**Control System Design**
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**Lecture - 47**
**Fundamental Properties of unstable systems**

Hello, in this clip we shall discuss the control of Unstable Plants and more to the point rather than discussing some special tricks or techniques that might be out there to control unstable plants. We shall be focusing more on some of the fundamental limitations that are imposed by unstable plants on our goals as control engineers.

So, therefore, this lecture would be of a very similar flavour as the lecture that I gave on a non minimum phase systems. In that we would be discussing some of the fundamental limitations imposed by a special structure of the plant. In the former case it was the fact that you had either time delays or non minimum phase 0 as part of the plants structure. In this case we are considering plants that have unstable dynamics.

So, what do we mean by unstable plants? Very simply unstable plants are those plants which have one or more of their poles on the right half of the complex plane. So, let us assume that we have a plant P which has one of it is poles on the right half of the complex plane.

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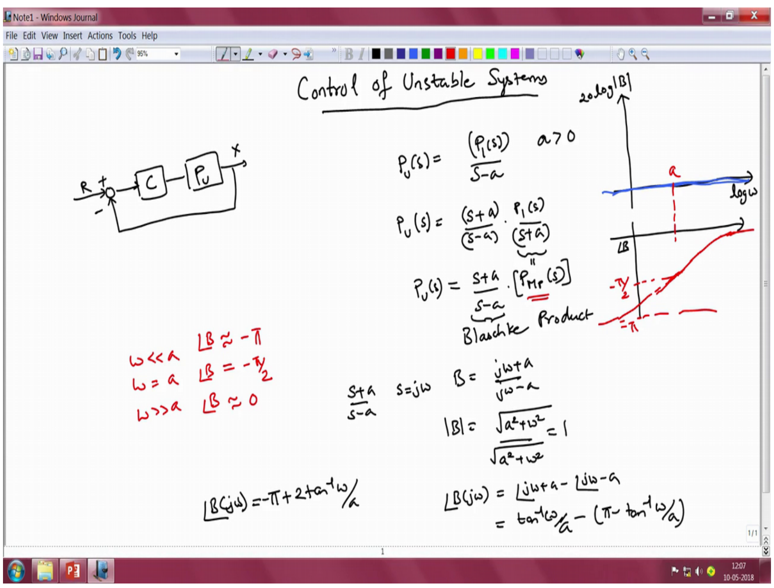

Then, we can write down the plants transfer function as P of s is equal to P 1 of s divided by s minus a; where the term a is greater than 0. So, if P 1 of s is a minimum phase transfer function or in other words all of the poles and zeros of P 1 of s are on the left half of the complex plain; then a plant P of s which is of the form P 1 s by s minus a has one of it is polls namely s is equal to a on the right half of the complex plane and hence and hence qualifies to be called an unstable plant. Just to underscore the fact that a plant of this structure is an unstable plant, we shall provide the plant P of s with the subscript u and called P u of s as the unstable plant.

Now to understand the kind of issues that one would encounter and one is trying to control an unstable plant. Let us first make a few algebraic manipulations and rearrangements, then take a look at the bode plots of loop gains that have unstable plants in them. And see the problems that are that we would first have to confront to decide upon the stability of a close loop system which has unstable open loop dynamics.

And subsequently see what fundamental limitations the loop gains of unstable open loop system would impose on our goals as control engineers. So, to start with let me write out the plant transfer function P u of s as s plus a by s minus a times P 1 of s by s plus a. So, in other words I have multiplied and divided the right hand side with the same term namely s plus a.

So, the right hand side remains unaffected, but now I can rearrange the right hand side as s plus a by s minus a times P m P of s; where P m P of s represents a minimum phase transfer function and it is essentially given by P 1 of s by s plus a. So, P 1 of s by s plus a essentially is the term P n P of s. So, we can write the unstable plants transfer function P u of s as s plus a and s minus a time P n P of s. And those of you who have looked at the lecture on non minimum phase plants will recognise this term s plus a by s minus a to be another blaschke product.

So, if one wants to plot the bode plot of the unstable plant we can view it as some of the bode plot of the blaschke product and of the minimum phase plant P m P of s. Once again the bode plot of P m P of s is something that we should be able to draw without any effort having viewed having looked at such plants all through these lectures; so let us focus on the bode plot of the blaschke product alone. So, the blaschke product is given by s plus a by s minus a and in order to draw it is bode plot we substitute s is equal to j omega so, that we would have the blaschke product to be equal to j omega plus a by j omega minus a. What is the magnitude of the blaschke product? It is going to be equal to square root of a square plus omega square divided by once again square root of a square plus omega square and that is going to be equal to 1.

So, just as in the case of non minimum phase systems the blaschke product of in this particular case for the unstable system also has unit magnitude over the entire frequency range independent of the frequency omega. What is the phase of the blaschke product?

We can show with some effort that the angle of the blaschke product as function of frequency is given by the angle of the numerator term namely the angle of j omega plus a minus the angle of j omega minus a. And we can show that this is going to be equal to tan inverse of omega by a minus the angle of j omega minus a can be shown to be equal to pi minus of tan inverse of omega by a and as a consequence of this the phase of the blaschke product B of j omega is going to be given by minus pi plus 2 tan inverse omega by a.

So, if one were to draw the magnitude characteristics of the blaschke product or in other words the x axis is log of omega the y axis is 20 log of magnitude of B. We would have a straight line that is co incident with the x axis. So, the gain is equal to 0 d B or in linear scale one unit a tall frequencies omega. If we come to the phase plot we note that where frequency omega is very small the phase of the blaschke product is close to minus pi radial.

