Control System Design
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Lecture – 50
Describing functions (Part 2/2)

Hello in the previous clip we introduced the notion of a Describing Function and talked about how it will help us to grapple with the issues that are introduced by non-linear elements that inevitably will be present in any control loop. So, we first catalogued the different nonlinearities that one generally encounters, we started from the electrical side of the Control System and looked at the presence of non-linearity such as, saturation and quantization and subsequently when we come to the electromechanical side, we saw that we had issues associated with relays and namely on off switching and then nonlinearity associated with relays itself in terms of hysteresis, subsequently views a different actuator and actuator other than the relay it might have a smooth in, but non-linear input output relationship.

The plant itself might exhibit nonlinearities, the mechanical elements are transfer power from the motor to the plant namely elements such as gears and so on, might also have nonlinearities in terms of backlash and dead zone and things of that kind.

So, therefore, we notice that there are commonly several nonlinearities to be encountered in a feedback control system, but up to the last clip we had resolutely you know decided to ignore the effect of these nonlinearities and assumed that there effect was small. In the last clip we asked ourselves how we could analyze the effect these nonlinearities on the control system that we have developed, the linear control system that we would have designed and analyzed and in search of a suitable common tool that can be employed to analyze a effect of all these desperate nonlinearities, which originate from fairly different physical you know influences. We came up with this idea of a describing function which attempted to extend the notion of a transfer function, but to a non-linear system.

So, to characterize a transfer function for instance, in the case of a linear system we apply any general input, a broadband input look at the response and take the ratios of the Laplace transforms of the response and the input and then obtain the transform function. But we discussed the difficulties associated with do in that in case of non-linear elements because, the response to 1 input might be substantially different from the response to a different input.

So, we could not apply any arbitrary input and get something similar to a transfer function for a non-linear element. So, we have to decide on what input we would provide and since as control engineers we are quite concerned about the stability of the closed loop system and that is and the effect of these non-linear elements on the stabilities far more important than there effects on the performance of the system.

We noted that a sinusoidal signal would be an appropriate input to our non-linear element because, if you have a control system that is on the threshold of instability then any input to the control system would result in a lot of transient response involving oscillations, which settle down in longer and longer times if our closed loop system is close to the threshold of instability.

So, since the sinusoidal signal therefore, is of importance when we have a control system which is on the threshold of instability, we decided to choose the sinusoidal signal as an input to our non-linear element. Then we discussed two other issues associated with characterizing this non-linear element namely; that the response will not be a sinusoid unlike in the case of a linear system.

In this case the response will be a distorted sinusoid, some non-linear function, but with a period that would be identical to the period of the sinusoidal input signal. And the second issue we discussed was that, the response of our non-linear element would also be dependent on the amplitude of the input, amplitude of the sinusoidal input that we provide, unlike in the case of a linear system where the response is independent of the amplitude of the input.

So, what we decided therefore, was that, you apply a sinusoidal input, look at the response of the system at a frequency of a the input signal. So, the response might be a periodic signal, but this periodic signal can be return out using Fourier series as a sum of sinusoidal signals of frequencies equal to the frequency of the input signal plus its higher harmonics.

So, we decided that we will look at only the component of the response at the frequency that matches the frequency of the input signal and then study the ratio or the relationship between this component and the input as function of both the frequency of the input as well as its amplitude. And that is what we called as the describing function of our non-linear system.

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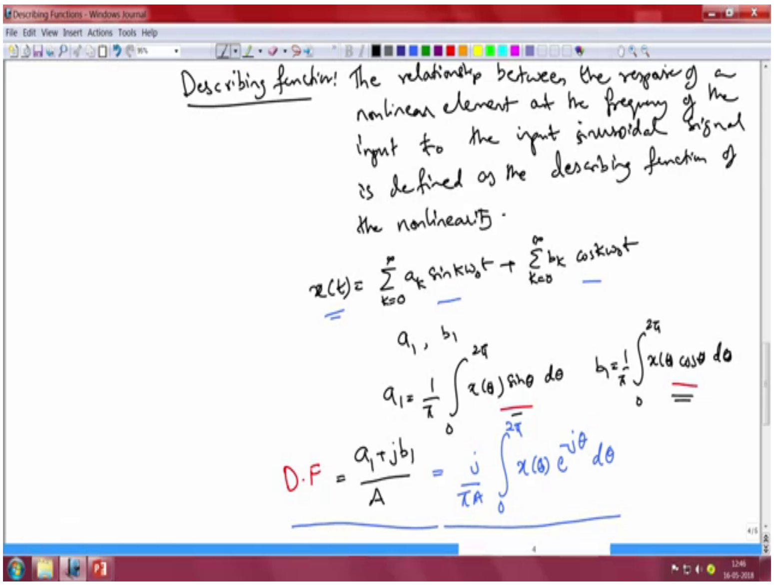

