Control System Design
Prof. G.R. Jayanth
Department of Instrumentation and Applied Physics
Indian Institute of Science, Bangalore

Lecture – 49
Describing functions (Part 1/2)

Hello, in this clip, we shall discuss the control of unstable plants. And more to the point rather than discussing some special tricks or techniques that might be out there to control unstable plants, we shall be focusing more on some of the fundamental limitations that are imposed by unstable plants on our goals as control engineers. So, therefore this lecture would be of a very similar flavor as the lecture that we that I gave on a non-minimum phase systems.

In that we would be discussing some of the fundamental limitations imposed by the special structure of the plant. In the former case, it was the fact that you had either time delays or non-minimum phase zero as part of the plants structure. In this case, we are considering plants that have unstable dynamics. So, what do we mean by unstable plants? Very simply unstable plants are those plants, which have one or more of their poles on the right half of the complex plane.

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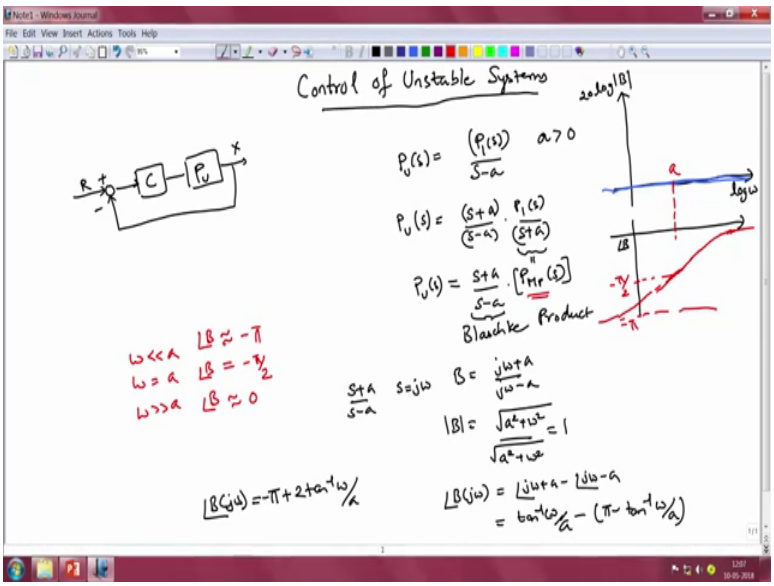

So, let us assume that we have a plant P which has one of it is poles on the right half of the complex plane, then we can write down the plants transfer function as P of s is equal to P 1 of s divided by S minus a, where the term a is greater than 0. So, if P 1 of s is a minimum phase transfer function or in other words all of the poles and zero's of P 1 of s are on the left of the complex plane. Then a plant P of s, which is of the form P 1 of s by S minus a has one of it is poles, namely S is equal to a on the right half of the complex plane. And hence qualifies to be called an unstable plant. Just to underscore the fact that a plant of this structure is an unstable plant, we shall provide the plant P of s with a subscript U and call P U of s as the unstable plant.

Now, to understand the kind of issues that one would encounter, when one is trying to control an unstable plant. Let us first make a few algebraic manipulations and rearrangements, then take a look at the bode plots of loop gains that have unstable plants in them. And see the problems that are that we would first have to confront to decide upon the stability of a closed loop system, which has unstable open loop dynamics. And subsequently see, what fundamental limitations the loop gains of unstable open loop systems would imposed on our goals as control engineers.

So, to start with let me write out the plant transfer function P U of s as S plus a by S minus a times P 1 of s by S plus a. So, in other words I have multiplied and divided the right hand side with the same term namely S plus a, so the right hand side remains unaffected. But, now I can rearrange the right hand side as S plus a by S minus a times P M P of S, where P M P of S represents a minimum phase transfer function and it is essentially given by P 1 of S by S plus a.

So, P 1 of s by S plus a essentially is the term P M P of S. So, we can write the unstable plants transfer function P U of s, as S plus a by S minus a times P M P of S. And those of you who have looked at the lecture on non-minimum phase plants will recognize this term S plus a by S minus a to be another Blaschke product.

So, if one wants to plot the bode plot of the unstable plant, we can give it as the sum of the bode plots of the Blaschke product and of the minimum phase plant P M P of S. Once again the bode plot of P M P of S is something that, we should be able to draw without any effort having you having looked at such plants all through these lectures. So, let us focus on the bode plot of the Blaschke product alone.

So, the Blaschke product is given by S plus a by S minus a and in order to draw it is bode plot, we substitute S is equal to j omega, so that we would have the Blaschke product to be equal to j omega plus a by j omega minus a. What is the magnitude of the Blaschke product? It is going to be equal to square root of a square plus omega square divided by once again square root of a square plus omega square and that is going to be equal to 1.

So, just as in the case of non-minimum phase systems, the Blaschke product of in this particular case for the unstable system also has unit magnitude over the entire frequency range independent of the frequency omega. What is the phase of the Blaschke product? We can show with some effort that the angle of the Blaschke product a function of frequency is given by the angle of the numerator term, namely the angle of j omega plus a minus the angle of j omega minus a. And we can show that this is going to be equal to tan inverse of omega by a minus the angle of j omega minus a can be shown to be equal to pi minus of tan inverse of omega by a.

And as a consequence of this the phase of the Blaschke product B of j omega is going to be given by minus pi plus 2 tan inverse omega by a. So, if one word to draw the magnitude characteristics of the Blaschke product or in other words, x axis is log of omega, the y axis is 20 log of magnitude of B. We would have a straight line that is coincident with the x axis. So, the gain is equal to 0 dB or a linear scale 1 unit at all frequencies omega.

