Control System Design
Prof. G. R. Jayanth
Department of Instrumentation and Applied Physics
Indian Institute of Science, Bangalore

Lecture – 05
Fourier transforms (Part 1)

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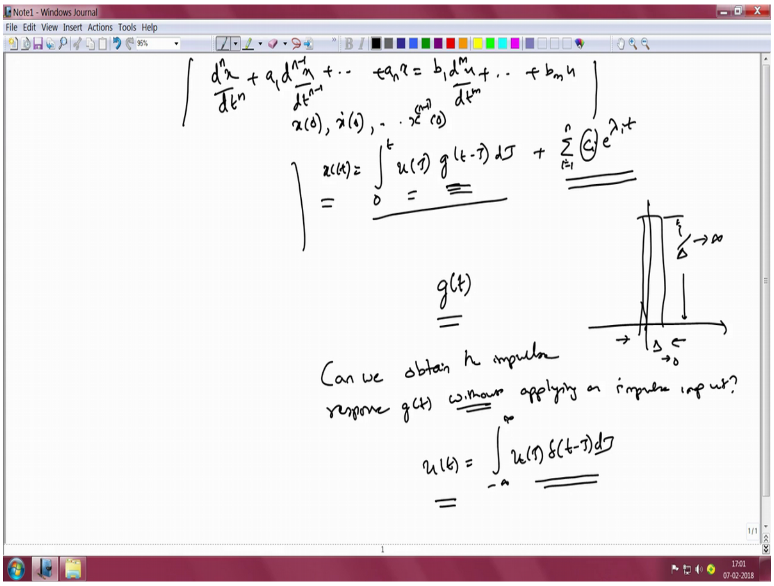

Hello. So, in the previous clip, we looked at solving non-homogeneous linear time invariant ordinary differential equations. In other words, equations that look as shown here. And so, nth derivative of x with respect to t plus a 1 times n minus 1 derivative of x with respect to t and so on and so forth, is equal to b 1 times mth derivative of u with respect to t and so on and so forth plus b m times u with the initial conditions x of 0, x dot of 0, and up to x n minus 1 derivative at time t equal to 0.

We found that the solution looks something like this. So, this part here, i going from 1 to n, is the solution to the homogenous differential equation. In other words, the response of the system to the n specified initial conditions. The coefficients c i are dependent on the n initial conditions and the response the input u is given by this equation here, this integral is called the convolution integral.

And if you have specified u of tau and if you obtain the impulse response g of t, then in principle, you can work out the response of the system x of t to any specified input u of T. So, as I emphasized, the beauty of linear system theory is that starting from very elementary principles with the combination of intuition and guess work and so on, we can solve this very fairly complicated case of a differential equation, where input has m derivatives on the right hand side, and the output has n derivatives on the left hand side, and you have n initial conditions and so on and so forth.

So, from a mathematical perspective, we are above done. There is nothing much more that one needs to know in order to solve differential equations of this particular kind. But often, although a problem might be mathematically solved from an engineer's point of view that solution might still have problems. In that for instance, the solution may not be easy to comprehend. As engineer we seek transparency and simplicity, in being able to understand the response of systems. And the response that we have obtained, for instance as it is indicated by this convolution integral has a problem in that, we have to compute this integral for us to obtain the response x of t.

And computation of a convolution integral is by no means, an intuitively obvious exercise. So, one has to learn quite a bit do quite a bit of convolution, examples in order to get comfortable with being able to predict how x of t would look like, if you are given a certain g of t, and if you are given a certain u of T. So, from a practical perspective, therefore one obvious problem that exists with this time domain based solution of the systems response is that, it is not very intuitive, I cannot easily predict how the response of the system would be for a given u of T, because it is not easy for me to anticipate what the output of this integral would look like. That is actually an important problem, but by no means the most important problem.

