Control System Design
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Lecture – 06
Fourier transforms (Part 2)

Hello, in the previous clip we defined what the Fourier Transform of a signal is. And we came up with this definition in our quest to represent signals using functions that are benign and not functions such as the delta function, which have you know which reach dangerously high magnitudes and could therefore, potentially destroy our system in our attempt to obtain the impulse response of that system.

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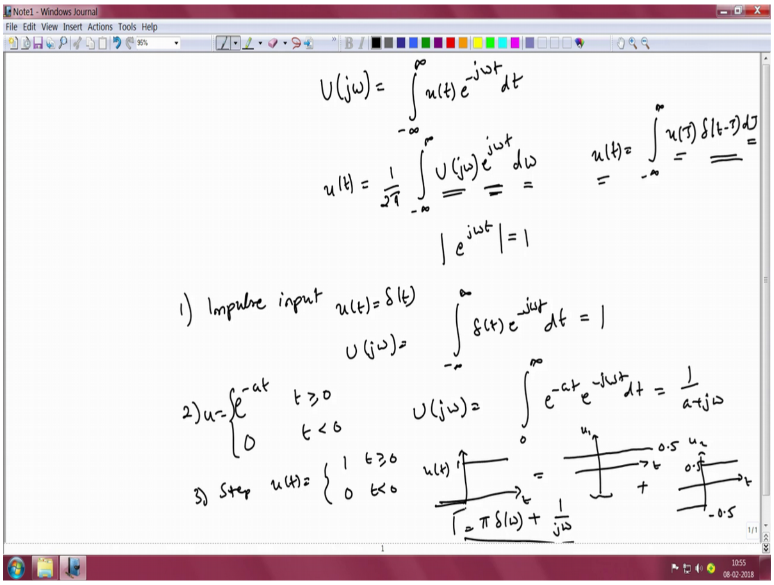

So, let us once again look at what the Fourier transform expression is. If I have a signal u of t then its Fourier transform U of j omega is given by u of j omega equal to integral minus infinity to infinity u of t e power minus j omega t d t. So, the utility of u of j omega is that it allows me to represent u of t in terms of complex exponential. So, I can write u of t as 1 by 2 pi integral minus infinity to infinity U of j omega e power j omega t d omega.

Now, I want to contrast the representation of u of t in terms of complex exponentials e power j omega t with what we had earlier, where we try to represent u of t in terms of delta functions. In that case, we had u of t equal to integral minus infinity to infinity u of tau delta of t minus tau d tau. So, what you see that we have done here is that, in the previous case we represented u of t as a sum of impulses. And these impulses how have magnitude u of tau delta tau or d tau.

And in this case, we have represented it as a sum of complex exponentials e power j omega t, and the quantity of each of these terms e power j omega t that we need to reconstruct u of t is given by U of j omega d omega. So, you see that, there is a continuity in terms of our effort to represent the signal using other elementary functions using elementary functions other than a delta function. And what we have succeeded in doing, now is that we have succeeded in representing it in terms of complex exponentials, which as I pointed out earlier are benign functions. Their magnitude never tends to infinity, their magnitude is always equal to 1. So, having looked at the utility of Fourier transforms in enabling us to represent u of t in terms of complex exponentials.

Let us briefly take a look at a Fourier transforms of some common signals. So, the first signal I consider is the impulse input, which is u of t equal to delta of t. For this case, the Fourier transform by definition is U of j omega equal to integral minus infinity to infinity delta of t e power minus j omega t d t, which from the definition of the delta function will be equal to 1.

Likewise, if I have a decreasing exponential e power minus e a t, and suppose this is exponential function starts at the time t equal to 0 t greater than or equal to 0 and for u is equal to e power minus a t when t is greater than or equal to 0 and e equal to 0 for t less than 0. Then for this function we can show that the Fourier transform u of j omega is integral 0 to infinity the lower limit becomes 0, because it exists only from time t greater than or equal to 0 e power minus a t e power minus j omega t d t. And we can show this to be equal to 1 by a plus j omega.

Likewise, if you have a step function, in other words u of t is equal to 1 for t greater than or equal to 0 and 0 for t less than 0. We can write the step function u of t which looks something like this. So, it is 1 for t greater than 0 greater than or equal to 0 and 0 for t less than 0. I can write this as a summation of a signal of constant magnitude 0.5 going from t minus infinity to t plus infinity, so u 1, plus I can write this as this plus a signal whose magnitude changes from minus 0.5 to 0.5 about time t equal to 0 I can write it this way.