If we come to the phase plot, we note that when frequency omega is very small, the phase of the Blaschke product is close to minus pi radian. So, if you were to draw the angle of B, we note that when omega is very small in other words when omega is much less than a, we would have the angle of B to be approximately equal to minus pi. When omega is exactly equal to a, we would have the angle of B to be equal to minus pi plus 2 times tan inverse of 1 and that is equal to minus pi by 2 so, B will be equal to minus pi by 2.

And omega is much greater than a, we note that the term tan inverse omega by a will will tend to pi by 2 and hence the overall phase of the Blaschke product will tend to 0. So, the angle of the Blaschke product will be close to 0 for frequencies omega that is much better than a. So, if we locate the frequency a on the x axis of the bode plot, then at very low frequencies the phase is close to minus pi. And at omega equal to a, the phase is equal to minus pi by 2. And as frequency tends to infinity, the phase assume partially approaches 0 radians. So, this is the bode plot of the Blaschke product.

And the unstable plant has the magnitude characteristic, it is going to be identical to the magnitude characteristic of the minimum phase plant namely P M P of S. Whereas, the phase characteristic of the unstable plant is going to be different from the phase characteristic of the minimum phase plant and on account of the phase added by the Blaschke product.

Now, it is quite common to come across, the phase characteristics of unstable systems to be as once that are quite different and unrecognizable from the phase characteristics that we have been looking that so far in this course. So, just to highlight the difficulties that the strange phase characteristics of a unstable loop gains would cost to us as control engineers and determining the stability of the close loop system. Let us take a specific example.

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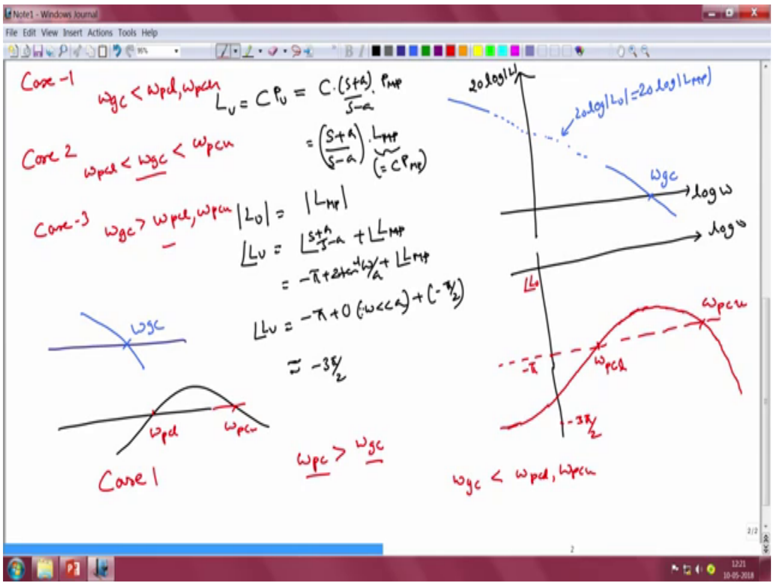

So, let us assume that we have a certain controller C, which multiplies our unstable plant P U. And we assume that a controller is a minimum phase controller, so I shall call the product of C times P U as L U, L subscript U. And from the previous slide, we note that this going to be equal to C times S plus a by S minus a times P minimum phase. And I shall combine the term C and P minimum phase and write rewrite this expression is S plus a by S minus a times L minimum phase, where by definition L minimum phase is equal to C times P minimum phase.

Now, we note that the magnitude of L U will be equal to the magnitude of L minimum phase, because the Blaschke product has a gain of 0 dB at all frequencies omega. Whereas, the angle of L U will be equal to the angle of the Blaschke product namely S plus a by S minus a plus the angle of the minimum phase transfer function or in other words it is going to be equal to minus pi plus 2 tan inverse omega by a plus angle of the minimum phase transfer function.

Now, let us assume that our minimum phase transfer function has an integrator as part of its structure presumably, because we are expecting good performance at very low frequencies. And in particular we would be expecting 0 tracking error for d c references or d c disturbances. So, if that is the than case, then our L minimum phase has an angle of minus pi by 2 around omega equal to 0. So, let us sketch the bode plot of our unstable loop gain, when our magnitude characteristics has an integrating characteristic at low frequencies. And see what kind of confusions the phase characteristic of such a loop gain would cause in our effort to determine the stability of the closed loop system.

So, on the right hand side, I am drawing the bode plot of L U. So, x axis is log omega, the y axis is 20 log magnitude of L so, at very low frequencies as we discussed if our L U has an integrator as part of its structure or if L minimum phase has an integrator as part of its structure, then the role of will be 20 dB per decade. And subsequently, we will have a certain role of depending on the kind of poles and zeros that the plant and the controller have. And finally, let us assume that the gain characteristics crossover at some frequency, which we shall of course call omega g c the gain crossover frequency.

Now, associated with this magnitude characteristic, we would have a certain phase characteristic. So, this is log of omega, I shall indicate the angle minus pi clearly in the phase plot. So, the second plot is going to be the phase of L U. This is the first plot is of course the 20 log magnitude of L U, it is going to be identical to 20 log magnitude of L minimum phase. As far as the phase lack of L U is concerned, we note that if you have an integrator as part of the controller structure of our open loop system that term adds a phase lag of minus pi by 2.

So, at very low frequencies in other words for omega is much less than a you would have the phase lag of L U to be equal to minus pi plus 0 that is because omega is much less than a plus the phase lag of L minimum phase is going to be equal to minus pi by 2, because we assume that we have an integrator. So, together it is going to be approximately equal to minus 3 pi by 2.