So, I have written down here once again the definition that we discussed in the previous clip, a describing function is the relationship between the response of a non-linear element at the frequency of the input to the input sinusoidal signal itself. So, mathematically we wrote out the expression that x of t can be written in terms of sinusoidal signals of the same frequency of excitation and its higher harmonics. So, it can be written out a sigma of a k sin k omega naught t plus b k cos k omega naught t. So, it is not sin k omega naught here. So, it is cos k omega naught t.

Then as far as describing functions definition is concerned we are interested only in the coefficients a1 and b1 and the coefficients a1 and b1 are given by the 2 expressions here, which have be already underlined using red color and a describing function is in general a complex entity which is defined as a1 plus jb1 divided by A. So, in general the describing function is a function of both the amplitude A as well as the frequency omega.

Then we went on to derive the describing function for a particular nonlinearity that we considered in the previous clip. And in this clip what we shall do is we shall first look at the describing function for some common nonlinearities that we come across and subsequently also discuss how a describing function can be put to use in order to analyze the stability of feedback systems and subsequently go one step further and see how it can even be used in the process of synthesizing a controller for our feedback system.

So, how can we incorporate the nonlinearity and design a suitable controller that functions well even in the presence of this nonlinearity. So, that is also what we will see in this clip.

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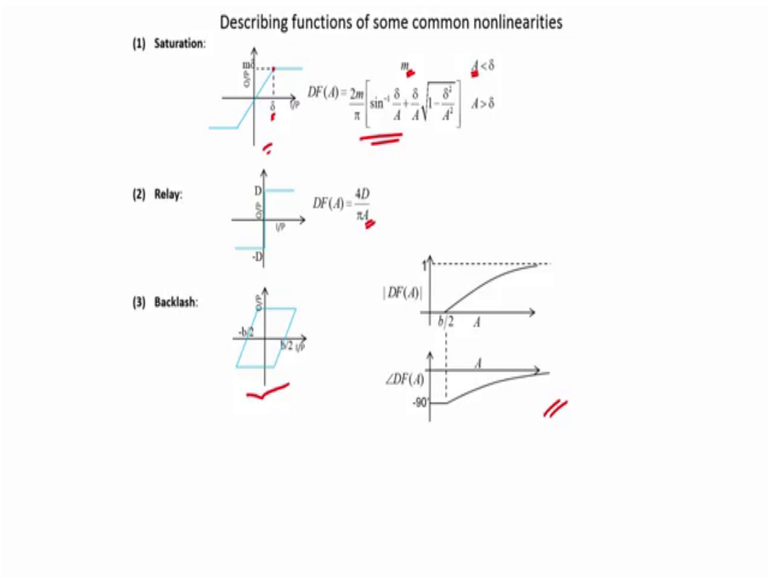

So, we shall first start with the saturation nonlinearity and the input output relationship has been shown here. So, the output is proportional to the input up to some either voltage or value which in this case has been called as delta. And if you write down the describing function, we note that sinusoidal signals of amplitude between 0 and delta, we would have the describing function to be equal to simply the slope of the straight line that relates the input to the output.

So, we note that for sinusoidal signals whose amplitude assumes values between 0 and delta, the describing function is going to be simply given by the slope of this linear characteristic because, it behaves as a simple linear amplifier between this range and if that slope happens to be m, then the describing function is going to be equal to m for amplitudes A of the input sinusoid less than delta.

Now, if the amplitude is greater than delta, then we note that the response of this or the output of this non-linear element would be a clipped sinusoid. And if we write out the first harmonic component of the clip sinusoid and take its ratio with respect to the amplitude of the input sinusoid, then we get the describing function in this range of amplitudes to be given by the second expression here namely, the describing function is going to be equal to 2 m by pi times sin inverse of delta by a plus delta by a times square root of 1 minus delta square by A square.