And, since, the Fourier transform of the sum of two signals there some of the Fourier transforms of these two signals. I can show that the Fourier transform of a step is equal to the Fourier transform of this signal, which I can calculate and show it to be equal to pi times delta of omega and plus the Fourier transform of this waveform and this I can show to be equal to 1 by j omega. So, the Fourier transform of a step is therefore, given by this particular expression.

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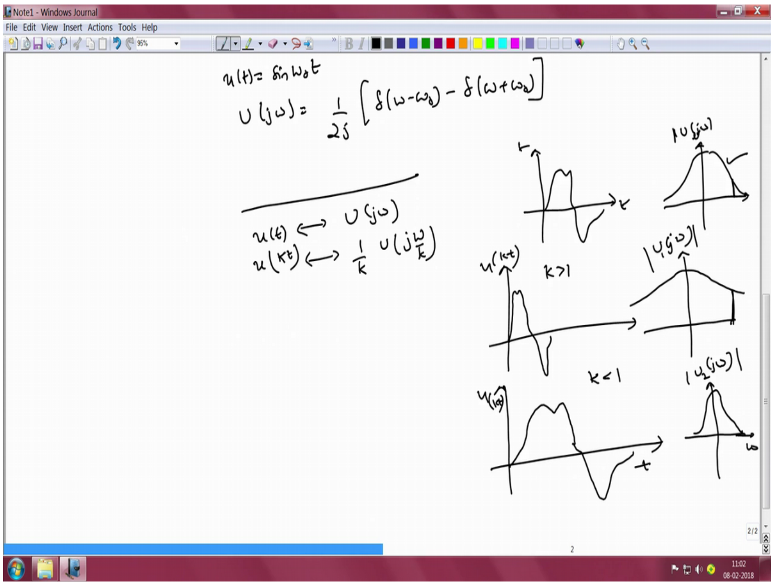

You can also derive the Fourier transform of other signals for instance, If you have a sinusoidal signal, u of t is equal to sin omega naught t, then its Fourier transform U of j omega can be derived to be equal to 1 by 2 j delta of omega minus omega naught minus delta of omega plus omega naught.

So, we will stop reviewing you know Fourier transforms of different signals at this point. We shall look at a couple of important properties of Fourier transforms. And the second property would in particular be very helpful to us in our quest which is to get the impulse response of a system without applying an impulse input, the first property is the effect of scaling in terms of time.

So, let us say, I have a signal u of t, and its Fourier transform is U of j omega. Suppose, I either speed up the signal or slow down the signal in other words I multiplying the independent variable with some constant k. Then I can show that the Fourier transform of u of k times t is 1 by k u of j omega by k. So, let me, just graph this relationship, and show you what I am what I mean. So, suppose u of t is some waveform that looks something like this as function of time. This has a certain Fourier transform, and, let us say, I am plotting the magnitude of U of j omega here, and it has some particular shape. Now suppose we take the signal u of k times t where k is greater than 1 which means that we are speeding up the signal what happens is u of k times t will be a signal that looks similar in shape to u of t, but would be compressed. So, it might probably look something like this.

And what this relationship says is that, the Fourier transform gets spread out. So, the Fourier transform of u of k times t where k is greater than one will be broader than the Fourier transform of u of t. So, u of U 1 of j omega, if I want to call it, that is going to look something like this, but it is going to be similar again in shape in comparison with U of j omega. So, intuitively what it means is that if you are speeding up the signal, you are you require many more of complex exponentials of higher frequencies to reconstruct that signal. And therefore, your Fourier transform also gets broadened, because at a particular frequency earlier you might not have needed as much of that complex exponential to construct u of t. But now because you are dealing with a faster signal you require much more of complex exponential at that particular frequency to reconstruct the signal.

So, in other words if you have a signal and you are represented in terms of complex exponentials, complex exponentials of higher frequencies can be used to reconstruct sharper and faster changing features of a signal. While complex exponentials at lower frequencies can be used to reconstruct parts of the signal that are changing slowly in terms of time. So, therefore, if I were to slow down the signal in other words, if I were to choose k to be less than 1, then once again in the time domain the shape of the signal is going to be very similar to u of t, but it is going to be stretched out in time, so this is how you going to look. And now, since we have slowed down the signal, we do not need as much quantity of complex exponentials to reconstruct this new signal as we did in the first case as we did in this case.

So, therefore, what happens is that the Fourier transform gets compressed. So, the Fourier transform would have the same shape as u of g omega, but it would have it would get compressed in the frequency domain. So, this is magnitude of U of j omega I will call this U 2 of j omega to represent the Fourier transform of u of k times t, when k is less than 1, and u 1 magnitude of U 1 of j omega to represent the Fourier transform of u of k times t when k is greater than 1. So, this helps us imagine what would happen to the frequency content of a signal when it gets speeded up or slowed down, and also highlights the fact that faster signals require much more of faster changing complex exponentials to for their reconstruction, while slower signals require much more of slower changing complex exponentials to reconstruct the signal.