So, the phase starts at minus 3 pi by 2, so somewhere here, I shall mark out the point minus 3 pi by 2 on the bode plot. So, the phase starts somewhere here and you notice that as the frequency omega increases the phase lag of the Blaschke product starts to reduce. So, it starts at a value close to minus pi, and then tends to 0.

So, assuming that our plants dynamics have not kit been yet, in this frequency range. What we will have is that the phase will be dominated the phase lag will be dominated by that of the Blaschke product alone. So, the phase will start to increase and there will be a frequency at which it crosses over. So, there will be one crossover, we shall we shall call omega phase crossover lower omega p c l. So, the phase will crossover here, because of the increasing phase of the Blaschke product. And then it will continue to rise, but then there will be a frequency at which the plants poles and zero's, the other poles and zero's of the plant is will start to contribute to phase lag.

So, this rise of phase as function of frequency will be arrested by the phase lag contributed by the other poles and zeros of the plant and the controller. And hence, the phase will start to once again decrease. And it will continue to decrease, because this phase lag contributed will increase with frequency. And that causes the phase characteristics to crossover once again, so there will be a second crossover frequency, which we shall call omega p c upper omega p c u and this phase lag will continue to increase.

And finally, the phase will asymptotically approach a certain phase lag that is determined by the relative degree of the loop gain. So, this is the phase characteristic and that is the magnitude characteristic. And here is where our problem begins as control engineers. How can we determine, whether a system an open loop system that has a bode plot of the kind that I have drawn here is stable or not. For one thing, this is the first time that we are coming across a system that has two phase crossover frequencies.

So, we have one phase crossover frequency, which happens because of the increasing phase of the Blaschke product as function of frequency and that is omega p c l. And we have another phase crossover frequency that happens because of a phase lag contributed by the other minimum phase terms in the loop gain namely in that of the plant and the controller and that we have called as omega p c u.

So, these kind of a phase characteristics is entirely unfamiliar to us. Assuming that we have confined ourselves to looking at the bode plots of minimum phase plants so, how do we determine, whether a system that has a phase characteristic of this kind is stable or unstable. Now, if we want sort of derive inspiration from the phase characteristics of the minimum phase systems, then for a minimum phase system to be stable.

We have noticed that firstly such a system would have a single phase crossover frequency not two phase crossover frequencies, but that one phase crossover frequency omega p c should be greater than the gain crossover frequency for the closed loop system to be stable or in other words the phase lag at the frequency at which the gain crosses over should be less than pi radians.

So, our omega p c been greater than omega g c is the necessary condition for a minimum phase system to have stable closed loop dynamics. If we were to uncritically apply the same criterion for stability in this particular case, we note that we have our gain crossover frequency in between omega p c l and omega p c u in the particular schematic that I have drawn. So, our gain crossover frequency is greater than the lower phase crossover frequency, but less than the upper phase crossover frequency.

Now, if you want to uncritically apply the lessons that we learned in case of minimum phase systems, where we wanted the phase crossover frequency to be greater than the gain crossover frequency. We might conclude that a system with this kind of a bode plot is actually unstable, because one of the phase crossover frequencies is actually less than the gain crossover frequency.

So, we might therefore we let to conclude that the rule that we need to apply in order for us to have a stable closed loop system is that the gain crossover frequency should be to the left of both the phase crossover frequencies. For in other words, we might be let to conclude that omega g c should be less than both omega p c lower as well as omega p c upper that may be our live conclusion. First guess about what is necessary in order for our closed loop system to be stable.

But, we have no logical arguments to back such a claim and that is because as I have repeated in the past. The bode plot is by no means the right tool for us to determine the stability of a closed loop system. In order for us to determine the stability of a closed loop system in the frequency domain, there is only one plot available to us and that is the Nyquist plot.

So, if you are given a certain unfamiliar magnitude and phase characteristic, all we need to do is to draw the Nyquist plot of this particular magnitude and phase characteristic. And look at the encirclement of the point minus 1, look at the number of unstable poles that you have for your system and open loop poles that you have for your system. And then come up with the rules for determining the stability of the closed loop system.

So, let us therefore not take shortcuts and try to guess about the stability of the closed loop system by simply looking at the bode plot of the open loop system. And trying to draw some week in inferences from the bode plots of minimum phase systems. Let us do it the right way, let us draw the Nyquist plot come up with the right rules for determining stability in the Nyquist plot. And then apply them to determine what conditions are necessary for our system to be stable.

So, before we do that we note that there are three possible cases. Here the first case is when we have the gain crossover frequency omega g c being less than both omega p c lower and omega p c upper, so which is what we thought was necessary for our closed loop system to be stable on the basis of what we saw in the case of minimum phase plants. So, omega p c lower, omega p c upper so, this is case 1, so, case-1 essentially is that omega g c is less than both omega p c lower and omega p c upper.

Now, there is a second case that is possible and that has what has been drawn on the schematic on the right namely, where omega g c the gain crossover frequency is between the two phase crossover frequencies. So, case-2 is 1, where omega p c lower is less than omega g c, but omega g c is also less than omega p c upper so, this is the second case. And the third case is the opposite of case-1 namely, where the gain crossover frequency is greater than both the phase crossover frequencies, in other words, when omega g c is greater than omega p c lower and omega p c upper.

The question is in which of these 3 cases will we have a stable closed loop system on our hands. Is it going to be the 1st case, which is what we think, it is now based on our live extension of what we observed in case of minimum phase loop gains or is it for a 2nd case or is it for a 3rd case. So, let us take each of these cases one by one apply the principles of Nyquist stability theory for these 3 cases and see for which case it is that the closed loop system would be stable.