And you note that if we substitute delta is equal to A, which is known to be a point that is at the very threshold of this change of the input output characteristics for this non-linear element, we get the describing function to be equal to m exactly as we expected to be.

So, what is worth noting in this case is that, this is a static non-linearity and hence the describing function is not a function of frequency, it is only a function of amplitude and that function has been written out here. Next we shall look at another a non-linear element namely that of a relay, whose output changes between values plus D and minus D when the input sign changes from a positive sign to a negative sign.

And for this case it can be shown that the describing function once again would be dependent only on the amplitude and not the frequency because, this is once again a static nonlinearity. B y a static nonlinearity I mean that, if you apply an input sinusoid and you change the frequency of the sinusoid, the nature of the response will not be dependent on the frequency.

So, we therefore, have in this case that the describing function to be equal to 4D by pi A, where A is the amplitude of the sinusoid and D represents the magnitude of the positive value or the negative value that the relays output would assume, when we provide a certain input to the relay. So, likewise we can also derive the describing function for backlash.

So, how does one derive all of these things, all one has to do is to provide a sinusoidal input to this non-linear element, look at the response it would be a periodic signal, but with distortion and now take its you know first harmonic or the look at its frequency content at the frequency of the input sinusoid and obtain the ratio between the 2 or the complex ratio between the 2.

In the case of backlash, whose input output characteristics has been shown here, I have not written down the mathematical expression for its describing function, but I have depicted it graphically. So, we note that when the input is of magnitude less than b by 2, which is the gap that needs to be traversed before for instance the output gear tooth can mesh with the input gear tooth, then we know that the output will be 0 up to the point that this contact happens.

So, up to an amplitude of b by 2, a sinusoidal signal of amplitude b by 2, the describing function will be equal to 0 in magnitude, but once the amplitude increases beyond b by 2, the describing function will gradually increase. And for very large amplitudes this little gap that one might have that is responsible for backlash especially, in case of gears will become very small compared to the amplitude of the inputs sinusoid and therefore, the driven gear will exactly follow the driving gear, at such large amplitudes and hence the describing function will tend to 1.

If you look at the phase of response, we note that at b by 2 the phase is going to be close to minus 90 degrees. We cannot really define phase for amplitudes less than b by 2 because, the response is actually 0, so assigning a phase 2 a complex number of 0 magnitude does not make much sense, but in the interest of ensuring continuity at the point b by 2, we shall assume that the phase is equal to minus 90 degrees even for amplitudes less than b by 2.

Now, as the input amplitude A increases beyond b by 2, the phase lag reduces and as we discussed when the input amplitude is very large much larger than the backlash that exist between the gear teeth for instance, then our output gear the driven gear will exactly follow, almost exactly follow the driving gear and hence the phase will asymptotically approach 0 degrees. What I want to underscore in the plot that I have shown here for the describing function of a backlash is that, the x axis is not frequency. So, this magnitude and phase is reminiscent of a bode plot. Hence one might be uncritically let to assume that the x axis is frequency, but in this case, the x axis amplitude.

So, it is not frequency, but it is amplitude and we are plotting the magnitude and the phase of the describing function as function of amplitude alone. Why have we not considered frequency? That is because, once again backlash is a static nonlinearity and the input output characteristics for backlash is independent of the frequency of the sinusoidal signal that we used to drive this element, that has drive this non-linear element that has backlash in it.

The next nonlinearity is an interesting nonlinearity. This is not something that is inherent to most control systems, but something that we might want to intentionally insert into a control system oven to the benefits that we as control engineers who accrue from it. So, this nonlinearity is what is known as a Clegg Integrator. So a Clegg Integrator.

