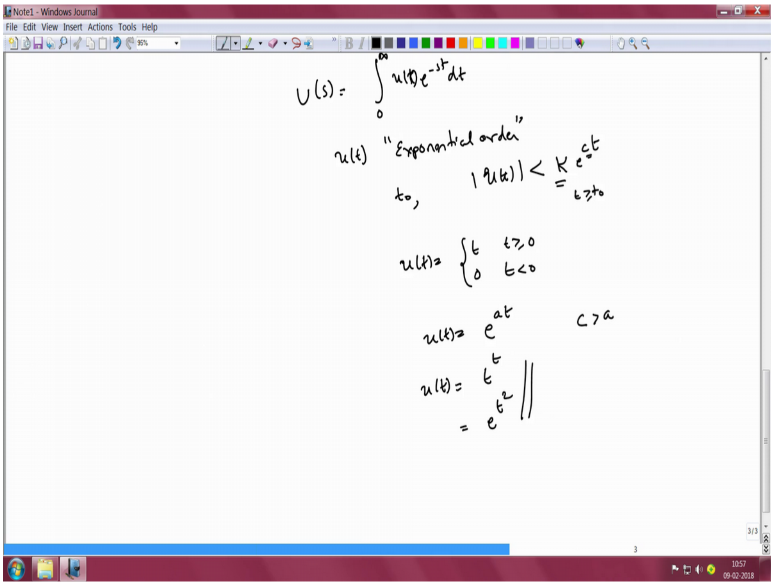

Now, the first couple of properties that I want to talk about or whether Laplace transforms exist for all signals u and for those signals for which it exists does it exist for all values of s.

So, to answer the first question Laplace transforms actually do not exist for all signals u. For Laplace transform to exist in other words we should be able to get a finite value for this integral; integral 0 to infinity u of tau e power minus s t u of t e power minus s t d t. This happens if only the signals u of t are of exponential order; what we mean by this, is that we should be able to find a time t naught beyond which the magnitude of this function u of t can be bounded by a term of the kind k e power c times t, where k is a constant and c is a constant; if a function u of t can be bounded by a function of the of the kind k e k e power c times t.

Where k and c are constants beyond the time t e greater than or equal to t naught, then such functions are said to be of exponential order. And it is easily verifiable that not all functions are of exponential order. So, if you take for instance a ramp input u of t equal to t; for t greater than or equal to 0 and 0 for t less than or equal to 0 less than 0 t less than 0.

Then you can easily show that the value of c here can be arbitrarily close to 0 and yet I will be able to bound u of t by using such a value of c. Likewise if I have a complex if likewise if I have u of t equal to e power a t ; then I can bound this function by choosing a c that is greater than a. So, even increasing exponentials or of exponential order; however, there are other functions that are not of exponential order. For example, if I take a signal of the kind u of t is equal to t to the power t or u of t is equal to e to the power t square.

These are all signals that are not of exponential order and for which the Laplace transform integral will not exist; in other words it explodes. So, we cannot define Laplace transform for signals of this kind, but fortunately for us, signals of this particular form are not commonly encountered in control system design and practice. And hence it is not of great concern to us that we cannot define Laplace transforms for these signals. The second question is does the Laplace transform exists for all values of s?.

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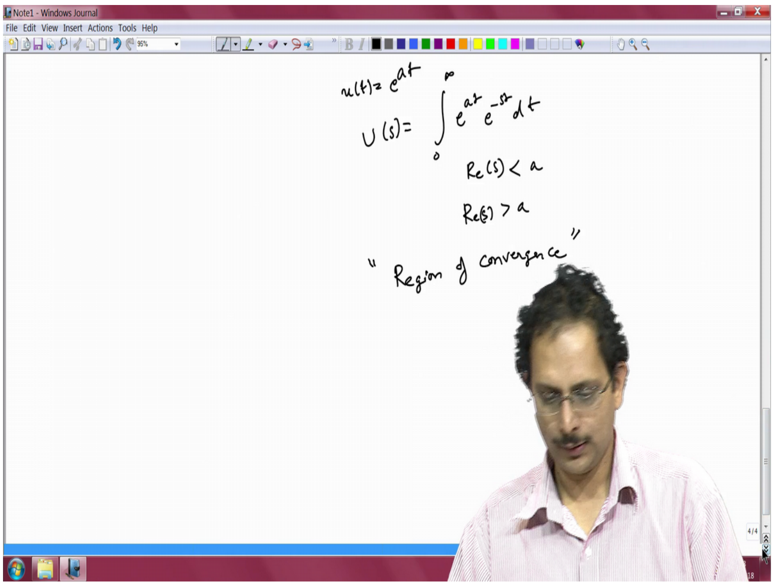