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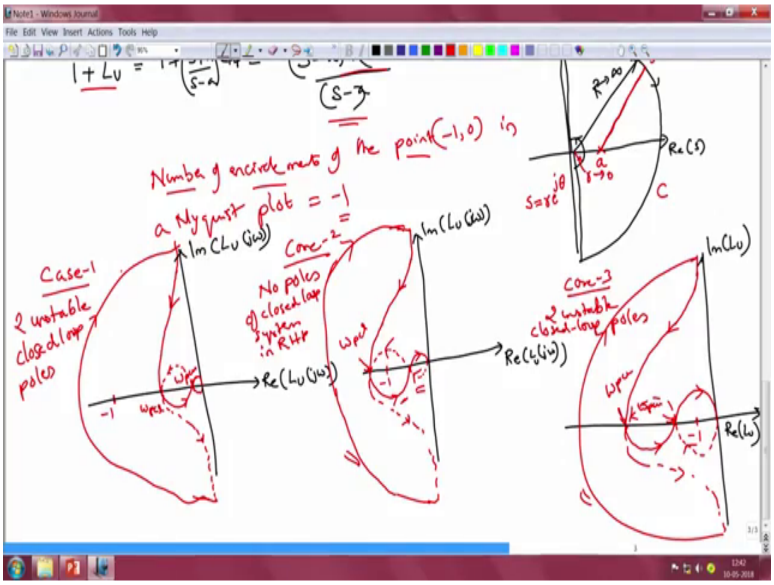

So, before we draw the Nyquist plot for the open loop system. Let us first examine what kind of rules need to be applied in order to determine the stability of the closed loop system. We note that the denominator transfer function for our closed loop system is going to be given by 1 plus L U. And if I go to replace the term L U by S plus a by S minus a times L minimum phase, then I would have that 1 plus L U would be equal to S minus a plus S plus a time L minimum phase divided by S minus a.

So, now let us first draw the complex plane. In other words, the real part of S versus the imaginary part of S, which is omega and our familiar D shaped contour would have it is straight edge coincident with the imaginary axis. And then, we have this semicircular arc of radius capital R that is tended to infinity and that encompasses the entire of the right half of the complex plane. So, this is our familiar D shaped contour along, which we would compute the complex number L of S and then draw its Nyquist plot.

Now, we note that in this particular case, we have an open loop pole namely S is equal to a within the D shaped contour, so S is equal to a is going to be within the D shaped contour. So, the complex number S minus a for points S on the contour that you have drawn here, on the contour C. The complex number S minus a is going to be given by this particular phasor. Now, when we take the S variable along this contour in the clockwise direction once, we note that the phasor S minus a will go round itself once or in other words, it will execute a change in angle of 2 pi radians.

But, since the term S minus a appears in the denominator of the transfer function 1 plus L U, we note that the net encirclement that will occur because of the term S minus a. When we take the variable S in the clockwise direction once is one counter clockwise encirclement, because 1 by s minus a will change it is angle in a sign in a manner that is opposite to the direction in which the angle of S itself changes.

Hence, if you want to zero's of 1 plus L U, which are going to be the closed loop poles of our system to not lie within the D shaped counter, then the rule that we need to apply for determining the stability of the closed loop system is that the number of encirclements of the point minus 1 comma 0 in a Nyquist plot should be equal to minus 1. Because, the term S minus a, which is in the denominator of 1 plus L U will result in one counter clockwise encirclement or the entire encirclement sign is negative, and hence that results in one counter clockwise encirclement.

And if none of the closed loop poles are within this D shaped contour or in other words if none of the zeros of 1 plus L U, which are essentially the zeros of the numerator polynomial of this transfer function 1 plus L U. If none of them are in the right of the complex plane, they will not contribute to any encirclement of critical point in the Nyquist plot. Hence, the only encirclement that will happen will be because of the term S minus a and that encirclement is counter clockwise.

Hence, the correct rule that we need to apply to determine the stability of the closed loop system in the Nyquist plot of an unstable system is that the number of encirclements of the critical point should be equal to minus 1. So, we should have one counter clockwise encirclement of the critical point by the Nyquist plot of the open loop system.

So, now let us return to the bode plot that we were just looking at, where we had three particular cases, one is when the phase crossover frequencies or both greater than the gain crossover frequency. The other is when the gain crossover frequency is between the 2 phase crossover frequencies. And the 3rd is one, where the phase crossover frequencies are both lesser than the gain crossover frequency. And see for which case it is that the Nyquist plot encircles the point minus 1, once in the counterclockwise sense.

Now, in order to draw the Nyquist plot for the particular example that were considered in the previous slide. We note that we had an integrator in the as part of our open loop transfer function. And hence, we need to introduce a tiny kink in the contour near the origin near S is equal to 0. The location where the integrator has it is pole, so we need to introduce a tiny kink in the origin. And modify our D shaped contour slightly in the manner that I have shown here, before we can draw the Nyquist plot of the open loop system.

So, now let us proceed to draw the Nyquist plot for the 3 cases that we discussed. The 1st case is one, where the two phase crossover frequencies, were both greater than the gain crossover frequency. Now, when both the phase crossover frequencies are greater than the gain crossover frequency, what we are essentially claiming is that at both the phase crossover frequencies, the gain of the open loop system is going to be less than 1.

So, if we were to draw the Nyquist plot for this case, the x axis will be real part of L U of j omega, the y axis will be the imaginary part of L U of j omega. We note that when omega is very small, the phase of L U starts at close to minus 3 pi by 2, so it will be somewhere here. This is the location where the Nyquist plots in the neighborhood of omega equal to 0.