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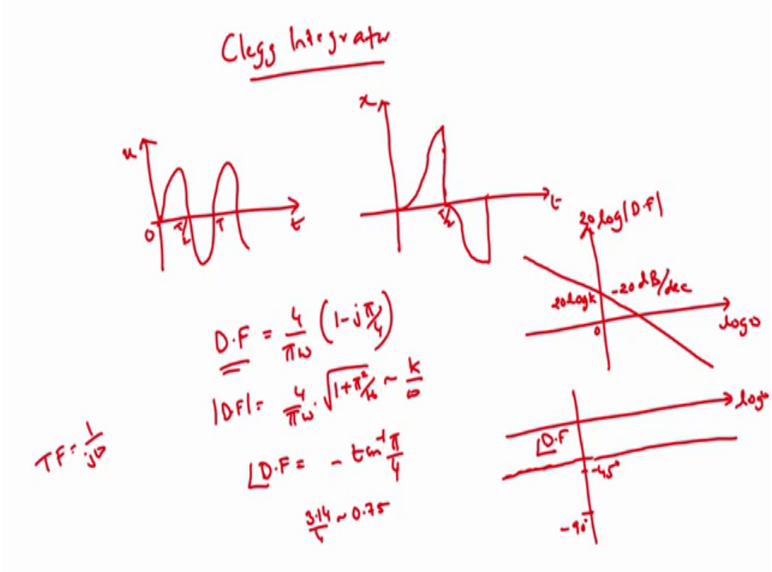

The input output characteristics for a Clegg Integrator are best explained graphically. So, if we provide a sinusoidal input to a Clegg Integrator; so let us say I have provided a sinusoidal input. What the Clegg Integrator does is, it integrates the signal up to the point that a signal crosses 0 either in the positive direction or in the negative direction, but the moment input signal crosses 0, the integrator resets the output. So, if 1 were to plot, the output of a Clegg Integrator. So, here the x axis is time, the y axis is the input u, if 1 were to plot x versus t for the integrator, Clegg Integrator then between the times 0 and t by 2, where capital T represents a period of this sinusoidal signal.

We have the signal being always greater than 0. So, this integrator therefore, integrates this signal between 0 and T by 2. Now at T by 2 what happens is that, the signal is crossing 0. So, the signal is going from a value slightly above 0 to a value slightly below 0. The moment a 0 crossing happens, the Clegg Integrator resets the output, which means that the output will abruptly come to 0 at T by 2.

Then once again between T by 2 and t the signal always remains negative and hence if one were to integrate the signal, it will look something like this. And again at capital T there is a 0 crossing for the input signal and the output of the Clegg Integrator gets reset. And this waveform repeats for all future time. So, this is the input output characteristic of a Clegg Integrator.

What is very interesting to note in this non-linear element is that, this non-linear element satisfies the scaling property, which is one of the test for a non-linear for linearity of a system, but it does not satisfy superposition property. So, if our to scale the input the output also gets scaled, but if I take two different sinusoids and add them up the output of the Clegg Integrator for the sum of the 2 sinusoids would not be equal to the sum of the outputs of the integrator for each of the 2 sinusoids.

Hence, this is a strange and interesting non-linear element which satisfies scaling, but not superposition. Now, why is this of interest to us as control engineers, it is of interest to us as of control engineers because, it has this particular describing function. So, if 1 were to look at the first harmonic of the response and then study it as function of the input frequency, one would note that the describing function for a Clegg Integrator is given by 4 by pi omega times 1 minus j pi by 4.

Now, what is so special about this particular describing function, let us just first sketch the describing function in a bode plot to see what is attractive about this describing function. So, let us first draw the magnitude plot, where we plot log omega versus log of 20 log magnitude of the describing function and what you notice is that, the describing function has this 1 over omega dependence of gain on frequency. So, the magnitude of the describing function as frequency are function of frequency omega is going to be given by 4 by pi omega times square root of 1 plus pi square by 16.

So, everything here is a constant, so it is on the of the form k by omega. And this is reminiscent of the magnitude characteristics of a conventional integrator. So, the roll off will be here minus 20 decibels per decade, exactly as in case of a conventional integrator. There will be some offset with the x and y axis depending on the particular value of gain that we have. So, for instance, if the gain is k then at log omega equal to 0 or in other words omega is equal to 1 the magnitude will be equal to 20 log k.

So, this is the magnitude characteristic of the Clegg Integrator. And you would notice if you plug in the numbers that this value of k is actually close to 1. So, the y intercept is actually quite close to the origin. Now what about the phase characteristics for the Clegg

Integrator, if our to plot the phase as function of frequency, so the angle of the describing function as function of frequency, we note that if this is the describing function for the Clegg Integrator by definition the angle, which is TAN inverse of the ratio of the imaginary part to the real part is given by the angle of DF is equal to minus TAN inverse of pi by 4.

