Control System Design
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Lecture - 46
Fundamental properties of nonminimum phase systems (part 2/2)

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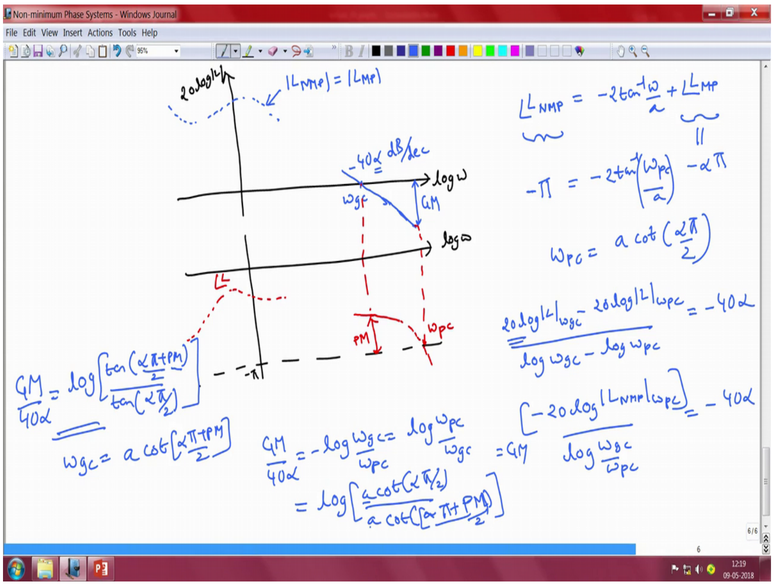

In order to analyze the best achievable gain crossover frequency when we have a certain gain margin specification; let us first assume that the magnitude characteristic of the loop gain continues with approximately the same slope namely minus 40 alpha decibels per decade between the gain crossover frequency omega gc and the phase crossover frequency omega pc.

So, therefore, by definition the gain of the open loop system at the frequency omega pc as the gain characteristic reduces at the specified rate of minus 40 alpha dB per decade will allow us to calculate the gain margin of our open loop system. Now in order to determine the maximum permissible value of omega gc in the presence of a certain specified amount of gain margin; let us first determine what omega pc what the phase cross over frequency would be for a specified value of alpha and subsequently determine the gain of the open loop system at that particular frequency.

So, we shall first return to the relationship between the angle of the non minimum phase loop gain and the angle of the minimum phase loop gain and that is given by angle of L

NMP is equal to minus of 2 times tan inverse of omega by a which is the phase lag of the Blaschke product plus the angle of the minimum phase part of the loop gain namely angle of L MP and if we assume that the slope of the magnitude characteristic near the phase cross over frequency is again equal to minus 40 alpha decibels per decade.

Then Bode's gain phase relationship tells us that angle of MP is approximately going to be equal to minus alpha pi exactly for the reasons that we discussed in the previous slide. Now at the phase cross over frequency by definition we would have angle of L NMP to be equal to minus pi radians; this is by definition.

So, we would have therefore, that minus pi is equal to minus 2 times tan inverse of omega pc by a minus alpha pi and this expression will help us to determine the phase cross over frequency in terms of a and alpha and that is given by omega pc is equal to a cot of alpha pi by 2. So, this is the value of the phase cross over frequency. Now we know that between omega gc and omega pc you have assumed that a magnitude characteristic has a slope of minus 40 alpha decibels per decade.

And at the gain cross over frequency we also know that the gain of the open loop system is going to be equal to 0 dB; hence we would have that 20 log of magnitude of l at omega gc minus 20 log of magnitude of l at omega pc divided by log of omega gc minus log of omega pc this is going to give us a slope of this curve and that is going to be equal to minus 40 alpha decibels per decade. Now we know that 20 log of magnitude of l at omega gc is equals to 0.