So, if you were to draw the angle of B we note that when omega is very small in other words when omega is much less than a we would have the angle of B to be approximately equal to minus pi; when omega is exactly equal to a we would have the angle of B to be equal to minus pi plus 2 times tan inverse of one and that is equal to minus pi by 2.

So, B will be equal to minus pi by 2 and omega is much greater than we note that the term tan inverse omega by a will tend to pi by 2 and hence the overall phase of the blaschke product will tend to 0. So, the angle of the blaschke product will be close to 0 for frequencies omega that is much greater than a. So, if you look at the frequency a on the x axis of the bode plot then at very low frequency the phrase is close to minus pi and at omega equal to a the phases equal to minus pi by 2 and the frequency tends to infinity the phase asymptotically approaches 0 radiance.

So, this is the bode plot of the blaschke product and the unstable plant has the magnitude characteristic which is going to be identical to the magnitude characteristics of the minimum phase plant mainly P m P of s where as the phase characteristics of the unstable plant is going to be different from the phase characteristics of the minimum phase plant and on account of the phase added by the blaschke product.

Now, it is quite common to come across the phase characteristics of unstable systems to be as once that are quite different and unrecognisable from the phase characteristics that we have been looking at so far in this course. So, just to highlight the difficulties that the strange phase characteristics of a unstable loop gains would cause to us as control engineers and determining the stability of the close loop system. Let us take a specific example.

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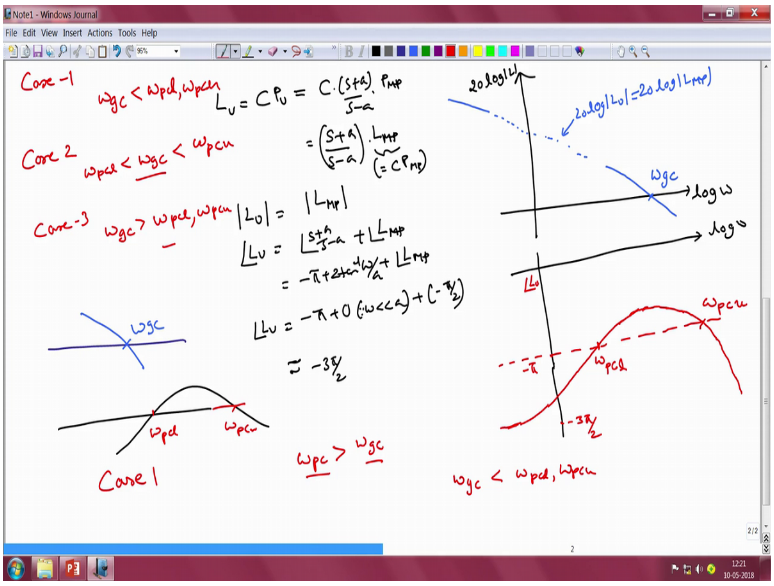


Let us assume that we have a certain controller C which multiplies our unstable plant P u and we assume that the controller is a minimum phase controller. So, I shall call the product of C times P u as L u L subscript u and from the previous slide we note that this is going to equal to C times s plus a by s minus a times P minimum phase. And I shall combine the terms C and P minimum phase and write rewrite this expression as a x plus a by s minus a times L minimum phase; where by definition L minimum phase is equal to C times P minimum phase.

Now, we note that the magnitude of L u will be equal to the magnitude of L minimum phase because the blaschke product has a gain of 0 d B at all frequencies omega. Whereas the angle of L u will be equal to the angle of the blaschke product namely s plus a by s minus a plus the angle of the minimum phase transfer function or another words it is going to equal to minus pi plus 2 tan inverse omega by a plus the angle of the minimum phase transfer function. Now, let us assume that our minimum phase transfer function has an integrator as part of it is structure presumably because we are expecting good performance at very low frequencies and in a particular we will be expecting 0 tracking error for dc references or for dc disturbances.

So, if that is the case then our L minimum phase has an angle of minus pi by 2 around a omega equal to 0. So, let us sketch the bode plot of our unstable loop gain; when our magnitude characteristics has an integrating characteristics at low frequencies. And see what kind of confusions the phase characteristic of such a loop gain could cause in our effort determine the satiability of the close loop system.

So, on the right hand side I am drawing the bode plot of L u. So, the x axis is log omega y axis is 20 log magnitude of l. So, at very low frequencies as we discussed if our L u has an integrator as part of it is structure or if L minimum phase has an integrator as part of it is structure then the role of will be minus 20 d B per decade. And subsequently we will have a certain role of depending on the kind of poles and 0 that the plant and the controller have. And finally let us assume that the gain characteristics cross over at some frequency, which we shall of course, call omega gc the gain cross over frequency.

Now, associated with this magnitude characteristics; we would have a certain phase characteristic. So, this is log of omega I shall indicate the angle minus pi clearly in the phase plot. So, the second plot is going to be the phase of L u this is the first plot is of course, the 20 log magnitude of L u. It is going to be identical to 20 log magnitude of L minimum phase. As for as the phase lag of L u is concerned we note that if we have an integrator as part of the controller structure of our open loop system that term adds a phase lag of minus pi by 2.

So, at very low frequencies in the other words when omega is much less than a we will have the phase lag of L u to be equal to minus pi plus 0 that is because omega is much less than a plus the phase lag of L minimum phase it is going to be equal to minus pi by 2 because we assume that we have an integrator.