Now, what is numerically the value of pi by 4? We note that pi is approximately 3.14. So, pi by 4 is 3.14 by 4, which is a little bit it is on the order of 0.75 because, if it was 3 by 4, it would be 0.75. So, 3.14 by 4 is the little bit more than 0.75. So, the angle of the describing function is going to be equal to minus of tan inverse of pi by 4, which is going to be on the order of minus of tan inverse of 0.75.

We note that tan inverse of 1 is pi by 4 hence; tan inverse of 0.75 is an angle less than pi by 4 or an angle less than 45 degrees. Therefore, we would have the angle of this describing function to be an angle that is less than 45 degrees and to be independent of frequency. So, if the angle minus 90 degrees is here and minus 45 degrees is here then the angle of describing function which is tan inverse of minus pi by 4 will be somewhere there.

And now you begin to appreciate the advantage of a Clegg Integrator. It has the magnitude characteristics of a regular integrator. So, therefore, it has the potential to give you good performance at low frequencies, but the transfer function of a regular integrator is of course, given by 1 by j omega and the phase associated with the regular integrator is minus 90 degrees. So, the phase lag is 90 degrees; however, in this case the phase lag is actually less than 45 degrees.

So, what this means is that, if we were to integrate a Clegg Integrator into a feedback system, you can reap the benefits at low frequencies as a consequence of the magnitude characteristics of the Clegg Integrator at low frequencies, but at high frequencies near the gain crossover frequency for instance, the phase performance gets improved because, unlike a regular integrator which supplies a phase lag of 90 degrees, a Clegg Integrator supplies a phase lag of less than 45 degrees. And hence the phase margin can be improved by an amount of 45 degrees or more as a consequence of using the Clegg Integrator.

So, this is another example of a non-linear element and in this case one might wish to deliberately introduce this non-linear element into a feedback system and we will see an example of that in the in a short while. In the interest of improving the performance of a feedback system beyond what can be accomplished by using linear controllers. So, having seen the describing function of several common nonlinearities and the describing function of one special nonlinearity that we might intentionally wish to have as part of our feedback system, let us now briefly see how we can employ the notion of a describing function.

To answer the question of how we can employ the notion of a describing function, let us first ask ourselves what a describing function of a linear system is, can we define something like a describing function for a linear system. In order to do that what do we need to do, we need to first provide a sinusoidal input to this linear system, look at the response of the linear system and in steady state we have to look at the first harmonic of the response. But it so happens that if our system is linear and stable then, if you provide a sinusoidal input to this linear system, then the response will also be a sinusoidal signal at in steady state and hence this going to be only 1 component and that is going to be at the frequency of the input sinusoid.

But the magnitude of this response will be larger than that of the input and there will be a phase shift between the input sinusoid and the output sinusoid. Therefore, if you look at the describing function of a linear element, it simply boils down to the transfer function of that element. Therefore, this describing function can be easily be viewed as an extension of the notion of a transfer function, but to a non-linear system. It elegantly reduces to simply a transfer function of the system when the system is a linear system, but if the system has nonlinearities then we also need to incorporate the dependence of the output on the amplitude of the input and hence it becomes a describing function in a manner that we defined earlier on in this clip.

So, for all practical purposes as control engineers we can view a describing function of a non-linear element as in some sense a quote unquote transfer function of that element. I am putting the phase transfer function between quotations because, transfer functions can be defined only for linear systems, but since we have defined a describing function in a manner very analogous to the definition of a transfer function, we shall choose to visualize the describing function in a manner similar to how we visualize a transfer function. And that immediately clarifies how we can put describing function to use as control engineers; either in analysis of control systems or in their designs.