So, this term is here going to be is equals to 0 so, we would have that minus 20 log of magnitude of L NMP at omega pc divided by log of omega gc by omega pc is going to be equal to minus 40 alpha and by definition we have that minus 20 log of the magnitude of L NMP at omega pc by definition this term here is going to be equal to the gain margin. So, we would have therefore, that the gain margin GM is divided by 40 alpha is going to be equal to minus of log of omega gc by omega pc or in other words is going to be equal to log of omega pc by omega gc. Now we know the expression for omega pc and omega gc in terms of alpha and the phase margin and the term a.

So, we would have therefore, that GM by 40 alpha is going to be equal to log of a times cot of alpha pi by 2 divided by a times cot of alpha pi plus PM by 2 this whole expression gets divided by 2. So, I shall rewrite the entire expression here we would have

GM by 40 alpha to be equal to log of we would have the term a in the numerator cancel the term a in the denominator and 1 by cot theta is essentially tan theta by definition hence you would have this to be equal to tan of alpha pi plus PM by 2 divided by tan of alpha pi by 2.

So, this is the equation that the variable alpha has to satisfy in order for us to get a specified phase margin PM and specified gain margin GM for our open loop system. If you notice this equation to discover that it is what is known as a transcendental equation and hence it is not possible in general for us to solve this equation by hand. However, we can solve this equation numerically and obtain the value of alpha that satisfies this equation for a specified value of the phase margin PM and gain margin GM. Now once we know that term alpha we can plug that back into the expression for the gain crossover frequency which is given by omega gc is equal to a times cot of alpha pi plus PM by 2.

So, we can plug that alpha which satisfies this particular equation back into this equation and determine the best possible gain crossover frequency that we can achieve for a specified gain margin and specified phase margin.

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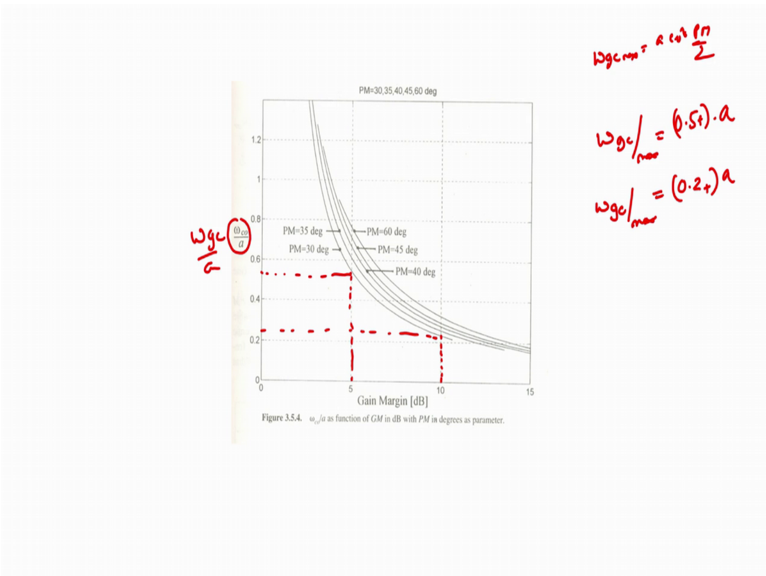

Now this solution has been numerically computed and the graph here shows the best possible gain crossover frequency this essentially is equal to omega gc by a

So, the gain crossover frequency has been normalised with respect to the location of the right half plane 0 a. So, it has been represented omega gc therefore, has been represented as a fraction of a and on the x axis we have our gain margin specification. So, what this reveals is that if we want a phase margin of 30 degrees and a gain margin of 5 dB then from the curve that has been shown here we can work out what is the best possible gain crossover frequency is. So, for a gain margin of 5 dB and a phase margin of 30 degrees, which corresponds to the lower most curve here in the set of curves we know that the best possible gain crossover frequency is somewhere between 0.6 and 0.4 times a.

So, omega gc max is going to be equal to 0.5 plus times a. So, I write the term 0.5 plus to indicate that the exact value is a little bit more than 0.5. So, our best possible gain of frequency is atmost 50 percent of a if we expect a gain margin of 5 dB. Of course, if we have a phase margin of 30 degrees and no gain margin expectation we know that omega gc max is going to be given by a cot PM by 2 According to the expression that we derived.