A more important problem is that our solution here depends on us obtaining the impulse response g of t of this system. How there is a small problem associated with obtaining the impulse response, namely that we have to apply an impulse input. And what is the problem associated with an impulse input, I pointed out that an impulse input is one, which has a unit area under its curve, so its width is vanishingly small delta tending to 0. And its height is 1 by delta, so its area is 1 unit, and 1 by delta therefore tends to infinity so it is a spike. So, to give a practical example an electrical spike would be an impulse input to an electrical system or if you were to strike a system with a hammer, that would be an impulse input a mechanic impulse input to the system.

You can imagine from both these examples, namely that of striking a system with a hammer or applying a spike to your electrical circuit, that clearly these are not safe inputs to apply to your system; it is very likely that you might cause irreversible damage to your physical system in the course of applying this input. So, we are therefore in this paradoxical situation, where we need to apply this impulse input to obtain the impulse response g of t, but the very act of application of this impulse input might possibly destroy our system, so how do we go about addressing this dilemma.

Given that g of t is of central importance, the way we would go about is to ask ourselves, if we can obtain g of t, namely the impulse response of the system, without actually applying an impulse input. So, can we obtain the impulse response g of t without applying an impulse input, it looks a little counterintuitive that this is possible, but by the end of this clip, hopefully you will be convinced that this is indeed possible. And this is enabled by a new tool, namely that of Fourier transforms.

Now, let us come back to this problem, why did we face this issue of why did we face this difficulty of obtaining x of t. Partially it is because we chose to represent our input u of t in terms of impulses, so we wrote u of t as u of tau delta of t minus tau d tau. So, we represented our input as a train of impulses and therefore, we then represented our response as therefore a set of impulse responses a summation of impulse responses.

So, to address this question of whether we can extract the impulse response without applying an impulse input, we shall start our investigation by asking ourselves. If we can represent the signal u of t in itself, not in terms of impulses, which are clearly dangerous signals to apply to your system.

But, rather in terms of other more benign functions, which do not have the spiky behavior and amplitude magnitudes going to infinity, and which are therefore likely to damage our system. So, we shall first see if we can first if we can represent u of t in terms of such benign functions, and subsequently see if those benign functions under associated mathematics will allow us to obtain g of t without applying an impulse input. So, this is the strategy that we would adopt.

Now, to start with let us ask ourselves, if we are already familiar with representing functions in terms of other fundamental building blocks. And here is where I assume that you have already come across the notion of Fourier series, so Fourier series allows you to represent any periodic signal in terms of sinusoidal signals of the same period and its higher harmonics.

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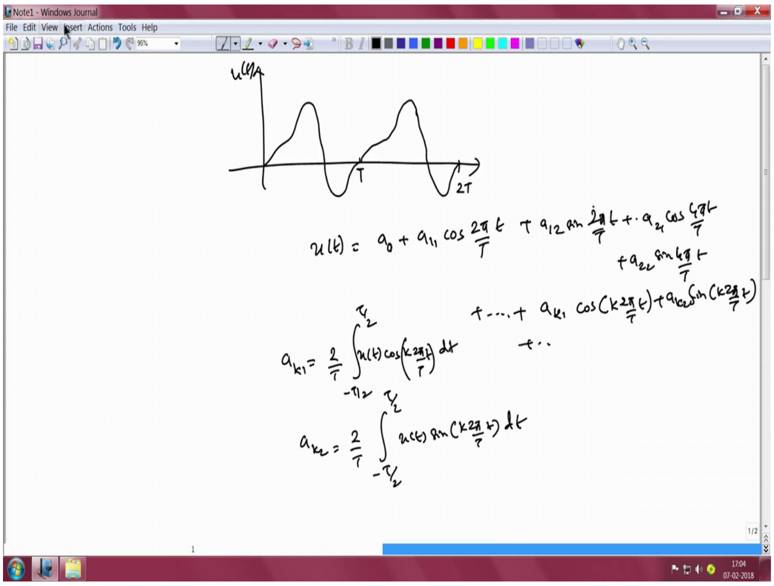

So, for instance, I shall graph here a periodic signals some u of t, so it may look something like this; it has a time period T. So, after this time period T, the same waveform repeats itself between T and 2 T and so on and so forth, between T and 2 T and so on and so forth.