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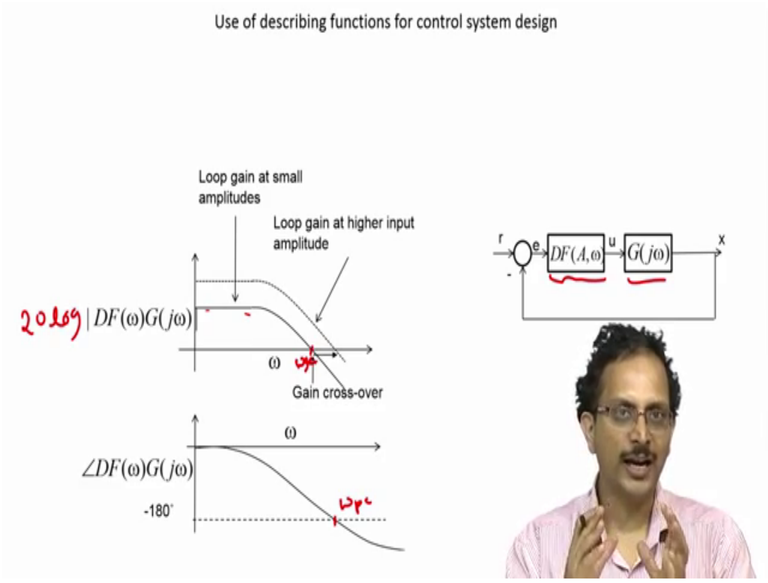

So, for instance, if we have the bode plot of the linear system looking as shown by the solid curve here. It has a certain magnitude plot and it has a certain phase plot, but there is also a certain linearity in our open loop system with a certain describing function DF, which is a function both the amplitude A, as well as the frequency omega. What does this describing function do to the open loop transfer function? Since a describing function is a generalization of the notion of transfer function to a non-linear element, this describing function DF of A comma omega can be viewed as a fancy transfer function. Hence, the open loop transfer function can be viewed as simply the product of DF of A comma omega and the existing open loop transfer function which I have called here as G of j omega.

So, if 1 were to now plot the magnitude and phase characteristics of the overall open loop system, the if we have a static nonlinearity for instance, this static nonlinearity will only affect the magnitude characteristics because, it adds no phase lag at any frequency, the phase lag associated with a static nonlinearity such as saturation or relay or a backlash element is 0. So, it only affects the magnitude plot and depending on the magnitude of nonlinearity that we might have, the magnitude plot by either get pushed up or pushed down. And what will the due to the gain crossover frequency; the gain crossover frequency will correspondingly get pushed either to the right or to the left.

Now, the describing function if it is meant to characterize a non-linear element, which shows some non-linear dynamics, there will also be a phase lag associated with this describing function, as function of frequency, just as we saw in case of the Clegg Integrator, in which case the describing function will also modify the phase characteristics of our linear open loop system.

So, if we were to now draw the bode plot of the overall open loop system where in we have incorporated both the linear dynamics as well as a non-linear dynamics, we would have the y axis of the bode plot to be 20 log of the magnitude of DF times G and the y axis of the of the phase plot to be the angle of the phase of DF times G, which is simply equal to the sum of the phases of the describing function plus the and that of the linear open loop system.

Now, we can apply the notions of stability exactly as we did earlier. So, if at the gain crossover frequency, if the open loop gain is greater than 0 dB, we can conclude that our closed loop system is going to be unstable.

So, our gain crossover frequency omega GC if we are working with minimum phase plants should be lesser than our phase crossover frequency omega pc for our overall system to be stable. So, if we have already designed our control system, ignoring the non-linear elements, now we can analyze the effect of non-linear elements by including the describing functions as part of one of the terms in our open loop system and considering that as another term that multiplies the linear loop gain of our open loop system.

And then drawing the bode plots for the cascade of the describing function and the linear loop gain and looking at where the gain crossover frequency and the phase crossover frequencies are. So, we can repeat this for the range of amplitudes that of response that we expect from our physical system and look at what is the possible variation in the magnitude characteristic and the phase characteristic of our open loop system.

And correspondingly fine tune our controller to ensure that even in the presence of this uncertainty in the magnitude characteristic and a phase characteristic brought about because of the non-linear element that we might have, whose response will also be dependent on the amplitude of the input. We can make sure that we have design such a controller that we have adequate phase margin regardless of the amplitude of input that we might have to this non-linear element.