In fact, you can show that if you take the simple example of u of t equal to e to the power a t; then U of S is equal to integral 0 to infinity e power a t, e power minus s t d t; you can clearly see by inspection that if the real part of s is less than a ; then you would have an increasing exponential on as the integrand. And therefore, U of S cannot be defined therefore, the real part of s has to always be greater than a; for its integral to exist and this can be observed even in case of other functions.

So, there are only certain values of s for which U of S results for which the integral converges and this set of values of s is called the region of convergence. So, it is only within the region of convergence that the Laplace transform of the signal is defined, but not outside it. Having looked at the fact that Laplace transforms; firstly, cannot be defined for all signals u of t and secondly, does not exist for all for any possible value of s; we shall now look at the Laplace transforms of some common signals.

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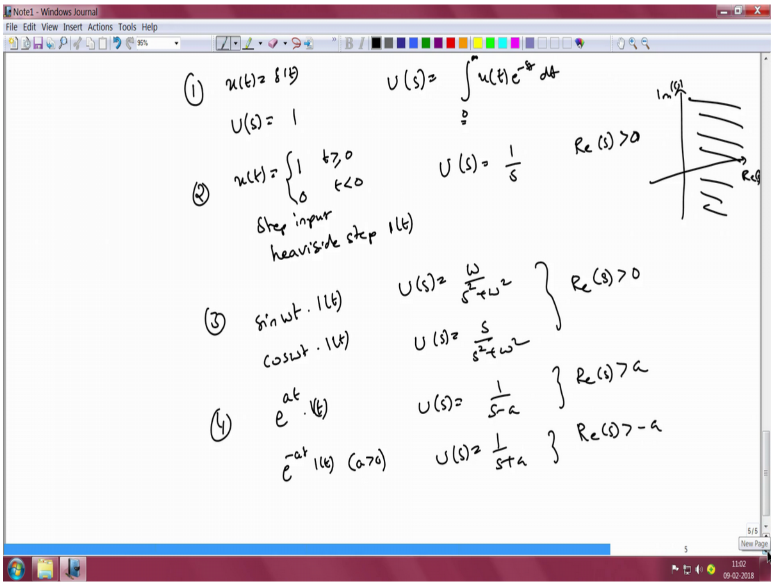

So, let us first take u of t is equal to delta of t; computing the Laplace transforms is a straightforward activity, where we simply evaluate the integral U of S is equal to integral 0 to infinity u of t e power minus s t d t. And if you do it for the case of u of t equal to delta of t you get U of S to be equal to 1.

So, this is the first signal one other common signal is u of t equal to 1 for t greater than or equal to 0 and equal to 0 for t less than 0; this is called as a step input. It is also often called as a Heaviside step and it is given a unique symbol, it is called 1 of t. So, what is the Laplace transform of 1 of t? You can show that the Laplace transform of 1 of t is given by 1 by s by simply computing this particular integral.

Third is a sinusoidal signal sine omega t; sine omega t times 1 of t because we are looking at signals that are starting only at time t equal to 0. And for this we can show that U of S is equal to omega by s square plus omega square. Likewise if you have cos omega t; times 1 of t U of S will be can be shown to be equal to S by S square plus omega square how about exponential functions? So, if I have e to the power a t times 1 of t; I can show that U of S is equal to 1 by S minus a.

But once again all these Laplace transforms have to be qualified by stating that these are the values of the Laplace transforms within the region of convergence for these particular signals. So, what is the region of convergence for a step input? You can show that the real part of s should be greater than 0. And which represents in the complex plane the entire of the right half of the complex plane.

So, this is the region of convergence for a step input; likewise also for sinusoidal input you can show that the region of convergence is real part of S is greater than 0. For an increasing exponential; however, you can show that the condition is that the real part of s should be greater than a. Likewise if you have a decrease in exponential so e to the power minus a t 1 of t, where a is greater than 0. You can show that U of S is equal to 1 by S plus a, but once again within the region of convergence real part of S greater than minus a.