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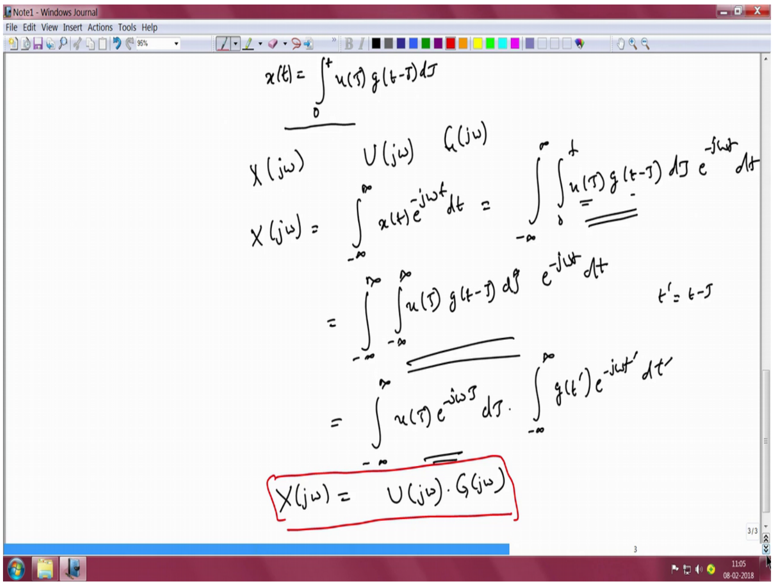

The second property that, I want to point out to which I said is particularly useful for us as engineers, control engineers is the property relating the Fourier transform of a convolution integral to the Fourier transforms of the individual functions. So, if I have x of t, as integral 0 to t u of tau g of t minus tau d tau. You will immediately recognize this as the response of my linear time invariant system to a certain input u of t given 0 initial conditions, what we are looking for is what is the Fourier transform of X let us say, we call it X of j omega what is it in relation to the Fourier transforms of U which is U of j omega and a Fourier transform of g, which is G of j omega how are these three terms related.

So, once again we go by the definition of the Fourier transform. So, X of j omega by definition is integral minus infinity to infinity x of t e power j omega e power minus j omega t d t, and that is equal to integral minus infinity to infinity integral 0 to t u of tau g of t minus tau d tau times e power minus j omega t d t. Now I can change the limits of integration for the expression u of tau g of t minus tau d tau to integration from minus infinity to infinity, because I know that u of tau is 0 for time t less than 0 and g of t minus tau is 0 for time t greater than for time tau greater than t. So, there is no change in the value of this integral if I were to replace t with infinity, and 0 with minus infinity, because in one limit this term goes to 0 in other limit the other term goes to 0. So, this is going to be u of tau g of t minus tau d tau times e power minus j omega t d t.

Now, after some algebraic manipulation I can show that this double integral can be written as integral minus infinity to infinity u of tau e power minus j omega tau d tau times integral minus infinity to infinity g of t prime e power minus j omega t prime d t prime, where t prime is equal to t minus tau. And you will recognize the first term, here as a Fourier transform of U of t which is U of j omega, and the second term as a Fourier transform of g of t which is G of j omega therefore, X of j omega is equal to U of j omega times G of j omega. This is a very important relationship is an important milestone for us, because we have very nearly come to the end of our quest. So, to remind you our quest was to obtain the impulse response of a system without applying an impulse input, and this particular relationship here shows us the way as to how it can be done. So, we shall elaborate a little bit more on this particular fact.

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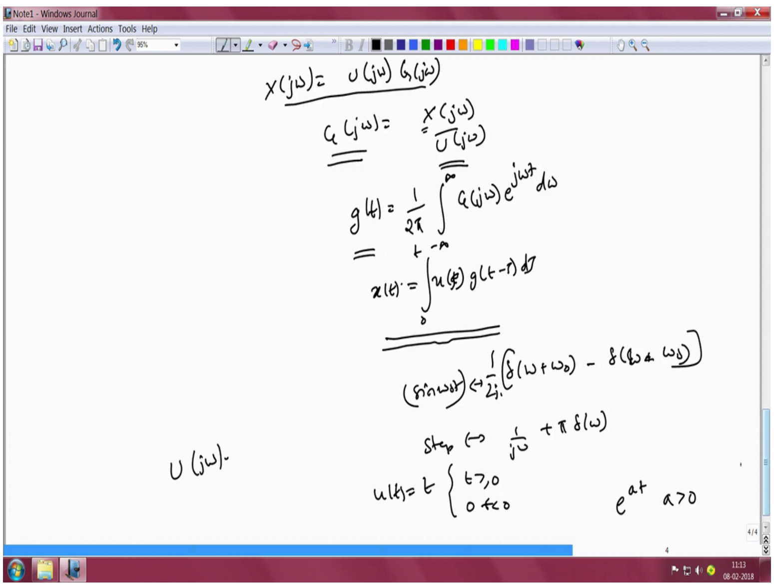