So, this is one straight forward way in which we can use a describing function for synthesis of feedback control systems. We just over design our linear controller in such a manner that, when we cascade the characteristics of the open loop linear open loop system with that of the describing function and we take into account the possible variation in the magnitude of the describing function and the phase of the describing function on account of the variation in the amplitude of the input signal to the describing function, we make sure that the controller still gives us the desired phase margin; regardless of all this variations that might be introduced by this non-linear element So, this is one simple way where by describing functions can be used during design of feedback control systems.

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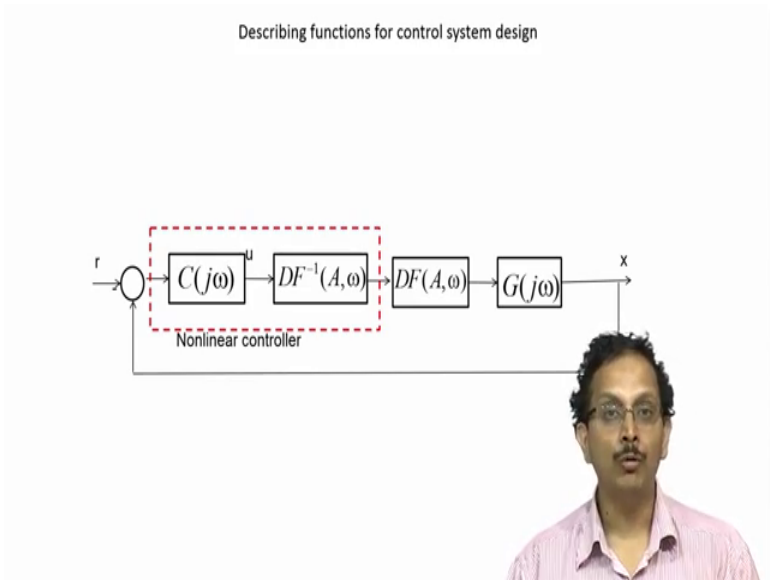

Another way to use it is, if we know something about if we know exactly how the describing function looks like then, we can come up with another non-linear element whose describing function is the inverse of the describing function of the non-linear element that might already be there in our physical system. Now it is other non-linear element that you come up with is actually going to reside in our computer. It is going to be a controller, it is going to be part of the controller and the non-linear element describing function would be the inverse of the describing function of a physical nonlinearity that already exist and that we have to contained with.

If this can be accomplished, then the overall loop gain from the reference to the output would have the effect of non-linearities substantially cancelled out. So, we can therefore, invert the model of the nonlinearity that we might have in our physical system and cancel its effect if it is possible. By coming up with a non-linear controller, whose describing function is going to be the inverse of the describing function that already exist in our feedback system.

So, in this case our controller would be the cascade of our linear controller C of j omega and this non-linear term which inverts the existing describing functions, so D F inverse of a comma omega and the cascade of the 2 is what is going to be implemented by us as control engineers; either in a computer or through some electronic circuitry. And hence, the cascade of the 2 comprises a non-linear controller because we have a non-linear element cascaded with our linear controller. So, this is a simplest way where by one can synthesize a non-linear controller in order to make sure that our overall closed loop system is relatively insensitive to the nonlinearities that exist already as part of the open loop system.

Now, having discussed the 2 ways in which we can employee the notion of describing functions and address the problems that are brought about as a consequence of having nonlinearities in our feedback system, let us now look at how we can actually exploit them to improve the performance of a feedback system.

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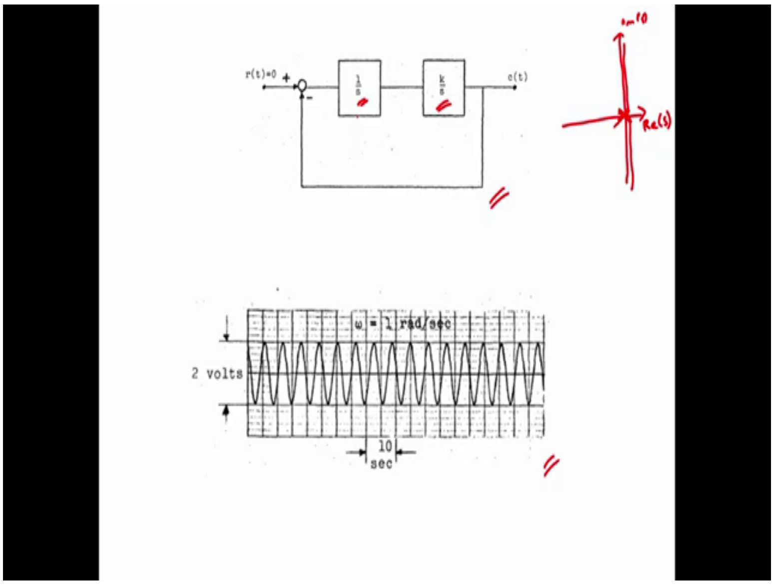