So, together it is going to approximately minus 3 pi by 2. So, the phase starts at minus 3 pi by 2 so somewhere here I shall mark out the point minus 3 pi by 2 on the bode plot. So, the phase starts somewhere here and you notice that as the frequency omega increases the phase lag of the baschke product starts to reduce. So, it starts at a value close to minus pi and then tends to 0.

So, assuming that our plants dynamics have not kicked in yet in this frequency range, what we will have is that the phase will be dominated the phase lag will be dominated that of the baschke product alone. So, the phase will start to increase and there will be a frequency at which it causes over. So, there will be one cross over we shall call omega phase cross over layer omega pc l. So, the phase will cross over here because of the increasing phase of the baschke product and then it will continue to rise, but then there will be a frequency at which the plants poles and zeros the other poles and zeros of the plants will start to contribute to phase lag.

So, this rise of phase as function of frequency will be arrested by the phase lag contributed by the other poles and zeros of plant and the controller. And hence the phase will start to once again decrease and it will continue to decrease because this phase lag contributed will increase with from with frequency and that causes the phase characteristics to cross over once again.

So, there will be second cross over frequency which we shall call omega pc upper omega pc u. And this phase lag will continue to increase and finally, the phase will asymptotically approach a certain phase lag that is determined by the relative degree of the loop gain. So, this is the phase characteristic and that is the magnitude characteristic. And here is where our problem begins as control engineers; how can we determine whether a system an open loop system that has a bode plot of the kine that I have drawn here is stable or not.

So, one thing this is the first time that we are coming across a system that has two phase cross over frequencies. So, we have one phase cross over frequency which happens because of the increasing phase of the blaschke product as function of frequency and that is omega pcl and we have another phase cross over frequency that happens because of the phase lag contributed by the other minimum phase terms in the loop gain namely in that of the plant and the controller and that you have called as omega pcu. So, this kind of phase characteristics is entirely unfamiliar to us assuming that we have confined ourselves to looking at the bode plots of the minimum phase plants. So, how do we determine whether a system that has a phase characteristics of this kine is stable or unstable.

Now, if you want to sort of derive inspiration from the phase characteristics of the minimum phase systems. Then for a minimum phase system to be stable we have noticed that firstly such a system would have a single phase cross over frequency not two phase cross over frequencies. But that one phase cross over frequencies omega pc should be greater than the gain cross over frequency for the close loop system to be stable or in other words the phase lag at the frequency at which the gain crosses over should be less than pi radius.

So, our omega pc being greater than omega gc is the necessary condition for a minimum phase system to have stable close loop dynamics. If we were to uncritically apply the same criteria for stability in this particular case; we note that we have our gain cross over frequencies in between omega pcl and omega pc u in the particular schematic that I have drawn. So, our gain cross over frequency is greater than the lower phase cross over frequency, but less than the upper cross over frequency. Now if you want to uncritically apply the lessons that we have learnt in case of minimum phase systems where we wanted the phase cross over frequency to be greater than the gain cross over frequency.

We might conclude that a system with this kind of bode plot is actually unstable because one of the phase cross over frequencies is actually less than the gain cross over frequency. So, we might therefore, we let to conclude that the rule that we need to apply in all that for us to have a stable closed loop system is that the gain cross over frequency should be to the left of both the phase cross over frequencies. Or in other words we might be let you conclude that omega gc should be less than both the omega pc lower as well as omega pc upper. That may be our live conclusion first guess about what is necessary in order for our close loop system to be stable.

But we have no logical arguments to back such a claim and that is because as I have repeated in the past; the bode plot is by no means the right tool for us to determine the stability of a close loop system. In order for us to determine the stability of close loop system in the frequency domain there is only one plot available to us and that is the nyquist plot.

So, if you are given a certain unfamiliar magnitude and phase characteristics all we need to do is to draw the nyquist plot of this particular magnitude and phase characteristics and look at the encirclement of the point minus 1 look at the number of unstable pose that you have for your system and open loop poles have to your system and then come up with the correct rules for determining the stability of the close loop system.

So, let us therefore, not take shortcuts and try to guess about the stability of the close loop system by simply looking at the bode plot of the open loop system. And trying to draw some weak inferences from the bode plots of minimum phase systems. Let us do it a right way let us draw the nyquist plots come up with a right rules for determining stability in the nyquist plot. And then apply them to determine what conditions are necessary for our system to be stable. So, before we do that we note that there are three possible cases here the first case is when we have the gain cross over frequency omega gc being less than both omega pc lower and omega pc upper.

So, which is what we thought it was necessary for our close loop system to be stable on the basis of what we saw in the case of minimum phase plants. So, omega pc lower omega pc upper so this is case 1. So, case one essentially is that omega gc is less than both omega pc lower and omega pc upper. Now there is a second case that is possible and that has what has been drawn on the schematic on the right namely where omega gc the gain cross over frequency is between the two phase cross over frequencies. So, case two is one where omega pc lower is less than omega g c, but omega gc is also less than omega pc upper so this is the second case.

And the third case is the opposite of case one namely where the gain cross over frequency is greater than both the phase cross over frequencies. And in another words that omega gc is greater than omega pc lower and omega pc upper. The question is which of these three cases will we have a stable close loop system on our hands? Is it going to be the first case; which is what we think it is now based on our live extension of what we have observed in case of minimum phase loop gains? Or is it for a second case or is it for a third case? So, let us take each of these cases one by one apply the principles of nyquist stability theory for these three cases and see for which case it is that the close loop system would be stable.