So, we have X of j omega to be equal to U of j omega times G of j omega, where G is the impulse, Fourier transform of the impulse response. So, therefore I would get G of j omega is equal to X of j omega by U of j omega. What does indicates is that, I can get the Fourier transform of the impulse response by applying any input U, and measuring the response to that input X. And taking the ratios of the Fourier transforms of the response to the Fourier transform of the input, which in turn implies that I do not necessarily have to apply a impulse input itself to obtain the impulse response G of j omega.

I could choose to apply for instance a step input. I could choose to apply sinusoidal inputs and sweep the sinusoidal frequency. I can choose to do all these things, and from the obtained time domain response x of t. If I take the Fourier transform (Refer Time: 16:03) time domain response, and divide that by the Fourier transform of the signal itself, I can obtain the Fourier transform of the impulse response. So, without having to apply impulses, without applying these dangerous inputs, which could potentially destroy our system.

The mathematics of Fourier transforms allows us to extract the impulse response well. We are not there yet, because we have managed to get the Fourier transform of the impulse response. The impulse response, of course is simply the inverse Fourier transform of G of j omega and that is given by integral minus infinity to infinity G of j omega e power j omega t d omega. So, this is one significant achievement. We started out by noting the centrality of g of t in our ability to obtain the response x of t for any specified input u of t. And that was in terms of the convolution integral. So, x of t was equal to integral 0 to t u of tau g of t minus tau d tau.

So, we noted two issues with this. One is how can we obtain g of t without applying an impulse input, and we have addressed that. And interestingly, the mathematics of Fourier transforms also helps us to address, the second issue that we discussed as far as this representation is concerned. We noted that in order to obtain the response to any input u of t, we had to compute this convolution integral. And this convolution integral is not easy to compute. One has to sit down take a pen and a paper, and actually go through the motion of solving this integral.

And that is a fairly non-intuitive activity. So, it is not easy as engineers for us to predict, how the response might be like for a specified u of t, for a given g of t. However, what the mathematics of Fourier transforms has allowed us to do is to replace the convolution integral operation, that we see in the time domain by a simple multiplication. So, the Fourier transform of x of t is simply the product of the Fourier transforms of u and that of g. And a product operation is a far more intuitive operation.

So, if we can train our intuition on how we can how the signals that appear in the time domain might appear in the frequency domain, then that is a worthwhile exercise to undertake. Because, in the frequency domain I just have to multiply the impulse Fourier transform of the impulse response, which is G of j omega with the Fourier transform of the input, to know what the Fourier transform of the response is going to be like. And with this intuition about how x of t might appear, if I know X of j omega, I can roughly predict how x of t might appear, after performing that multiplication. So, in our attempt to represent signals using benign functions, such as complex exponentials.

We have firstly found that this mathematics, namely the mathematics of Fourier transforms allows us to obtain g of t. But, having obtained g of t, we also discover that it is not necessary for us to know go back to the time domain. If we can build our intuition, on how signals appear in the frequency domain. And if we do that, then we can stick in the frequency domain, and understand the response of systems to various inputs in a far more transparent manner.

As useful as the mathematics of Fourier transforms is there are some certain, still certain very important drawbacks associated with it. As far as our requirements as control engineers is concerned. So, for instance if you look at the Fourier transform of a sinusoidal signal or a d c input, you see that you have delta functions. So, for example the Fourier transform of sine omega naught t had a delta function at plus omega naught and minus omega naught, delta omega plus omega naught delta omega minus omega naught, and 1 by 2 j of this was the Fourier transform of sin omega t.

Similarly, the Fourier transform of a step was 1 by j omega plus pi times delta of omega. So, once again you had delta functions. And delta functions by their very nature are difficult to deal with, because they have magnitude tending to infinity over a time duration that is tending to 0. So, they are difficult to imagine, difficult to sketch, and therefore are difficult to work with. So, this is one problem but, a bigger problem is that there are many signals of interest to us as control engineers for which you cannot define a Fourier transform at all.