And then as omega increases, the gain of the open loop system is start to reduce, the gain will start to reduce. And then there be one frequency at which the phase crosses over, the first time it crosses over, it is at the frequency omega pc lower. And subsequently, it will cross over again at another frequency at a higher frequency omega p c upper. And finally, the magnitude characteristics will tend to 0 in some particular fashion. And the gain of the transfer function L U at both omega p c lower and omega p c upper will be less than 0 dB for the case namely case 1, when the gain crossover frequency is to the left of both omega p c lower and omega p c upper. In other words, the critical point minus 1 will be located to the left of both omega p c lower and omega p c upper.

Now, if we complete the Nyquist plot, the Nyquist plot for the complex conjugate of the imaginary axis or the Nyquist plot for the negative imaginary axis, we will look something like this. And that will tend to minus infinity along the negative imaginary axis. So, this is going to be the mapping of the positive imaginary axis and the negative imaginary axis in the g of as a consequence of this transformation namely L of j omega L U of j omega.

The big D shape contour will collapse to the origin. And what about this small semicircular thing that we have introduced that avoids the integrator at the origin. We can show that by substituting the expression that along this contour, we would have the complex number to be of the form S is equal to small r e power j theta, where small r is a radius of this kink and that is tended to 0 of course.

And we see that theta here goes from minus pi by 2 to plus pi by 2. We can show that this particular curve gets mapped to a big semicircular arc, I am sorry my semicircle does not look that good. But, it is it would be a semicircle, when we were to draw it correctly of radius r that is tended to infinity. So, the radius r here is going to be on the order of 1 by small r and that would go to infinity, when small r tends to 0. So, our Nyquist plot would look something like this for the first case.

Now, if you look at this Nyquist plot, we note that the parts that we have to the right of the point minus 1. The two curves that we have here do not encircle the critical point at all, but it is big curve here encircles the point minus 1 once, but in the clockwise direction. Now, if we have one clockwise encirclement of the point minus 1, what it indicates is that two of the zeros of the transfer function 1 plus L U are within the D shaped contour.

Because, if two zeros are within the D shaped contour, these two zeros together contribute to two clockwise encirclements of the point minus 1. And when combined with one counter clockwise encirclement of the point minus 1 due to the term S minus a in the denominator. You would have a net of 1 clockwise encirclement of the point minus 1 and that is precisely what we are seeing in this particular case. So, in case-1 therefore we would have 2 unstable closed loop poles so, unlike what our naive intuition? Led us to guess, it is the case-1 that results in an unstable closed loop system and not a stable closed loop system.

Now, let us take a look at what happens for case 2. So, in case-2 we note that the gain crossover frequency is between omega p c lower and omega p c upper. So, if I were to plot the Nyquist plot again is going to be real part of L U of j omega is going to be equal to imaginary part of L U of j omega. Once again, we would have the Nyquist plot starting at this location, because a phase lag would be equal to 3 pi by 2 as omega tends to 0 and then, it reduces with frequency.

Now, the first time the phase crosses over or the first time the phase of the loop gain reaches minus phi radians. The gain is going to be greater than minus 1 so, the phase the loop gain will cross at a point that is to the left of the point minus 1 comma 0. So, this will be the frequency omega p c lower. And then subsequently, the 2nd time it crosses the phase crosses over or the second time the phase lag assumes a value of pi radians. The gain would have dropped below 0 dB so, it would cross somewhere to the right of the point minus 1. And subsequently, it will go to 0 in some particular manner.

The Nyquist plot along the negative imaginary axis will be the mirror image of this Nyquist plot about the real axis and would therefore look something like this. And the D shaped, and a small D shaped contour near the origin gets mapped to this huge D shaped contour that starts at the negative imaginary axis at close to minus infinity. And goes towards the positive imaginary axis close to plus infinity so, once again this is supposed to be a semicircle, but owning to shortage of space and my limited abilities as an artist, it looks like a distorted circle.

So, if you were to draw the arrow for the direction in which the loop gain moves, when the variable S is taken around the contour D shaped contour in the clockwise sense, we know that it moves down this way and then that way, finally this way. And along the negative imaginary axis, this is the way in which the loop gain changes as we traverse the D shaped contour.

So, if we focus on this particular Nyquist plot, we see that in this case this big loop does not encircle the point minus 1 at all, unlike in the previous case. And we have one loop here that encircles the point minus 1. The other loop that is close to the origin, once again does not encircle the point minus 1. So, this guy does not encircle the point minus 1, this guy also does not encircle the point minus 1, it is only the middle loop that encircles the point minus 1. And in what sense is it encircling it if you notice here, it encircling it in the counter clockwise sense.

And if we go back to the rule that we need to apply, in order to determine stability of the closed loop system, when you have one unstable pole, we note that this particular encirclement satisfies that requirement. The number of encirclements to the point minus 1 in the Nyquist plot has to be minus 1 or there should be one counter clockwise encirclement for the closed loop system to be stable. And to not have any of the zeros of 1 plus L U, which are the closed loop poles of our system to be on the right half of the complex plane.

So, it is for case-2 that we end up with a stable closed loop system. So, no poles of closed loop system in the right half plane RHP. So, contrary to what our intuition might have lead us to believe, it is the 2nd case namely the schematic namely the schematic that I have drawn here, where omega g c is in between omega p c l and omega p c u that results in a stable closed loop system.

Now, let us consider the 3rd case, where the gain crossover frequency is greater than both the phase crossover frequencies, what is implies then is that at both the phase crossover frequencies our loop gain will be greater than 0 dB. So, if one were to draw the Nyquist plot once again, the x-axis once again is real part of L U and the y axis is imaginary part of L U. We will have that the loop gain once again start somewhere here as frequency increases, the gain reduces.