But if you have a gain margin specification of 5 dB this numerical analysis reveals that our maximum gain cross over frequency is utmost going to be about half of a. Now let us say our gain margin requirement is even higher let us say we execrated 10 dB gain margin then for such a gain margin requirement we note that omega gc by a is actually just a little bit above 0.2 for a phase margin of 30 degrees.

So, what this reveals therefore, is that our gain crossover frequency maximum for the case when we have a gain margin expectation of 10 dB is going to be just a little bit above 0.2 so, it is going to be 0.2 plus times a. So, in general what is reveals is that our gain crossover frequency is going to be always a small fraction of a if you have specifications of a certain phase margin and gain margin that are close to what is generally assumed for minimum phase transfer functions.

Hence although the maximum possible gain crossover frequency is a cot PM by 2 if you have a certain gain margin specification this maximum actually reduces further and it is generally going to be a small fraction of a; where a is the location of our non minimum phase 0 and this happens despite the fact that we might have a plant all of (Refer Time: 10:56) poles and 0s are very fast.

So, even if our plant has poles that are on the left half of the complex plane that are 10 times of 100 times greater than a and therefore, the minimum phase part of the plant or loop gain can respond very quickly to whatever command might be provided to it; the closed loop system gets slope down dramatically because of the presence of the small minimum phase 0.

And the close loop bandwidth is going to be determined predominantly by the non minimum phase characteristics of the overall system and not by the minimum phase transfer function of our open loop system. So, the analysis of the kind that we undertook in case of the non minimum phase 0 can also be extended to determine the best possible gain crossover frequency and phase crossover frequency in case of a time delay.

So, let us first do that and subsequently look at the kind of limitations on loop gain the magnitude of the loop gain that these particular theoretical limits on the gain crossover frequency would impose.

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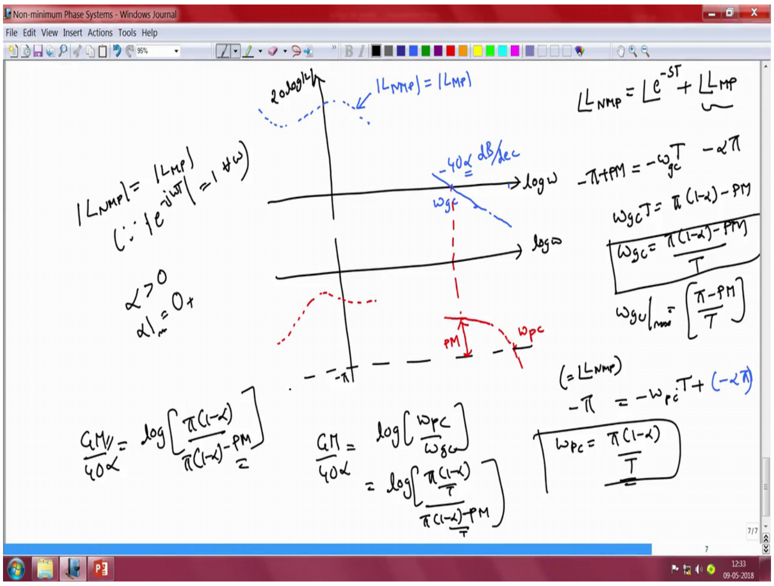

Now, just as we derived the fundamental limits on the gain crossover frequency when we have a non minimum phase 0 we can also derive the fundamental limits on gain crossover frequency when we have a time delay by using the exact same steps as he undertook for the previous case. So, once again let us assume that our loop gain the non minimum phase loop gain L NMP has a certain magnitude characteristic and it crosses over at some frequency omega gc and at the gain crossover frequency let us assume that the slope of the magnitude characteristic in the bode a plot is minus 40 alpha decibels per decade.

And suppose we want to certain phase margin for our overall open loop system of PM then we assume that the phase characteristics of our open loop system does something in the low frequency range as determined by the other poles and 0s of our plant and the controller and finally, achieves a value of PM at omega gc. Now we can write that the angle of L NMP at omega gc is going to be given by the angle of time delay e power minus ST at omega gc plus the angle of the minimum phase part of the loop gain.