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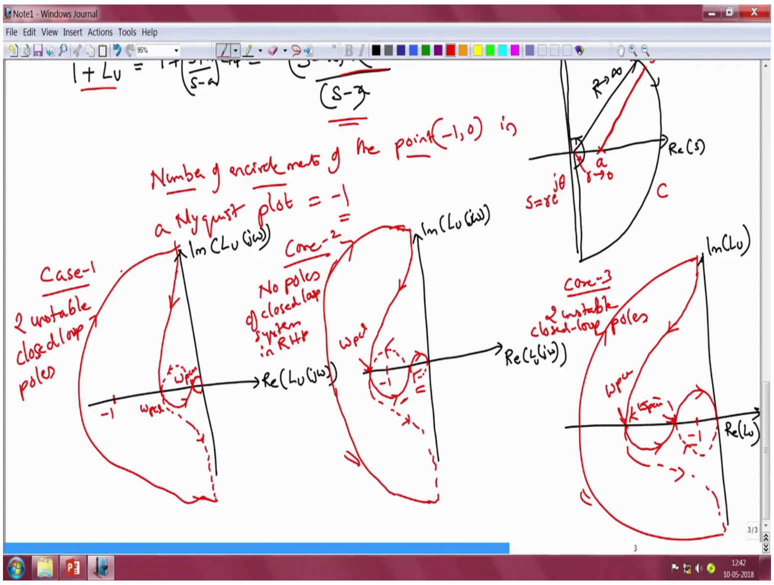


So, before we draw the nyquist plot for the open loop system; let us first examine what kind of rules need to be applied. In order to determine the stability of the close loop system, we note that the denominator transfer function for our close loop system is going to given by 1 plus L u and if I were to replace the term L u by s plus a by s minus a times L minimum phase, then I would have that 1 plus L u would be equal to s minus a plus s plus a times L minimum phase divided by s minus a. So, now let us first draw the complex plane in other words the real part of s versus the imaginary part of s which is omega.

And our familiar d shaped contour will have it is straight edge co incident with the imaginary axis. And then we have this semi circular arc of radius capital arc that is tended to infinity and that encompasses the entire of the right half of the complex plane. So, this is our familiar d shaped contour ah; along which we would compute the complex number L of s and then draw it is nyquist plot. Now we note that in this particular case we have an open loop pole namely s is equal to a within the d shaped contour. So, s is equal to a is going to be within the d shaped counter. So, the complex number s minus a for point s on the counter that we have drawn here on the contour C the complex number s minus a is going to given by this particular phasor.

Now, when we take the s variable along this contour in the clock wise direction once; we note that the phasor s minus a will go round itself once or in other words it will execute a change in angle of 2 pi radiance. But since the term s minus a, appears in the in the denominator of the transfer function 1 plus L u. We note that the net encirclement that will occur because of the term s minus a. When we take the variable s in the clockwise direction once is one counter clockwise encirclement. Because the term 1 by s minus a will change it is angle in a sign in a manner that is opposite to the direction in which the angle of s itself changes.

Hence if you want zeros of 1 plus L u which are going to the close loop poles of our system; to not lie within the d shaped contour then the rule that we need to apply for determining the stability of the close loop system, is that the number of encirclements of the point minus 1 comma 0 in a nyquist plot should be equal to minus 1. Because the term s minus a which is in the denominator of 1 plus L u will result in one counter clockwise encirclement or the encirclement sign is negative and hence that results in one counter clockwise encirclement.

And if none of the close loop poles are within this d shaped contour or in other words if none of the zeros of one plus L u; which are essentially the zeros of the numerator polynomial of this transfer function 1 plus L u. If none of them are in the right half of the complex plane, they will not contribute to any encirclement of the critical point in the nyquist plot. Hence, the only encirclement that will happen, will be because of the term s minus a and that encirclement is counter clockwise. And hence the correct rule that we need to apply to determine the stability of the close loop system in the nyquist plot of the unstable system is that the number of encirclements of the critical point should be equal to minus 1.

So, we should have one counter clockwise encirclement of the critical point by the nyquist plot of the open loop system. So, now, let us return to the bode plot that we were just looking at. Where we had three particular cases one is when the phase cross over frequencies or both greater than the gain cross over frequency. The other is where the gain cross over frequency between the two phase cross over frequencies. And the third is one were phase cross over the frequencies are both lesser than the gain cross over frequencies. And see for which case it is that the nyquist plot encircles the point minus 1 once in the counter clockwise sense.

Now, in order to draw the nyquist plot for the particular example that we considered in the previous slide; we note that we have an integrator in the as part of our open loop transfer function. And hence we need to introduce a tiny king in the contour near the origin here s is equal to 0 the location where the integrator has it is pole. So, we need to introduce the tiny king in the origin and modify our d shaped contour slightly in the manner that I have shown here before we can drawn the nyquist plot of the open loop system.

So, now let us proceed to draw the nyquist plot for the three cases that we discussed. The first case is one where the two phase cross over frequency were both greater than the gain cross over frequency. Now when both the cross over frequencies are greater than the gain cross over frequency, what we are essentially claiming? I s that at both the phase cross over frequencies the gain of the open loop system is going to be less than 1. So, if we were to draw the nyquist plot for this case the x axis will be real part of L u of j omega, the y axis will be the imaginary part of L u of j omega. We note that when omega is very small the phase of L u starts at close to minus 3 pi by 2. So, it will be somewhere here this is the location where the nyquist plot starts in a neighbourhood of omega equal to 0.