So, these are the Laplace transforms of some of the commonly encountered signals; let us now look at some of the general properties of Laplace transforms that are applicable to all functions for which we can define Laplace transforms.

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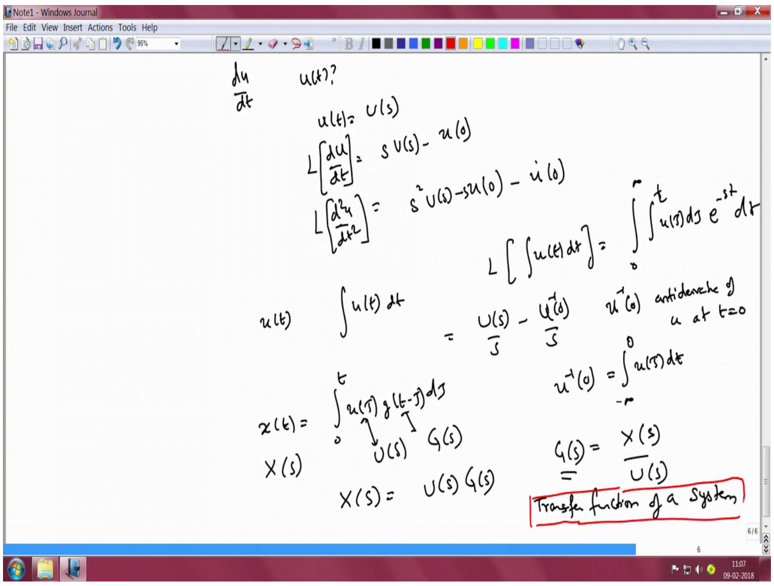

One is the property of differentiation of the signal; how is the Laplace transform of the time derivative of a signal namely d u by d t, how is a Laplace transform of d u by d t related to the Laplace transform of u of t? So, how are these 2 related?

So, you can show that the laplace if the Laplace transform of u of t is equal to U of S, then the Laplace transform of d u by d t is equal to S times U of S minus small u of 0; small u of 0 is a value of the signal time domain signal u of t at the time t equal to 0. So, likewise you can generalize this further if you have d square u by d t square, the Laplace transform of d square u by d s d t square; I will represent L of d u by d t as a Laplace transform of d u by d t.

So, like likewise a Laplace transform of d square u by d t square is equal to S square U of S minus s times u of 0 minus u dot of 0. So, this is something that you can show as well and so on and so forth. Likewise if I have any signal u of t can we derive the Laplace transform of integral of u of t in terms of the Laplace transform of u of t; it is indeed possible to do this.

So, we do this by computing the Laplace transform of integral u of t d t as integral 0 to infinity; integral u of tau d tau, the upper limit of integration being t; e power minus s t d t and you can show by integration by parts that; this is going to be equal to U of S by S minus u inverse of 0 by S; where u inverse of 0 represents the anti derivative of u at time t equals u. Or in other words it is integral up to time t equal to 0 u of tau d tau; this is equal to 2 inverse of c next is the convolution. So, if x of t is equal to integral 0 to t, u of tau, G of t minus tau d tau.

What is the Laplace transform X of S in terms of U of S and G of S; in terms of the Laplace transforms of the input and the impulse response. You can show using mathematics very similar to what was employed in case of Fourier transforms that X of S will be equal to U of S times G of S or in other words G of S that is a Laplace transform of the impulse response is a ratio of the Laplace transform of any response of the response to any input X of S divided by the Laplace transform of the input itself namely U of S.

So, this term G of S which is a Laplace transform of the impulse response plays as important a role in frequency domain and Laplace domain analysis; as the impulse response itself plays in the time domain. So, there is a special name for G of S, it is called the transfer function of a system.