So, you know that for Fourier series allows you to represent u of t in terms of sinusoidal functions of the same period capital T and its higher harmonics. So, in other words, I can write u of t as a 0 plus a 1 1 cos 2 pi by T times small t plus a 1 2 sin 2 pi by T times small t plus a 2 1 cos 4 pi by T times t plus a 2 2 sin 4 pi by T times t plus and so on and so forth. And in general, the kth term will be a k 1 cos k times 2 pi by T times t plus a k 2 sin k times 2 pi by T times t and so on and so forth. You might have infinite terms or a finite number of terms depending on the specific appearance of this waveform u of t.

Now, what are the coefficients, a k 1 and a k 2. In general, a k 1 is given by 2 by T integral minus T by 2 to T by 2 u of t cos k times 2 pi by T dt; a k 2 is equal to 2 by T integral minus T by 2 to T by 2 u of t sin k times 2 pi by T dt. So, this is something that I assume, you have already come across in the course of your undergraduate education. Now, the same expression for u of t can be written in a slightly more compact manner, if we express cos 2 pi by T times t and sin 2 pi by T times t in terms of complex exponentials. So, let us do that next.

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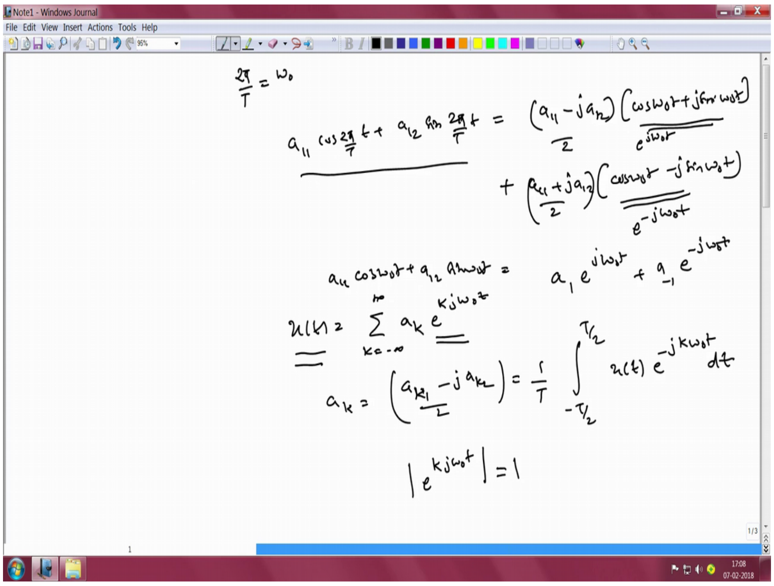

I shall first define 2 pi by T capital T as the frequency omega naught. And we notice that I can write a 1 1 cos 2 pi by T times t plus a 1 2 sin 2 pi by T times t as a 1 1 minus j a 1 2 by 2 cos omega naught t plus j sin omega naught t plus a 1 1 plus j a 1 2 by 2 times cos omega naught t minus j sin omega naught t. This is just an algebraic manipulation that I have done. I did this manipulation in order to write the long expression in a slightly more compact form.

What you notice is that cos omega naught t plus j sin omega naught t is e to the power j omega naught t, likewise cos omega naught t minus j sin omega t is minus e to the power minus j omega naught t. Therefore, I can write this expression of a 1 1 cos omega naught t plus a 1 2 sin omega naught t as, a 1 1 cos omega naught t plus a 1 2 sin omega naught t is equal to a 1 e power j omega naught t plus a minus 1 e power minus j omega naught t.

Now, I can do this for the other terms also, for higher harmonics of a omega naught. And in general therefore, I can write u of t as sigma k going from minus infinity to infinity a k e power k times j omega naught t, where the coefficient a k is given by a k 1 minus j a k 2 by 2. And that you can easily verify is equal to 1 by T integral minus T by 2 to T by 2 u of t e power minus j k omega naught t d t. So, this therefore, allows us to represent a periodic signal u of t in terms of other functions, in this case complex exponentials e power k times j omega naught t.