And once again from bodes gain phase relationship we note that when the minimum phase loop gain L MP which has to remind you once again the exact same magnitude characteristics as L NMP even for the case when we have a time delay cascaded with our system, this we discussed in our previous clip because a magnitude of e power minus j omega t is going to be equal to 1 at all frequencies.

Hence we would have that the magnitude of L non minimum phase is going to be equal to the magnitude of L minimum phase and that is because magnitude of e power minus j omega t is going to be equal to 1 for all omega.

So, given that we would have that the magnitude characteristics of L MP will be identical to the magnitude characteristics of L NMP. So if the magnitude characteristics of L NMP roles down at slop with minus 40 alpha decibels per decade near the gain crossover frequency and so, does the magnitude characteristics of L MP; however, Bode's gain phase relationship tells us that, if you have a minimum phase loop gain which is rolling of at minus 40 alpha decibels per decade or which is reducing at the slope of minus 40 alpha decibels per decade then the approximate phase lag associate with it is going to be equal to minus alpha pi.

And we know that if you have a time delay then the phase lag associated with the time delay is going to be equal to minus omega t and if we are talking about the gain crossover frequency then we are talking about the phase lag at the frequency omega gc and that by definition by net phase lag by definition is going to be the phase lag of L NMP which is going to be equal to minus pi plus PM.

So, what this tells us therefore is that omega gc times T is given by pi times 1 minus alpha minus PM or in other words omega gc is going to be given by pi times 1 minus alpha minus PM divided by capital T. Now to determine the maximum possible value for the gain crossover frequency we once again note that, it would happen when alpha assumes its smallest value and the smallest value that alpha can assume in order for the gain to crossover is some value that is slightly greater than 0.

So, alpha has to be always greater than 0 for the magnitude characteristic to cross over. So, alpha minimum can be equal to 0 plus where the term 0 plus represents a small value above 0. So, if that is the value that our alpha assumes then we would for that case we would get the maximum possible gain crossover frequency and that is given by omega gc max equal to pi minus PM by capital T.

So, exactly as in the case of the non minimum phase 0 our time delay also imposes a fundamental upper limit on the achievable gain crossover frequency. Now we can show with very similar arguments as what we have done here that the phase crossover frequency for a non minimum phase system with time delay can be obtained by setting the angle of L non minimum phase to be equal to minus pi.

So, this going to be the angle of L non minimum phase that is going to be equal to minus omega pc times capital T plus the angle of L minimum phase if we assume that our loop gain continues down with the same rate namely minus 40 alpha decibels per decade between the frequencies omega gc and omega pc then you know the phase lag associated with L MP at omega pc is also going to be equal to minus alpha pi. Hence we would have that our omega pc would be given by pi times 1 minus alpha by capital T.

So, this is going to be the phase crossover frequency and this expression here is going to be the gain crossover frequency for a specified phase margin and if you have specified a certain gain margin. Then we know that gain margin GM by 40 alpha is going to be equal to log of omega pc by omega gc or in other words it is going to be equal to log of pi times 1 minus alpha by T divided by pi times 1 minus alpha minus PM divided by T. So, if you rewrite this equation we would have that GM by 40 alpha is going to be equal to log of pi times 1 minus alpha divided by pi times 1 minus alpha minus PM.

So, for a specified phase margin and a specified gain margin this equation tells us the kind of rate at which the loop gain has to reduce between the frequencies omega gc and omega pc for the overall open loop system to have the specified values of GM and PM. So, this equation once again is a transcendental equation and it has to be solved numerically, but if you do solve it numerically you would get a set of curves very similar to the one that I showed some time back in the context of non minimum phase 0s and what that once again indicates is that the maximum possible gain crossover frequency is generally going to be a small fraction of 1 by T where capital T represents the time delay of our non minimum phase loop gain.