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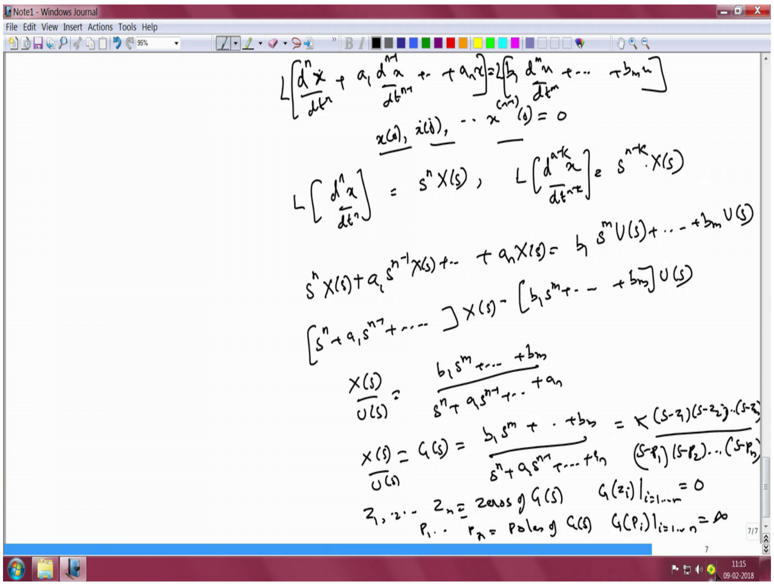

Having defined the transfer function let us look to look at how the transfer function appears for a system that is of interest to us.

So, to remind you the system that is of interest to us is the a system with the 1 by n th order differential equation relating the output to the input. So, d nth derivative of x with respect to time plus a 1 times n minus 1 th derivative of x with respect to time and. So, on and. So, forth plus a n x is equal to b 1 times m th derivative of u with respect to time and so on and so forth plus b m u with n initial conditions x of 0 x dot of 0 and so on and so forth; up to x n minus 1 of 0.

So, we are interested to obtain the transfer function for the system. So, to; so we are therefore, exclusively interested in focusing on the input output relationship; so, for the sake of simplicity at this point we shall not consider the effect of the initial conditions and we shall set all these initial conditions to be equal to 0. So, what we do not have to carry them over in our calculations and in our quest to obtain appearance of the transfer function for the system.

So, with all initial conditions equal to 0; I shall now apply Laplace transform both to the left hand side as well as the right hand side of this equation. So, I apply the Laplace transform of this and that should be equal to the Laplace transform of that. The Laplace transform of n th derivative of x with respect to time when the initial conditions are 0 is simply equal to S power n times the Laplace transform of X.

Similarly for any other index Laplace transform of n minus kth derivative of x with respect to time can be shown to be simply equal to S power n minus k times the Laplace transform of X. So, utilizing this fact we can write out the Laplace transform of the terms on the left hand side to be S power n times X of S plus a 1 times S to power n minus 1 times X of S plus and so on and so forth plus n times X of S to be equal to b times S power m times capital U of S, where capital of U of S represents the Laplace transform of u of t plus so on and so forth up to b m times U of S.

So, if I were to simplify this expression I can take out X of S from this from the left hand side and write this out as a polynomial in S. So, S power n plus a 1, S power n minus 1 and so on times X of S will be equal to b 1 S power m plus and so on and so forth up to b m times U of S. So, what we would get is therefore, that X of S by U of S is equal to b 1 S power m and so on and so forth up to b m divided by S power n plus a 1; S power n minus 1 and so on and so forth up to a n.

And from our previous slide we have shown that X of S by U of S is nothing, but G of S which is the Laplace transform of the impulse response. And therefore, by comparing these 2 equations we conclude that G of S is essentially equal to this expression. b 1 is power m and so on and so forth up to b m plus S power n plus a 1; S power n minus 1 and so on and so forth up to a n.

So, what we see therefore, is that for the kind of systems that we are interested in this course; the Laplace transform of the system is going to be a ratio of 2 polynomials in S. So, a certain numerator polynomial and a certain denominator polynomial; now what more can we tell about the appearance of the transfer function for systems of this kind; linear time invariant systems.

Before we uncover the other properties of the transfer functions; let me just define the notion of poles and zeros which emerge from this particular equation. Since I have the ratio of 2 polynomials as a transfer function, I can factorize these 2 polynomials and I can essentially write my G of S as some constant k times S minus Z 1, times S minus Z 2 and so on and so forth; up to S minus Z m divided by S minus P 1, times S minus P 2 and so on and so forth up to S minus P n.