And the first (Refer Time: 38:48) first time it crosses over namely at omega p c u, the gain is greater than 0 dB. So, the point minus 1 comma 0 will be to the right of this particular location. And then subsequently, it will crossover once again. So, there will be the phase will once again assume a value of minus pi at omega p c upper, so this is omega p c upper. And this point is also a point that is to the left of the point minus 1, because in this particular case, the gain of the open loop system would be greater than 0 dB even at omega p c upper.

And subsequently, the phase does it is particular, the loop gain does it is particular thing depending on the higher order dynamics of the plant and the controller and finally goes to 0. So, this is how the Nyquist plot will look for the positive imaginary axis. For the negative imaginary axis, it will be a reflection of this plot and it will therefore look something like this. And the small D shaped contour, we will get mapped to this big D shaped contour and the Nyquist plot will look something like this.

So, for this case-2 we note that the big loop here does not encircle the point minus 1 at all. The 2nd the middle loop also does not encircle the point minus 1, it is the loop that is closest to the origin that encircles the point minus 1. But, unfortunately if you look at the sign in which this loop is traversed by the term L of j omega, we see that the loop is traversed in the clockwise sense. And therefore, exactly as in the first case as in case-1 we have one clockwise encirclement of a point minus 1 in case-3, when the gain crossover frequency is greater than both the phase crossover frequencies.

And what this indicates to us is that. Once again in case-3 just as with case-1, we would have two unstable closed loop poles or two zeros of the transfer function 1 plus L U on the right half of the complex plane. So, 2 unstable closed loop poles. Hence, this analysis reveals how contour intuitive it can be for us to determine the correct rules for stability of a closed loop system, when we have fairly unfamiliar bode plots that are given to us. So, when the phase characteristics look as unfamiliar and unsettling, as what we saw in the previous slide, where you had two phase crossover frequencies and so on.

The best and indeed the only thing that we need to do is the first come to the Nyquist plot and depending on the structure of the open loop transfer function namely whether the open loop transfer function has any unstable poles or not. We first have to come up with the right rules for determining the stability of the closed loop system. And subsequently, plot the Nyquist plot for the open loop gain. And then see whether the conditions thought for stability are satisfied or not.

So, more generally if we have n open loop poles of the plant on the right half of the complex plane, for our closed loop system to be stable, the number of encirclements of the critical point should be minus n or in other words you should have n counter clockwise encirclements of the critical point for our closed loop system to be stable. So, having discussed this issue associated with determining correctly, the stability of a closed loop system, which has unfamiliar phase characteristics. Let us now see what kind of fundamental limitations, the magnitude and phase characteristics of unstable systems force to us as control engineers.

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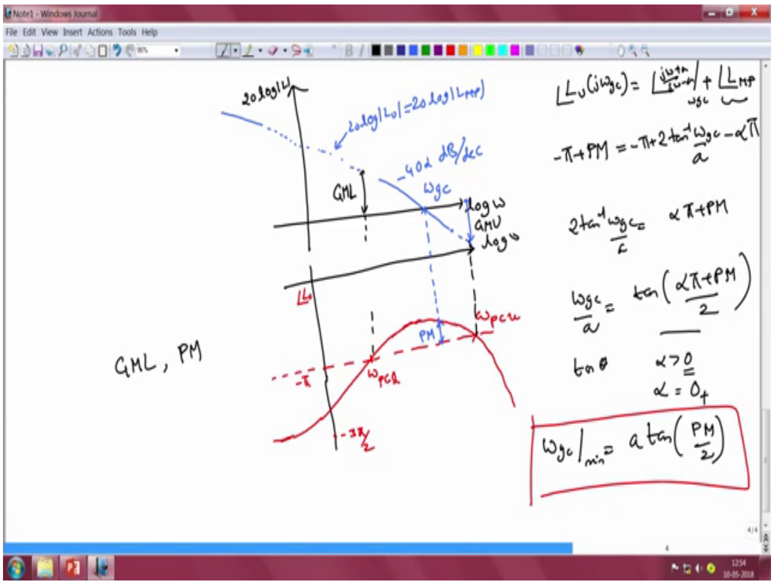

So, let us return to the bode plot that we have just drawn for which case our closed loop system was stable. So, we have the gain crossover frequency between omega p c lower and omega p c upper assuming that both these phase crossover frequencies exist. So, let us assume that the slope of the magnitude characteristic near the gain crossover frequency is minus 40 alpha dB per decade.

So, let us assume that we want a certain phase margin for our open loop system. So, let assume that we have achieved that phase margin in this particular case. So, we have the phase margin P M that we want. So, we can once again write, the angle criterion near the gain crossover frequency as angle of L U at j omega g c is going to be equal to the angle of the Blaschke product, which is j omega plus a by j omega minus a at omega g c plus the angle of the minimum phase part of loop the of the loop gain.

And from the graph here, we note that the angle of L U at omega g c is given by minus pi plus the phase margin that we have specified. And we know that the angle of the Blaschke product or the phase of the Blaschke product at any particular frequency w is given by minus pi plus 2 times tan inverse omega by a. And at the gain crossover frequency, it will be 2 times tan inverse omega g c by a. And the phase of the minimum phase loop gain is determined by the magnitude characteristics thanks to Bode's gain phase relationship.

So, as we saw in the previous clips as well. If the magnitude characteristic of the minimum phase loop gain rolls off at minus 40 alpha decibels per decade, as it is happening in this case, because the magnitude for characteristics of L M P is identical to the magnitude characteristics of L U. Then the phase lag associated with this magnitude characteristic is going to be alpha pi or in other words the phase is going to be equal to minus alpha pi radians. So, this we get from the approximate bodes gain phase relationship.