So, just as in case of non minimum phase 0 the time delay also introduces fundamental upper limit to the achievable gain crossover frequency and this in turn has implications on the achievable performance at frequencies that are less than the gain crossover frequency. So, let us know spend the last part of the clip looking at the fundamental upper limits to the achievable magnitude of the loop gain at frequencies below the gain crossover frequency.

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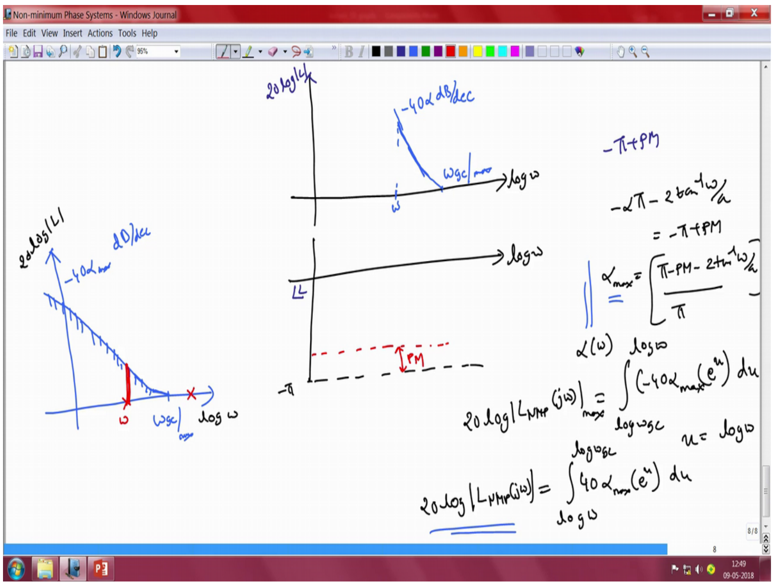

So, now let us return to the Bode plot of our non minimum phase loop gain and in this Bode plot I have indicated the maximum possible gain crossover frequency along the x axis along the frequency axis and in the interest of stability let us assume that the maximum permissible phase lag for the non minimum phase loop gain for frequencies below omega gc max is given by minus pi plus PM or in other words the phase lag should be such that the difference between the phase lag minus pi should always be greater than or equal to a specified phase margin PM for all frequencies below omega gc max.

Now the question that you are trying to answer is if you pick a particular frequency omega which is less than omega gc max what is the best possible magnitude for the loop gain that we can achieve at this frequency. Now the reason this question is important is because it is the magnitude of the loop gain that decides the performance of the close loop system in terms of its ability to reject disturbances or track references or achieve robustness to plant parameter variations. The higher is the possible gain that the loop gain at a particular frequency the better is our control objective met at that particular frequency.

So, the question that we are trying answer in this case is what is the best possible loop gain that we can achieve at some frequency omega, that is less than omega gc max. Now in order to get the highest possible loop gain at this particular frequency we need to note that our magnitude characteristic should have as large a slope as possible in magnitude between the frequencies of omega gc max and omega. So, every frequency between omega gc max and omega if our magnitude characteristic has the largest permissible slope in magnitude the slope is of course, negative in sign. But if the magnitude is as large as it is permissible between these two frequencies then the gain at the frequency omega will be the maximum that is possible.

So, if we have a certain magnitude characteristic with a certain slope we know that from Bode's gain phase relationship there a certain phase lag associated with it. Now this gain characteristic cannot be arbitrarily large in magnitude because if it is arbitrary large in magnitude then the phase lag associated with such a large negative slope will be so large that our phase margin specification will not be met at frequencies omega less than omega gc max.

So, if the slope at any particular frequency omega is given by minus 40 alpha decibels per decade then we know that from bodes gain phase relationship the minimum phase loop gain L MP will have a phase lag of minus alpha pi radians. And since we have a non minimum phase term that is cascaded with the minimum phase transfer function, if at this point in time assume that we have a non minimum phase 0 then the phase lag associated with that is going to be given by minus 2 tan inverse of omega by a. And the 2 together is going to be equal to minus pi plus PM because this is the maximum permissible phase lag at any particular frequency omega.