Where P 1, P 2, P n and so on are the roots of the denominator polynomial, Z 1, Z 2 and up to Z m are the roots of the numerator polynomial. Now there is a special name for these roots. Z 1, Z 2 etcetera up to Z m are called the zeros of G of S for a very simple reason because at these particular values; if you if I were to compute G of Z i, where i goes from 1 to m; I would get G of Z i to be equal to 0 likewise P 1, P 2; P 1, P 2 and so on up to P n are called the poles of G of S.

Because at these particular locations; in order to compute G of P i going from one to n I would get infinity. So, at these particular locations G of S blows up; so these are Z 1 to Z m are called zeros and P 1 to P n are called the poles of the transfer function. Now is there anything more that we can say about the structure of the transfer function for linear time invariant systems?

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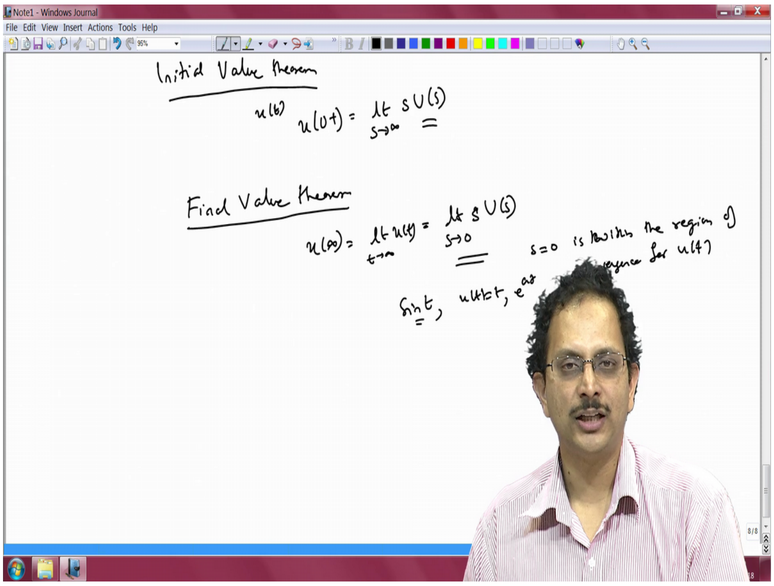

It is possible to do so but to uncover these additional properties; we have to appeal to a couple of theorems that relate the time domain response to the transfer function of the system. The first theorem is called as the initial value theorem; I state the theorem here without proof because you are expected to have come across this theorem in one of your preliminary courses on control systems.

If you have not come across this theorem the proof has been provided in the notes and is also indicated and is also available in the textbooks to which we have referred in the notes. So, the initial value theorem states that if I have a signal u of t; then in the vicinity of time t equal to 0 u of 0 plus, I have this relation that the value of the signal at the time u of at a time t equal to 0 plus is equal to limit; S tends to infinity S times capital U of S.

So, when capital U of S is the Laplace transform of the signal u of t; likewise there is one more theorem that allows us to predict the final value of a signal from its Laplace transform U of S, without actually having to obtain the inverse Laplace transform and that is called the final value theorem. I state this theorem also without proof; so, u of infinity which essentially means that limit t tends to infinity u of t is equal to limit S tends to 0; S times U of S where U of S once again is the Laplace transform of u of t. However, the final value theorem needs to be taken with a pinch of salt because this theorem is valid only for those signals u of t for which the origin s equal to 0 is within the region of convergence of that signal. In other words the point S equal to 0 should be a point at which the integral the Laplace integral has a finite value; is within the region of convergence for u of t. Hence it is not applicable for signals such as sine t or for signals such as u of t is equal to t or e power a t and so on and so forth.

If you were to apply it you get paradoxical results which do not make sense because these signals either blow up or in other words tend to infinity as time t tends to infinity or they are undefined as in the case of sinusoidal signals. But the reason we are not we get such contradictory and counterintuitive results for these cases is because these signals do not include the origin as part of their region of convergence.

Now let us focus on the initial value theorem and look at its consequences as far as the structure of the transfer function is concerned. The initial value theorem sheds important light on the structure of the transfer function G of S.