So, I can replace the angle of L minimum phase approximately by the term minus alpha pi. So, with this equation we can determine the gain crossover frequency in terms of the parameter a as well as the roll of rate alpha. So, to do that we rearrange the terms and see that 2 times tan inverse of omega g c by a will be equal to alpha pi plus P M or in other words omega g c by a will be equal to tan of alpha pi plus P M by 2. So, what this indicates is that the gain crossover frequency can be readily predicted approximately. If we know the rate at which the magnitude characteristic is rolling of near the gain crossover frequency and that rate is given by minus 40 alpha decibels per decade. And we have a certain specified phase margin P M.

Now, this equation can be employed to deduce the lower limit to the gain crossover frequency that is because, if we look at the term on the right hand side, we have a tan of some particular variable. And we know that tan of theta is an increasing function of theta. So, around theta equal to 0, tan theta is 0 and theta as theta tends to pi by 2 tan theta tends to infinity. Since, the term alpha has to be a positive number, because our gain has to crossover or in other words the slope of the magnitude characteristic has to be some negative number alpha has be greater than 0, so alpha has to be greater than 0.

And therefore, what this indicates is that the gain crossover frequency is going to be an increasing function of alpha. So, the larger is the value of alpha or the steeper is the slope in the vicinity of a gain crossover frequency; the larger will be the gain crossover frequency itself. So, there is a lower limit to the gain crossover frequency that happens when alpha assumes the smallest value that, it is permitted to assume. And if we note the fact that alpha has to always be greater than 0, in order for the gain to cross over or in other words in order for the magnitude of the loop gain to change from some value greater than 0 dB to some value less than 0 dB. We note that the minimum possible value for alpha is going to be some value close to 0, but slightly larger than 0, which I have indicated as 0 plus.

So, for this particular value of alpha, the gain crossover frequency will assume it is minimum value. And that is given by omega g c minimum is equal to a times tan of P M by 2, this is because if alpha is very close to 0, we can ignore the term alpha pi in relation to the term P M. And hence conclude that the minimum gain crossover frequency is given by a tan P M by 2.

So, if you note this expression, once again this is for the first time in these lectures that you are coming across an a lower limit to the gain crossover frequency for this does not happen in case of minimum phase plants, it does not happen even in case of non-minimum phase plants in the presence of time delay or non-minimum phase 0. So, there is a lower limit to the gain crossover frequency, which implies that we have to maintain a certain minimum bandwidth for our close loop system, no matter what in order to make sure that we achieve a certain specified phase margin P M for our close loop system.

Now, in contrast to the non-minimum phase 0 in whose case, we had an upper limit to the gain crossover frequency. And that prevented us from tracking certain references and rejecting some disturbances. In the case of the unstable plant, we have a lower limit to the achievable gain crossover frequency. So, even if we do not desire performance at frequencies up to omega g c minimum, we are still force to make sure ensure that the closed loop bandwidth is at least going to be equal to a tan P M by 2.

So, in the case that we considered here, we assume alpha to be very close to 0. And for this case, the gain margin is going to be also close to 0 dB. Now, for the kind of bode plot that the unstable system possesses, we note that there are two gain margins. One is the gain margin at omega p c lower and we can call this gain margin as GML. And then another is the gain margin at omega p c upper so, the negative of the gain at the frequency omega p c u, and we can call that as gain margin upper.

So, there are two gain margins and we need to make sure that both is gain margins assuming that they exist are greater than 0 dB. So, gain margin upper by definition is the negative of the loop gain at omega p c u. And gain margin lower by definition is the magnitude not the negative, but directly the magnitude of the loop gain in the decibel scale at omega p c l. And both of these have to be greater than 0 for our closed loop system to be stable. But, it may happen that from for some plants omega p c l may not exist or omega p c u may not exist in which case, we will have only one of these gain margins that we need to pay attention to.

So, let us undertake an analysis, where we have been specified a certain gain margin lower. A certain gain margin lower has to be achieved and the certain phase margin has to be achieved. And let us see, what kind of fundamental limitations or restrictions there would be on the gain crossover frequency in the presence of these two specifications.

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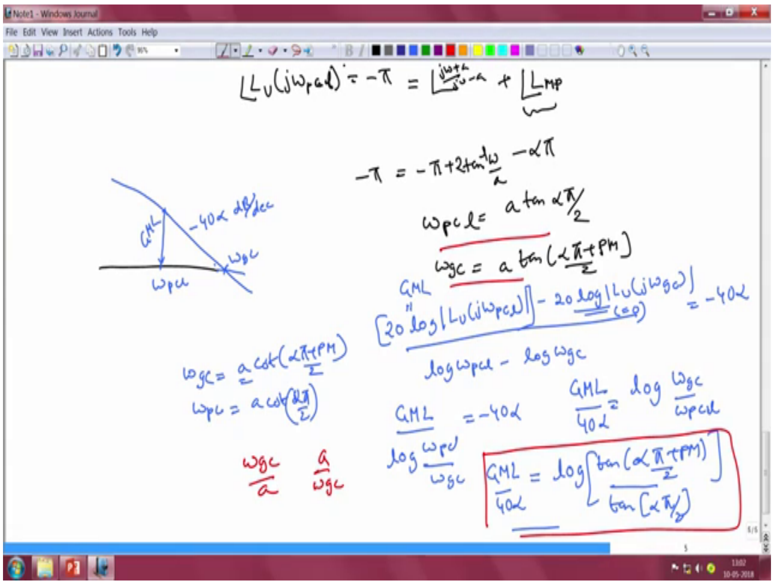

Now, since we want a gain margin of GML omega p c l, we first need to write down the frequency at which the phase crosses over the first time. So, the phase crosses over the first time, when the phase lag of the loop gain at omega p c l g of j omega is p c u is going to be equal to minus pi radians. Now, in order to undertake this derivation of determining the gain crossover frequency, when we have specified a certain GML and a certain phase margin, we assume that the magnitude characteristic between the frequencies omega p c l and omega g c rolls off at the same constant rate of 40 alpha decibels per decade.