So, this tells us what the maximum possible value of alpha is at any particular frequency omega and that is given by alpha max is equal to pi minus PM minus 2 tan inverse omega by a divided by pi. So, this is the maximum possible value of alpha and that therefore, determines the maximum magnitude of the slope of the magnitude characteristic at in the neighbourhood of a frequency omega and we note from this equation that alpha is in general a function of omega.

Therefore, the maximum possible loop gain at some particular frequency omega is obtained by integrating the maximum permissible slope between the frequencies omega gc max and omega and hence is given by 20 log magnitude of L NMP at some frequency omega is this the maximum possible value is given by integral from log omega gc to log omega of minus 40 alpha max of omega. Since we are talking of the Bode plot we will convert omega to log of omega so, alpha max of e to the power u du where u is essentially equal to log of omega. So, this integral can be simplified a little bit by exchanging the limits of integration.

And written as is equal to log integral log omega to log omega gc of 40 alpha max of e to the power u du and this is going to be equal to 20 log of magnitude of L NMP at some frequency omega. Now if one word to graph this magnitude characteristic for the case of a non minimum phase 0 then the x axis would of course, p log omega the y axis will be 20 log magnitude of l. We note that at a frequency omega gc max we would have alpha to be equal to 0 because by definition at that frequency the gain will crossover with a slope that is very close to 0 decibels per decade.

So, the magnitude characteristic would start with a slope that is close to 0 and finally, increase and when omega is very small we note that alpha max is going to be given by pi minus PM by pi from this particular expression here.

And therefore, there will be a constant negative slope of minus 40 alpha max decibels per decade for frequencies omega there is much less than the corner frequency a of the non minimum phase 0. So, this is going to be the magnitude characteristic that we would obtain by solving this particular integral for different values of frequency omega.

Now what is indicates to us is that this is the best achievable loop gain at any particular frequency in order for the non minimum phase system to have a net phase lag of atmost minus pi plus PM radians over the entire frequency range from 0 to omega gc max and what is also implies therefore, is that if we have a disturbance at a frequency greater than omega gc max our non minimum phase system cannot reject this disturbance.

So, this is an important fact no matter how much money we are willing to spend on a fancy controller it is not possible to reject disturbances the a correct frequencies greater than omega gc max, likewise if we consider a frequency that is less than omega gc max where we have a certain loop gain that is greater than 0 dp, we note that the loop gain can only be at most this particular value if our loop gain has to be greater than this particular value then the phase lag of our loop gain in the frequency range between this particular frequency omega gc max has to be greater than minus pi plus PM and our stability specification will be compromised.

So, this is the best achievable magnitude for the loop gain at this particular frequency omega. So, suppose we have a disturbance in the neighbourhood of this particular frequency omega that needs to be rejected, what we notice that this is the magnitude of the loop gain L at this particular frequency decides the maximum extent by which this disturbance can be rejected. If as control engineers; we desire that the disturbance rejection needs to be even better than whatever is possible for this particular magnitude of the loop gain then what we have to settle for is that it is impossible to achieve the desired extents of disturbance rejection.

So, there are upper limits to what can be accomplished as control engineers in terms of disturbance rejection, robustness to plant parameter variation and to tracking of references and that upper limit is what is indicated by the curve that we obtain here and by solving this particular integral. So, our loop gain are limited to values below this curve and corresponding to this limit in the magnitude of the achievable loop gain at every frequency omega less than omega gc max our abilities as control engineers to reject disturbances, to track references and to achieve robustness to plant parameter variation we also be correspondingly limited.

So, once again I want underscore the fact that this problem is not something we have come across so, far in this course. This problem does not exist for the case of minimum phase plants because there does not exist such a thing as a maximum possible gain crossover frequency in the case of a minimum phase plant.

It is only when you have non minimum phase plants that we discover that they are upper limits to what we can have as the gain cross over frequency and associated with the upper limit if you want a certain stability specification in terms of the phase margin then this is the upper limit to the magnitude characteristic at each frequency less than the maximum permissible gain crossover frequency.

Thank you.