So, for instance if you have u of t is equal to t, so this is a ramp input. Let us say for t greater than or equal to 0, it is equal to t and is equal to 0 for t less than 0. You can show that you cannot define a Fourier transform for the signal, because when you compute that integral U of j omega that integral diverges, it explodes. And similarly, a rising exponential e to the power a t, where a is greater than 0.

Similarly, also does not have a Fourier transform, because that expression the integral for U of j omega, once again explodes. But, all these signals a ramp signal an increasing exponential, and even sinusoidal signals, and step signals and so on, are signals that we very commonly encounter, during our you know work as control engineers. So, it is therefore a big handicap that we cannot apply Fourier transforms for these specific signals. So, how do we salvage the situation, we salvage the situation by wondering whether we can obtain the Fourier transform of these signals by suitably weighing them down.

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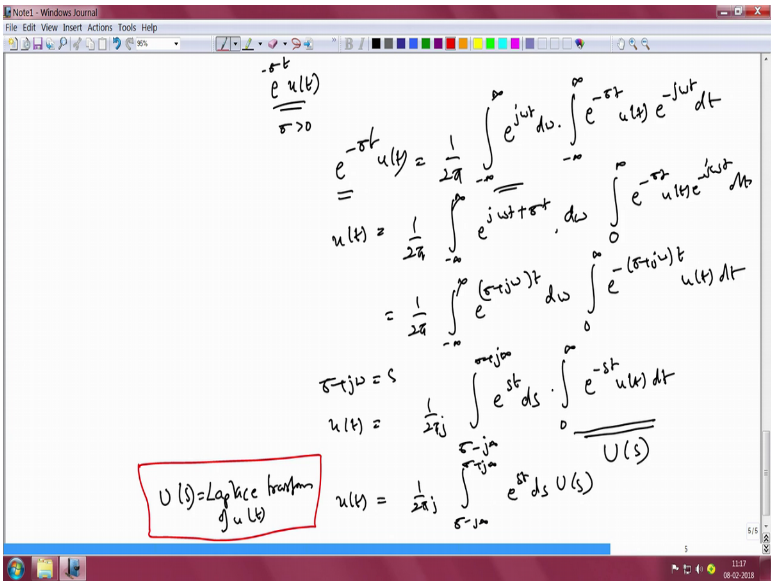

So, if I want to multiply, the signal for which we cannot define a Fourier transform. So, let us a step input or a increase in exponential or some other such function, with a decreasing exponential e power minus sigma t, where sigma is greater than 0, in which case these signals are not really tending to infinity, as time t tends to infinity ok. Is it possible, then that this signal could have a Fourier transform. It is quite likely that it would have a Fourier transform, because such a signal might then be square integrable.

So, if that is so then I can write out the Fourier transform of I can write down this signal e power minus sigma t u of t as 1 by 2 pi integral minus infinity to infinity e to the power j omega t d omega times the Fourier transform of this signal, which is given by integral minus infinity to infinity e to the power minus sigma t u of t e to the power minus j omega t d t.

Now, I shall shift e power minus sigma t to the right, so I would have u of t to be equal to 1 by 2 pi integral minus infinity to infinity, since this function is not a function of omega. I can take it within this integral, and write this a e power j omega t plus sigma t times d omega integral e power minus infinity to infinity e power minus sigma t u of t e power minus j omega t d t.

Now, firstly we shall note that the lower limit of this integral can is not minus infinity, because my time my signal u of t assumed to start at time t equal to 0. So, I shall therefore replace the lower limit of this integral with 0. And I shall now write this as 1 by 2 pi integral minus infinity to infinity e power sigma plus j omega times t d omega, times integral 0 to infinity e power minus of sigma plus j omega times t u of t d t. And I shall define sigma plus j omega as some complex number as in which case I can write u of t as 1 by 2 pi. The limits of this integration will now change, it will be if s is defined as sigma plus j omega and omega is going from minus infinity to infinity, then s will go from sigma minus j infinity to sigma plus j infinity e power s t d s times integral 0 to infinity e power minus s t u of t d t.

Now, I shall call e power minus s t u of t d t as U of s, and write u of t capital U of s, and I shall write u of t s 1 by 2 pi sorry there is a j here. So, I shall write this as 1 by 2 pi j integral sigma minus j infinity sigma plus j infinity e power s t d s U of s. And U of s is called the Laplace transform of u of t. You see therefore that there is a real valid reason for us to introduce Laplace transforms, when we are dealing with control systems, because we are having to deal with signals for which you cannot define Fourier transforms. Even though the language of Fourier transforms is very intuitive, and easy to apply. We shall look at other properties of Laplace transforms in the next video clip.