And then as omega increases the gain of the open loop system will start to reduce the gain will start to reduce. And then there will be one frequency at which the phase crosses over. The first time it crosses over it is at the frequency omega pc lower. And subsequently it will cross over again at another frequency at a higher frequency omega pc upper. And finally, the magnitude characteristics will tend to 0 in some particular fashion. And the gain of the transfer function L u at both omega pc lower and omega pc upper will be less than 0 d B for the case namely case 1.

Then the gain cross over frequency is to the left of both omega pc lower and omega pc upper. In other words the critical point minus 1 will be located to the left of both the omega pc lower and omega pc upper. Now if we complete the nyquist plot the nyquist plot for the complex conjugate of the imaginary axis or the nyquist plots of the imaginary axis; will look something like this and that will tend to minus infinity along the negative imaginary axis.

So, this is going to be the mapping of the positive imaginary axis and the negative imaginary axis in the g of as a consequence of this transformation; namely L of j omega L u of j omega. The big d shaped contour will collapse to the origin and what about the small semi circular king that we have introduced that avoids the integrator at the origin. We can show that by substituting the expression that along this contour we would have the complex number of form s is equal to small r e power j theta. Where small r is the radius of this king and that is tended to 0 of course.

And we see that theta here goes from minus pi by 2 to plus pi by 2. We can show that this particular curve gets mapped to a big semi circular arc I am sorry semi circle does not look that good, but it would be semi circle when we want to draw it correctly, of radius r that is tended to infinity. So, the radius r here it is going to be on the order of 1 by small r and that would go to infinity when small r tends to 0. So, our nyquist plot would look something like this for the first case.

So, if we look at this nyquist plot we note that the parts that we have to the right of the point minus 1. That two curves that we have here do not encircle the critical point at all. That is big curve here encircles the point minus 1 ones, but in the clockwise direction. Now if we have one clockwise encirclement of the point minus 1; what it indicates is that 2 of the 0's of the transfer function 1 plus L u are within the d shaped contour. Because if 2 zeros are within the d shaped contour these 2 zeros together contribute to 2 clockwise encirclements of the point minus 1 and when combined with one contour clockwise encirclement of the point minus 1 due to the term s minus a in the denominator we would have a net of one clockwise encirclement of the point minus 1. And this precisely what we are seeing in this particular case. So, in case one therefore, we would have 2 unstable closed loop poles.

So, unlike what our naive intuition let us to guess it is the case one that results in an unstable close loop system and not a stable close loop system. Now let us take a look at what happens for case two; so in case two we note that the gain cross over frequency is between omega pc lower and omega pc upper. So, if I were to plot the nyquist plot again it is going to be real part of L u of j omega is going to be equal to imaginary part of L u of j omega.

Once again we would have the nyquist plot starting at this location because the phase lag would be equal to 3 pi by 2 as omega tends to 0. And then it reduces with frequency for the first time the phase crosses over or the first time the phase of the loop gain reaches pi minus pi radians the gain is going to be greater than minus 1.

So, the phase the loop gain will cross at a point that is to the left of the point minus 1 comma 0 so this will be the frequency omega pc lower. And then subsequently the second time it crosses the phase crosses over or the second phase lag assumes the value of pi radians; the gain would have dropped below 0 d b. So, it would cross somewhere to the right of the point minus 1 and subsequently it will go to 0 in some particular manner.

The nyquist plot along the negative imaginary axis will be the mirror image of this nyquist plot about the real axis and would therefore, it look something like this. And the D shaped and the small d shaped contour near the origin gets mapped to this huge d shaped contour that starts at the negative imaginary axis at close to minus infinity and close towards the positive as many axis close to plus infinity.

So, once again this is supposed to be a semi circle, but go into shortage of space and by limited abilities as an artist it looks like a distorted circle. So, if you were to draw the arrow for the direction in which the loop gain moves when the variable s is taken around the contour d shaped contour in the clockwise sense. We know that it moves from this way and then that way finally, this way and along the negative imaginary axis this is the way in which the loop gain changes as we traverse the d shaped contour.

So, if you focus on this particular nyquist plots; we see that in this case this big loop does not encircle the point minus 1 at all unlike in the previous case. And we have one loop here that encircles the point minus 1 the other loop that is close to the origin once again does not encircle the point minus 1. So, this (Refer Time: 36:46) does not encircle the point minus 1 this (Refer Time: 36:47) also encircle the point minus 1 this only the middle loop that encircle the point minus 1.

And in what sense is it encircling it if you notice here it is encircling it in the contour clockwise sense. And if we go back to the rule that we need to apply in order to determine the stability of the close loop system when you have one unstable pole we note that this particular encirclement satisfies that requirement.

The number of encirclements of the point minus 1 in a nyquist plot has to be minus 1 or there should be 1 counter clockwise encirclement for the close loop system to be stable and do not have any of the 0's of 1 plus L u which are the close loop poles of our system to be on the right half of the complex plane. So, it is for case two that we end up with a stable close loop system. So, no poles of close loop system in the right half plane RHP. So, contrary to what our intuition might have let us to believe it is the second case namely the schematic. Namely the schematic that I have drawn here where omega gc in between omega pcl and omega pc u that results in a stable close loop system.

Now, let us consider the third case where the gain cross over frequency is greater than both the phase cross over frequencies. What it implies then is that at both the phase cross over frequencies our loop gain will be greater than 0 d B. So, if one were to draw the nyquist plots once again the x axis once again is the real part of L u and the y axis is imaginary part of L u.

We will have that the loop gain once again starts somewhere here as frequency increases the gain reduces and the first time it crosses over namely at omega pc u the gain is greater than 0 d b. So, the point minus 1 comma 0 will be to the right of this particular location.