So, what you see in the block diagram at the top is a conventional linear control system with the plant being an integrator namely k by s and the controller also being the an integrator 1 by s. So, we have just 2 integrators as part of our open loop dynamics. Now if 1 were to plot the root locus for a system which has 2 integrators as part of its open loop dynamics, when in other word from plotting the real part of s verses the imaginary part of s, we have 2 open loop poles at the origin.

And what this implies is that root locus will always be along the imaginary axis. So, the 2 branches of root locus that we would have for this case, we will always coincide with the imaginary axis and that in turn means that our closed loop system will always be on the threshold of instability or in other words the closed loop poles of our system will always be imaginary.

And that in turn implies that if you release the system with some initial condition, then its response will always be oscillatory, depending on the magnitude of the initial condition that we provide. Likewise, if we provide some reference, change of reference that will be a transient associated with it, which will never died on because, this because of the presence of poles exactly on the imaginary axis and that is what has been shown at the bottom here. In this case the reference r of t has been set to 0 and the system has been released from some initial condition and we note that the output has this oscillatory waveform which will persists for all future time. The frequency of this oscillation is depend on to the gain k of the plant.

Now, suppose we were to replace the integrator here, the linear integrator that we see here, with a Clegg Integrator, we note that our Clegg Integrator supplies a phase lag; that is significantly less than the phase lag of a conventional integrator. The phase lag of a conventional integrator is 90 degrees whereas, the phase lag of a Clegg Integrator is actually less than 45 degrees. So, in principle if our feedback control system was on the threshold of instability, when we used a regular integrator as part our feedback loop. If you were to replace the regular integrator with the Clegg Integrator in principle it should become stable because, the phase margin has improved by more than 45 degrees as a consequence of introducing a Clegg Integrator. And that is what has been done in this next slide.

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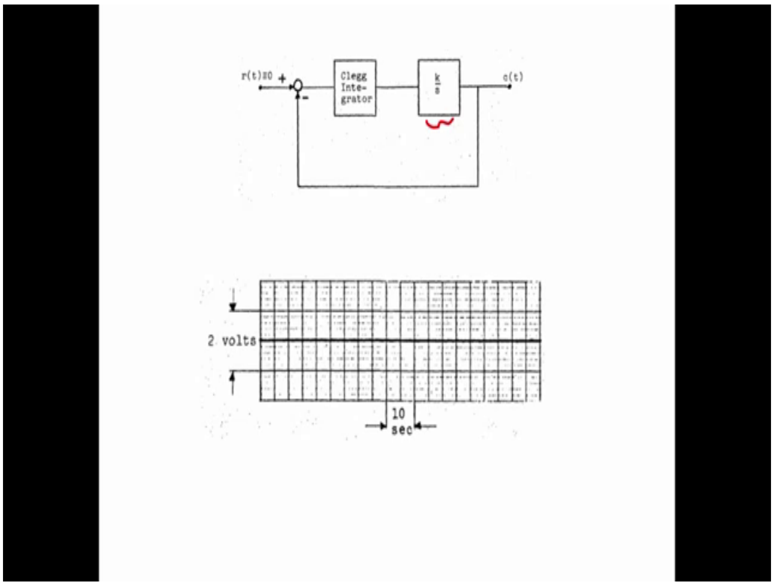

So, we have replaced the conventional integrator with the Clegg Integrator and we are trying to control the same plant k by s. Once again the reference has been set to 0 and the system has been released from some initial condition. And what we note is that for all future time after some initial brief transience, the output is going to be exactly equal to 0; in other words our closed loop system is going to be stable.

So, this is one example where, an intentionally introduced nonlinearity in this case a Clegg Integrator can be employed to improve the performance of the feedback control system, beyond what could be accomplished using a linear control system.

Thank you.