Now you want to notice that, we have made some small progress here. Earlier we had represented our input in terms of delta functions. Here we are representing our input u of t in terms of complex exponentials. And the improvement accrues on account of the fact that, the magnitude of this complex exponential e power k times j omega naught t is always equal to 1. So, unlike a delta function, which goes tends to infinity for a brief period of time, but no point in time are these complex exponentials of infinite magnitude, they are always finite under magnitude is always equal to 1.

So, we have partially therefore, succeeded in our attempt to represent a signal in terms of benign elementary signals. But, the problem with this entire analysis is that it is limited to periodic signals, and not to a periodic signals. Before we understand, how this can be extended to a periodic signals. Let us take a couple of numerical examples to illustrate the principles behind Fourier series.

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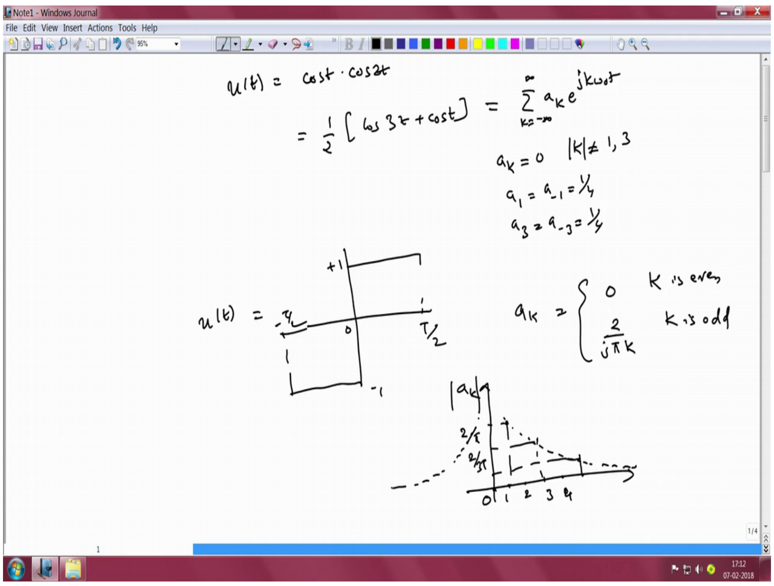

So, suppose I have the signal u of t to be cos t times cos 2 t, then I can easily show that this signal can be written as half of cos 3 t plus cos t. And if I were to represent this in terms of complex exponentials a k e power j k omega naught t, where k goes from minus infinity to infinity. What I will discover is that all the a k are 0; for k not equal to 1 or 3; and a of for magnitude of k not equal to 1 or 3; and a of 1 is equal to a of minus 1 equal to 1 by 4; a of 3 equal to a of minus 3 equal to 1 by 4. I can easily verify this by evaluating the coefficients a k, using the expression that I gave a few minutes back.

Likewise, if we consider another signal u of t, which is a pulse; so, u of t is of value plus 1 from 0 to plus T by 2, and assumes value minus 1 between minus T by 2 to 0. Then, we firstly notice that this function is a discontinuous function, and is also an odd function of time. So, therefore, when we write it out in terms of sinusoidal signals and cosine signals, what we will what we expect to see is that, we would have only sin components and not the cosine components, because cosine omega t is an even function of time, whereas sin omega t is an odd function of time. What we have as the actual signal is an odd function of time. Therefore, when we represent it in terms of its elementary functions, we cannot have even functions of time in the representation. So, we would have only sin terms.