And then subsequently it will cross over once again so there will be the phase will be once again assume a value of minus pi at omega pc upper. So, this is omega pc upper and this point is also a point that is to the left of the point minus 1; because in this particular case the gain of the open loop system would be greater than 0 d B even at omega pc upper.

And subsequently the phase does it is particular the loop gain does it particular thing depending on the higher order dynamics of the plant and the controller and finally, goes to 0. So, this is how the nyquist plot will look for the positive imaginary axis for the negative imaginary axis it will be reflection of this plot and it will therefore, look something like this. And the small d shaped contour will get mapped to this big d shaped contour and the nyquist plot will look something like this.

So, for this case to we note that the big loop here does not encircle the point minus 1 at all. The second the middle loop also does not encircle the point minus 1 it is the loop that is closes to the origin that encircles the point minus 1. But unfortunately if you look at the sign in which this loop is traversed by the term L of j omega we see that the loop is traversed in the clockwise sense.

And therefore, exactly as in the first case as in case one we have one clockwise encirclement of the point minus 1 in case three. When the gain cross over frequency is greater than both the phase cross over frequencies and what this indicates towards is that once again in case three just as with case one we would have 2 unstable close loop poles or 2 zeros of the transfer function one plus L u on the right half of the complex plane so, 2 unstable closed loop poles..

Hence, this analysis reveals how contour intuitive it can be for us to determine the correct rules for stability of a close loop system. When we have fairly unfamiliar bode plots that are given to us. So, when the phase characteristics look as unfamiliar and unsettling as what we saw in the previous slide where you had two phase cross over frequencies and so on.

The best and indeed the only thing we have to do is first come to the nyquist plot and depending on the structure of the open loop transfer function namely whether the open loop transfer function has any unstable poles or not. We first have to come up with the right rules for determine the stability of the close loop system and subsequently plot the nyquist for the open loop gain and then see whether the conditions that for stability are satisfied or not.

So, more generally if you have n open loop poles of the plant on the right half of the complex plane for our close loop system to be stable the number of encirclements of the critical point should be minus n or in other words you should have n counter clockwise encirclements of the critical point for our close loop system to be stable. So, having discussed this issue associated with determining correctly the stability of a close loop system which has unfamiliar phase characteristics. Let us now see what kind of fundamental limitations the magnitude and phase characteristics of unstable systems poles to us as control engineers.

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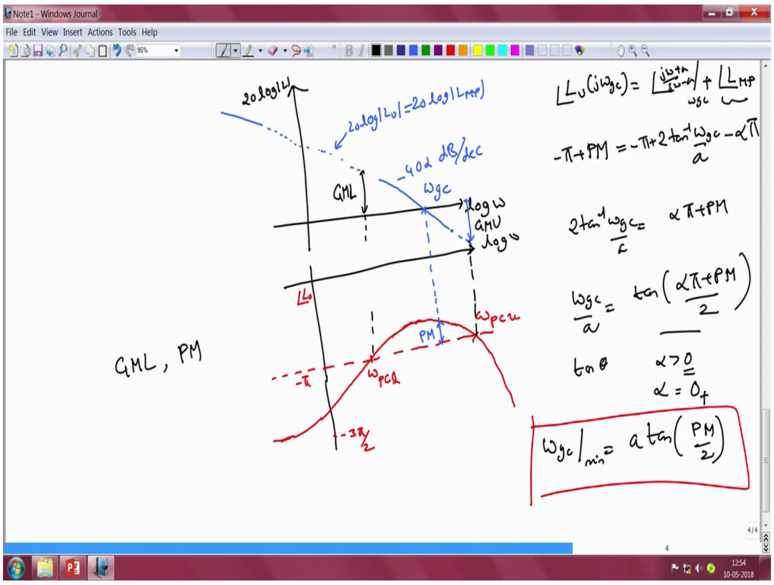

So, let us return to the bode plot that we have just drawn for which case our closed loop system were stable. So, we have the gain cross over frequency between omega pc lower and pc upper; assuming that both these phase cross over frequencies exist. So, let us assume that the slope of the magnitude characteristics near the gain cross over frequency is minus 40 alpha d B per decade. So, let us assume that we want a certain phase margin for our open loop system.

So, let us assume that we have achieved the phase margin in this particular case. So, we have the phase margin P m that we want. So, we can once again write the angle criterion near the gain cross over frequency as angle of L u at j omega gc is going to be equal to the angle of the blaschke product which is j omega plus a by j omega minus a at omega gc plus the angle of the minimum phase part of the loop gain.

And from the graph here we not e that the angle of L u at omega gc is given by minus pi plus the phase margin that we have specified. And we know that the angle of the blaschke product or the phase of the blaschke product at any particular frequency omega is given by minus pi plus 2 times tan inverse omega by a. And at the gain cross over frequencies it will 2 times tan inverse of omega gc by a.And the phase of the minimum phase loop gain is determined by the magnitude characteristics thanks to bodes gain phase relationship.

So, as we saw in the previous clips as well if the magnitude characteristic of the minimum phase loop gain roles off at minus 40 alpha decibels per decade. As it is happening in this case because the magnitude for characteristics of L n P is identical to the magnitude characteristics of L u.

Then the phase lag associated with this magnitude characteristic is going to be alpha pi or in other words the phase is going to be equal to minus alpha pi radians. So, this we get from the approximate bodes gain phase relationship. So, I can replace the angle of L minimum phase approximately by the term minus alpha pi. So, with this equation we can determine the gain cross over frequency in terms of the parameter a as well as the role of rate alpha.