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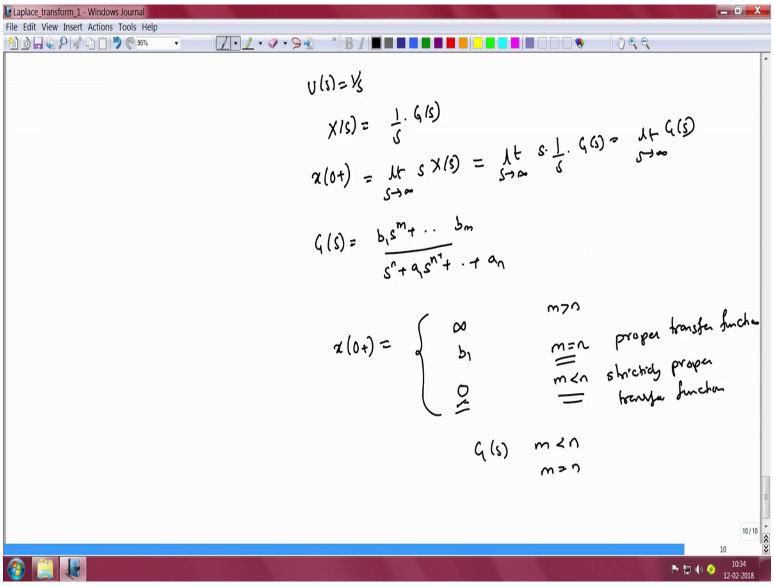

So, if you let us say assume you that you apply a step input to your system in other words U of S is equal to 1 over S; then the response of your system is going to be given by X of S is equal to 1 over S times G of S, where G of S is the transfer function of your system.

So, if you apply initial value theorem to this response X of S; then we would have that x of 0 plus would be equal to limit its tends to infinity S times X of S. And if I were to write it out in terms of G of S; I would have limit S tends to infinity, S times 1 over S times G of S which is equal to limit s tends to infinity G of S.

Now we know that G of S has this particular form; G of S is equal to some con some b one s power m plus so on and so forth up to b m divided by S power n plus a 1, S power n minus 1 so on so forth up to a n; this is a structure of our G of S. Now when we apply the limit S tends to infinity for this function G of S; we have 3 possible cases.

So, we would have x of 0 plus which is the value of the output S at a at a time t which is just a little bit after the application of the step input to be equal to; in the case when m is greater than n when m is greater than n you can show that in the limit S tends to infinity G of S would also tend to infinity. So, x of 0 plus would be equal to infinity if m is greater than n; however, if m is equal to n; in other words the degree of the numerator polynomial is equal to the degree of the denominator polynomial, then we would have, but G of S would be equal to b 1.

And in the third case when m is less than n we would have G of S to be equal to 0. Now what this tells us is that if we apply a step input to this physical system; if m is greater than n the output of the physical system an instant after application of the step input is equal to infinity; when m is greater than n is equal to b 1, when m is equal to n and is equal to 0 when m is less than n. Now it remains for us to judge which of these 3 cases is a practically reasonable case.

I am sure you would agree with me that just after application of a step input the response of a system which is initially at rest has to be close to 0. This is because all systems have inertia and it takes some time before any system can track sudden changes in input. Therefore, all physical systems have to satisfy this condition that m should be less than or equal less than n. However, there are occasions in which the timescale in which we expect our system to respond is much larger than the actual time that the system takes to respond.

In which case we can model such a system approximately as one where m is equal to m; in other words the response system is almost instantaneous in comparison with the time scale in which we expected to respond. So, transfer functions where m is equal to n or called proper transfer functions and transfer functions where m is less than n are called strictly proper transfer functions. Since we are dealing with physical systems for controlling physical systems, using electrical circuitry which is again physical systems and so on all such systems have a transfer function G of S where m is strictly less than n.

So, all our transfer functions are going to be strictly proper; however, as I said in the course of analysis sometimes we can get away by approximating the transfer function to be one where m is equal to n. Or in other words a transfer function is a proper transfer function, but this is only an approximation of the physical system which might be valid for the frequency range within which the system operates.

Therefore, the initial value theorem has shed an important light on the structure of the transfer function namely that the degree of the numerator polynomial m has to be less than or equal to the degree of the denominator polynomial. Having looked at the structure of G of S; thus far in the next clip we shall see what further simplifications are possible in the appearance of G of S.