And we can show that for this particular case a k is 0, when k is even; and is equal to 2 by j times pi times k, when k is odd. So, if I were to represent the values of a k, so (Refer Time: 17:43) when k is equal to 0, its magnitude is 0; when k is equal to 1, the magnitude we would represent the magnitude of a k. When k is equal to 1, the magnitude is 2 by pi that is the magnitude; this will be 2 by pi. And k equal to 2, it is again 0; k equal to 3, it is 2 by 3 pi a little bit lesser. And k is equal to 4, it is again 0; and k is equal to 5, it is 2 by 5 pi, which is even lesser. So, if I were to connect all these non-zero of points by a dotted line, it will resemble a rectangular hyperbola. So, this is how it will look for positive values of k and you will have exactly the similar shape also for negative values of k.

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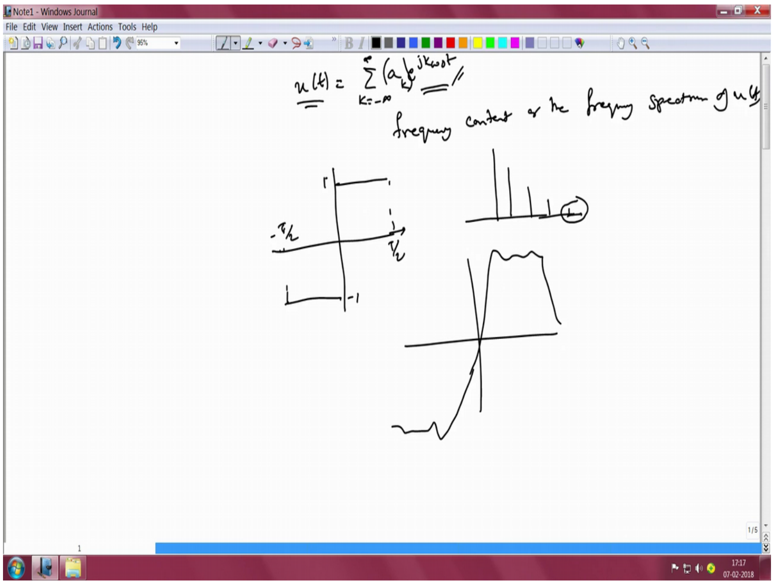

So, what we have done, if we go back to the representation, is that we are represented the signal u of t in terms of basic building blocks e power j k omega t omega naught t, where k goes from minus infinity to infinity. And the quantity of each of these blocks that you require is in some sense represented by a k. And therefore, these coefficients a k are also called the frequency content or the frequency spectrum of u of t.

Because, there if you want to think of this in terms of construction of some building, so u of t can be imagined to be the building you are trying to construct. And the basic building blocks are the terms t power j times k omega naught t, where or in other words e power omega j omega naught t e power 2 j omega naught t and so on and so forth. And the quantity of each of these that you need to construct u of t is specified by a of k.

So, for instance, if we take the case of a step like wave form, which is which assumes value minus 1 between 0 minus T by 2 and 0 and plus 1 between values 0 between time 0 to T by 2, and this waveform repeats in time. We saw that, we have the odd coefficients being non-zero, and the even coefficients all being 0.

Now it may happen that, we may not be able to properly reproduce the coefficients a k, when our k is very large. So, physically what we mean is that, you have these complex exponentials e power j times k omega naught t, which are changing very rapidly in time, and physically you can imagine that a system cannot track such inputs very well. So, one can imagine therefore, that when you apply this kind of an input to your physical system, it will not end up tracking the components a k for very large values of k.

And what that means is that, when I reconstruct the response of the system, those components which are at a very high frequencies contribute to the sharp features of this signal u of t. And the component that are at low frequencies contribute to the slow varying features of u of t. So, if my system is not able to track these fast varying components very well, in other words it cannot reconstruct the signal at high frequencies, then I will not be able to represent this signal accurately at the locations, where its changes are very fast. So, if I were to therefore represent this signal with only a finite number of terms, where k is not going to infinity, my response might look something like this.

So, therefore, inevitably what we will have to do as control engineers is to accept the fact that the physical systems out of which we build our controllers and so on and so forth, will not be able to track you know signals that are changing very fast in time. Therefore, what we should accept as control engineers is that, our system cannot reconstruct fast changing complex exponentials in time, and therefore our response also cannot track fast changing references in reference commands that we might choose to provide to our plant.