So, to do that we rearrange the terms and see that 2 times tan inverse of omega gc by a will be equal to alpha pi plus P m or in other words omega gc by a will be equal to tan of alpha pi plus P m by 2. So, what this indicates is that the gain cross over frequency can be readily predicted approximately if we know the rate at which the magnitude characteristic is rolling of near the gain cross over frequency. And that rate is given by minus 40 alpha decibels per decade and we have a certain specified phase margin P m.

Now, this equation can be employed to reduce the lower limit to the gain cross over frequency. That is because if we look at the term on the right hand side we have tan of some particular variable and we know that tan of theta is an increasing function of theta. So, around theta equal to 0 tan theta is 0 and theta as theta, theta tends to pi by 2 tan theta tends to infinity. Since the term alpha has to be a positive number because our gain has to cross over or in other words the slope of the magnitude characteristics has to be some negative number alpha has to be greater than 0 so, alpha has to be greater than 0.

And therefore, what is indicates is that the gain cross over frequency is going to be an increasing function of alpha. So, the larger is the value of alpha or the steeper is the slope in the vicinity of the gain cross over frequency the larger will be the gain cross over frequency itself. So, there is a lower limit to the gain cross over frequency that happens when alpha assumes the smallest value that it is permitted to assume. And if we note the fact that alpha has to always be greater than 0 in order for the gain to cross over or in other words in order for the magnitude of the loop gain to change from some value greater than 0 d B to some value less than 0 d B.

We note that the minimum possible value for alpha is going to be some value close to 0, but slightly larger than 0 which are indicated as 0 plus. So, for this particular value of alpha the gain cross over frequency will assume it is minimum value and that is given by omega gc minimum is equal to a times tan of PM by 2. This is because if alpha is very close to 0 we can ignore the term alpha pi in relation to the term PM.

And hence conclude that the minimum gain cross over frequency is given by a tan PM by 2. So, if you note this expression once again this is for the first time in these lectures that you are coming across a lower limit to the gain cross over frequency. For this does not happens in case of minimum phase plants, it does not happens even in case of non minimum phase plants in the presence of time delay or non minimum phase 0. So, there is a lower limit to the gain cross over frequency which implies that we have to maintain a certain minimum band width for our close loop systems no matter what in order to make sure that we achieve a certain specified phase margin P m for our close loop system.

Now, in contrast to the non minimum phase 0 in whose case we had a upper limit to the gain cross over frequency and that prevented us from tracking certain differences and rejecting some disturbances. In the case of the unstable plant we have a lower limit to the achievable gain cross over frequency. So, even if you do not desire performance at frequencies up to omega gc minimum; we are still force to make sure ensure that the close loop band width is at least going to equal to a tan PM by 2.

So, in the case that we have considered here we assume alpha to be very close to 0. And for this case the gain margin is going to be also close to 0 d B. Now for the kind of bode plot that the unstable system processes we note that there are two gain margins. One is the gain margin at omega pc lower and we can call this gain margin as GML and then another is the gain margin at omega pc upper. So, the negative of the gain at the frequency omega pc u and we can call that as gain margin upper. So, there are 2 gain margins and we need to make sure that both these gain margins assuming that the exits are greater than 0 d b.

So, gain margin upper by definition is a negative of the loop gain at omega pc u and gain margin lower by definition is the magnitude not the negative, but directly the magnitude of the loop gain in the decibel scale at omega PCL. And both of these have to be greater than 0 for our close loop system to be stable, but it may happened that some for some plants omega PCL may not exist or omega pcu may not exist in which case we will have one of these gain margin that we need to pay attention to.

So, let us undertake an analysis where we have been specified a certain gain margin lower. A certain gain margin lower has to be achieved and a certain phase margin has to be achieved. And let us see what kind of fundamental limitations or restrictions there would be on the gain cross over frequency in the presence of these two specifications.

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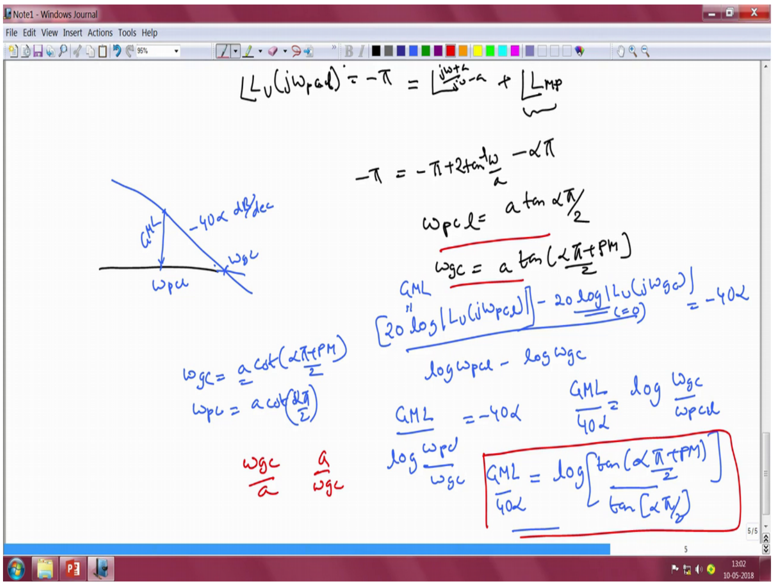


Now, since we want a gain margin of GML omega pcl we first need to write down the frequency at which the phase crosses over the first tile. So, the phase crosses over the first tile when the phase lag of the loop gain at omega pcl g of j omega pcl is pc u is going to be equal two minus pi radians. Now, in order to undertake this derivation of determining the gain cross over frequency when we have specified a certain GML and a certain phase margin, we assume that the magnitude characteristic between the frequencies omega pcl and omega gc rolls of at the same constant rate of minus 40 alpha decibels per decade.