So, under that assumption if we were to write the angle of the loop gain L U in terms of the Blaschke product and that of the minimum phase loop gain, we would have this to be equal to the angle of the Blaschke product angle of j omega plus a by j omega minus a plus the angle of L minimum phase. Now, if we make the assumption that I just stated namely, that the loop gain is rolling off at the same rate of minus 40 alpha dB per decade at omega p c l as it rolls off at at omega g c. Then we can write the angle of minimum phase from Bode's gain phase relationship to be approximately equal to minus alpha pi, so that is going to be the phase lag of the minimum phase part of the loop gain.

And the Blaschke product of course will have a phase of minus pi plus 2 tan inverse omega by a and that is going to be equal to minus pi at the lower phase crossover frequency. So, from this equation, we note that the lower phase crossover frequency omega p c l. So, here also it should be l omega p c l is given by omega p c l is equal to a times tan of alpha pi by 2. So, omega g c to remind you is going to be given by a times tan of alpha pi plus P M by 2.

Now, since we have assumed that the magnitude characteristic is rolling off at the same constant rate of minus 40 alpha dB per decade. So, if this is the magnitude characteristic at omega p c l, we have a certain gain that is given by GML. And then we have omega g c, we assume that in between these two frequencies the slope is minus 40 alpha decibels per decade, which implies that 20 times log of magnitude of L U at the frequency omega p c lower minus 20 times log of magnitude of L U at the frequency omega g c divided by log of omega p c lower minus log of omega g c is going to be equal to minus 40 alpha decibels per decade.

Now, we note that 20 log of L U at j omega g c by definition is going to be equal to 0. So, this term is going to be equal to 0 and by definition the term 20 log of L U of j omega p c l is going to be equal to GML. So, we would have that GML divided by log of omega p c l by omega g c is going to be equal to minus 40 alpha or in other words GML by 40 alpha is going to be equal to log of omega g c by omega p c l.

Now, we have the expressions for omega p c l here and omega g c here. In terms of alpha and a and the phase margin, so if we substitute that here, we would get that GML by 40 alpha is going to be equal to log of tan of alpha pi plus P M by 2 divided by tan of alpha pi by 2 that is going to be equal to GML by 40 alpha.

Now, once again this is a transcendental equation. And we have to solve it numerically for obtaining the value of alpha that satisfies it for a specified value of GML in a specified phase margin P M. Now, there is one simple trick that we can adopt in order to quickly obtain the value of alpha. If we go back to our discussion on non-minimum phase systems in particular on systems, which have non-minimum phase zero. We noted in that case that the gain crossover frequency omega g c was given by a cot alpha pi plus P M by 2, where a was the location of the non-minimum phase zero in that case. And omega p c we had just a single phase crossover frequency there was given by a cot alpha pi by 2.

And we therefore, got an equation that look very similar to what we have here for the case of the unstable system with the exception that instead of having tan of alpha pi plus P M by 2 divided by tan of alpha pi by 2, as we have in this case. We had the opposite of it, we had cot of alpha pi plus P M by 2 divided by cot of alpha pi by 2. Hence, we have we end up with the same expression in case of the unstable system as what we had in case of the non-minimum phase 0.

So, if we were to replace omega g c by a, which was the vertical y axis of the graph that we showed in the previous clip, which plotted the best achievable gain crossover frequency as function of the gain margin for different phase margin values. If we were to replace that with the inverse of it namely a by omega g c, then we can use the exact same graph to predict, what the minimum necessary gain crossover frequency is for the case of the unstable system.

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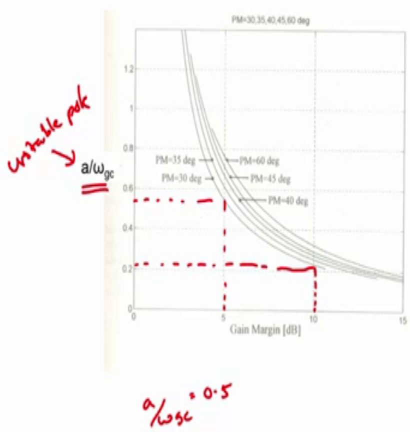

So, we shall revisit the same plot, but this time the y-axis of the plot has been changed to a by omega g c, where a is the location of the unstable pole; so a is the location of unstable pole. And we can use the same graph to tell us what is the minimum gain crossover frequency necessary for a specified phase margin, and a specified gain margin?

For instance, if you specify a gain margin of 5 dB and a phase margin of close to 30 degrees, then your a by omega g c has to be at least equal to 0.5 or in other words a little bit more than 0.5 or in other words the minimum necessary gain crossover frequency has to be almost double of a. Likewise, if you want a phase margin of 30 degrees and a gain margin of 10 dB, then we note that a by omega g c will be close to 0.2; or in other words, our the minimum necessary gain crossover frequency. When you desire a gain margin of 10 d B and a phase margin of 30 degrees, we will be at least 5 times a.

So, unlike in the case of the minimum phase plant, where there was an upper limit to the gain crossover frequency, which was a small fraction of the location of the non-minimum phase 0. In case of the unstable plant, there is a lower limit to the necessary gain crossover frequency in the interest of stability, which is going to be several multiples of the location of the unstable pole. Now, there are interesting consequences to these realizations when we have both unstable poles as well as unstable zeros in our open loop system. This we shall look at in the next clip.

Thank you.