Thus far, we have looked at how we could represent a signal in terms of benign signals, in this case complex exponentials. But, as I said, the limitation of this of the theory of Fourier series is that it is applicable only to periodic signals. How do we extend these notions to a periodic signals, signals that do not repeat in time.

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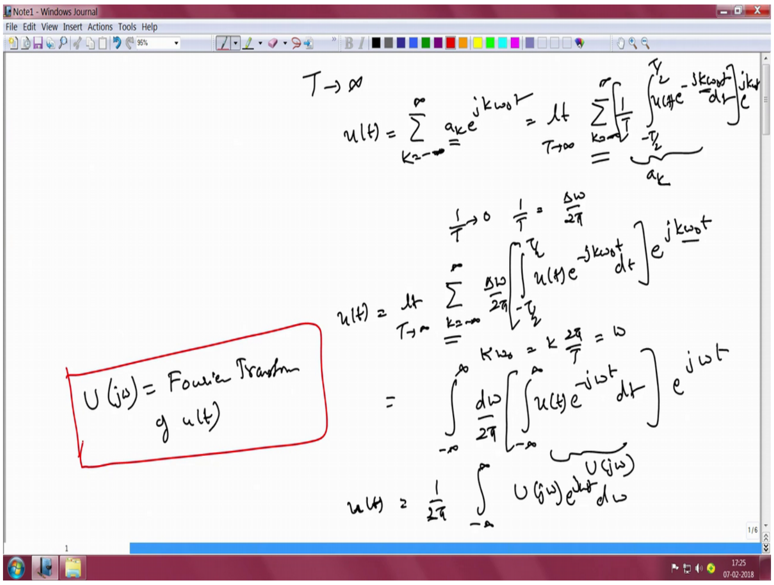

There is one nice trick; we can employ in order to extend these concepts to a periodic signals. And it is by assuming that these a periodic signals are actually periodic signals, but their periodicity, is their time period is so large, that they do not repeat themselves within the time scale of interest to us. In other words, the time period t of our periodic signals, we shall tend them to infinity.

In other words, I shall write out u of t as sigma a k e power j k omega naught t, where k goes from minus infinity to infinity. And I shall now explicitly expand a k. So, we know that our time period T is tended to infinity, so this will look like this, k going from minus infinity to infinity 1 by T integral minus T by 2 to T by 2 u of t e power minus j k omega naught t dt. So, this is my coefficient a k. And this gets multiplied with e power j k omega naught t. So, this is going to be my signal ok.

Now, I have tended my T to infinity to capture the fact that the signal is not is actually a periodic signal, but it is not repeating in a time scale, that is of importance to me, as an engineer. So, I can if T is tended to infinity, then 1 by T tends to 0. So, I can write 1 by T as some delta omega by 2 pi. So, I can now write u of t as u of t equal to limit, T tends to infinity, sigma k going from minus infinity to infinity 1 over t, which I shall replace with delta omega by 2 pi integral minus T by 2 to T by 2 u of t e power minus j k omega naught t dt. And this gets multiplied with e power j k omega naught t.

Now, in the when k varies from minus infinity to infinity k times omega naught, which is equal to k times 2 pi by T, would be some frequency omega on the frequency axis. And once again, when k varies from minus infinity to infinity and delta omega tends to 0, I can replace this summation here by an integral.

And I can write u of t as integral minus infinity to infinity d omega by 2 pi; and the term within the bracket can be written as integral minus infinity to infinity, because in T tends to infinity, the limits of this go to infinity and minus infinity. So, integral minus infinity to infinity u of t e power minus j k omega naught (Refer Time: 26:43) some frequency omega t d t.

And this multiplies e power j. Once again, k omega naught here is some frequency omega, for j omega t. I shall call this term within the bracket as u of j omega. And write u of t as 1 by 2 pi integral minus infinity to infinity u of j omega e power j omega t d omega. And u of j omega is called the Fourier Transform of u of t.