So, under that assumptions if you want to write the angle of the loop gain L u in terms of the blaschke product and that of the minimum phase loop gain we would have this to be equal to the angle of the blaschke product angle of j omega plus a by j omega minus a plus the angle of L minimum phase. Now if we make the assumption that I just stated namely that the loop gain is rolling of at the same rate of minus 40 alpha d B per decade at omega pcl as it roles of at omega g c. Then we can write the angle of that minimum phase from bode gain phase relationship to be approximately equal to minus alpha pi.

So, that is going to be the phase lag of the minimum phase part of the loop gain and the blaschke product of course, will have a phase of minus pi plus 2 tan inverse omega by a. And that is going to be equal to minus pi at the lower phase cross over frequency. So, from this equation we note that the lower phase cross over frequency omega pcl. So, here also it should be L of q omega pcl is given by omega pcl is equal to a times tan of alpha pi by 2. So, omega gc will remind you is going to be given by a times tan of alpha pi plus PM by 2.

Now, since we have assumed that the magnitude characteristic is rolling of at the same constant rate of minus 40 alpha d B per decade. So, if this is the magnitude characteristics at omega pcl we have a certain gain that is given by g m L and we have omega gc we assumed that in between these two frequencies the slope is minus 40 alpha decibels per decade, which implies that 20 times log of magnitude of L u at the frequency omega pc lower minus 20 times log of magnitude of L u at the frequency omega gc, divided by log of omega pc lower minus log of omega gc is going to equal to minus 40 alpha decibels per decade.

Now we note that 20 log of L u at j omega gc by definition is going to be equal to 0 this term is going to equal to 0. And by definition the term 20 log of L u of j omega pcl is going to be equal to GML.

So, we would have that GML divided by log of omega pcl by omega gc is going to equal to minus 40 alpha or in other words GML by 40 alpha is going to equal to log of omega gc by omega pcl. Now we have the expression for omega pcl here and omega gc here in terms of alpha and a and a phase margin. So, if you substitute that here we would get that GML by 40 alpha is going to equal to log of tan of alpha pi plus PM by 2, divided by tan of alpha pi by 2. That is going to equal to GML by 40 alpha. Once again this is the transcendental equation and we have to solve it numerically for obtaining the value of alpha that satisfies it for a specified value of GML in the specified phase margin PM.

Now, there is one simple trick that we can adopt in order to quickly obtain the the value of alpha. If we go back to our discussion on non minimum phase systems in particular on systems which have non minimum phase 0, we noted in that case that gain cross over frequency omega gc was given by a cot alpha pi plus PM by 2; where a was the location of the non minimum phase 0 in that case. And omega pc we have just a single phase cross over frequency there it was given by a cot alpha pi by 2. And we therefore, got an equation that look very similar to what we have here for the case of unstable system; with the exception that instead of having tan of alpha pi plus PM by 2 divided by tan of alpha pi by 2. As we have in this case we had the opposite of it; we had cot of alpha pi plus PM by 2 divided by cot of alpha pi by 2.

Hence we have we end up with the same expression in case of the unstable system as what we had in case of the non minimum phase zero. So, if we want to replace omega gc by a which was the vertical y axis of the graph that we shown in the previous clip, which plotted the best achievable gain cross over frequency as function of the gain margin for different phase margin values. If you want to replace that with the inverse of it namely a by omega gc and then we can use the exact same graph to predict what the minimum necessary gain crossover frequency is for the case of the unstable system.

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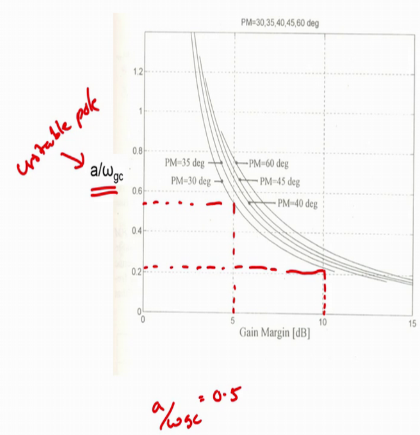


So, we shall re visit the same plot, but this time the y axis of the plot has been changed to a by omega gc; where a is the location of the unstable pole. So, a is the location of unstable pole and we can use the same graphs to tell us what is the minimum gain cross over frequency necessary for a specified phase margin and a specified gain margin. For instance if you specifies a gain margin of 5 d B and phase margin of close to 30 degrees.

And then your a by omega gc has to be at least equal to 0.5 or in other words a little bit more than 0.5 or in other words the minimum necessary gain cross over frequency has to be almost double of a. Likewise if you want a phase margin of 30 degrees and a gain margin of 10 d B. And then we note that a by omega gc will be close to 0.2 or in other words our minimum necessary gain cross over frequency. When you desire a gain margin of 10 d B and a phase margin of 30 degrees will be at least 5 times a.

So, unlike in the case of the minimum phase plant where there was an upper limit to the gain cross over frequency which was a small fraction of the location of the non minimum phase 0. In case of the unstable plant there is a lower limit to the necessary gain cross over frequency in the interest of the stability, which is going to be several multiples of the location of the unstable pole. Now, there are interesting consequences or to these realizations. When we have both unstable poles as well as unstable zeros in our open loop system, this we shall look at in the next clip.

Thank you.