Control System Design

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Lecture - 41
Fundamental properties of the loop gain phase (part2/2)

The next important constraint that we would talk about is the relationship between the real and imaginary parts of the loop gain or equivalently the relationship between the magnitude and the phase plots of the open loop gain l in a Bode a plot. Now, the question that we want to ask is whether it is possible to independently design the magnitude characteristic and a phase characteristic of the loop gain or equivalently is it possible to independently design the imaginary part and the real part of the loop gain. Well, this question is of obvious attraction interest as a engineers because, in terms of performance what we need to pay attention to is a magnitude characteristic.

We need to make sure the magnitude is high enough in the frequency ranges where we want to achieve robustness or disturbance rejection or tracking of a certain reference or something like that. So, if you could independently engineer the magnitude characteristic and the phase characteristic, then we can first design the magnitude characteristic to satisfy whatever performance specification have been given to us. And, subsequently look at the phase characteristic, determine the phase margin and if the phase margin is not adequate, then separately shape the phase characteristic in order to achieve whatever phase margin we desire.

So, therefore, the problem of stability can be greatly minimized by adopting the systematic procedure for control design, provided we can independently realize the real and imaginary parts of the loop gain or equivalently the magnitude and phase of the loop gain. But, what we will see in this discussion is that unfortunately this is not possible. The magnitude and phase characteristics are intimately link and if you for instance specify the real part of the loop gain over the entire frequency range, the imaginary part of the loop gain also gets fixed over the entire frequency range and vice versa. So, let us see how that can be arrived at.

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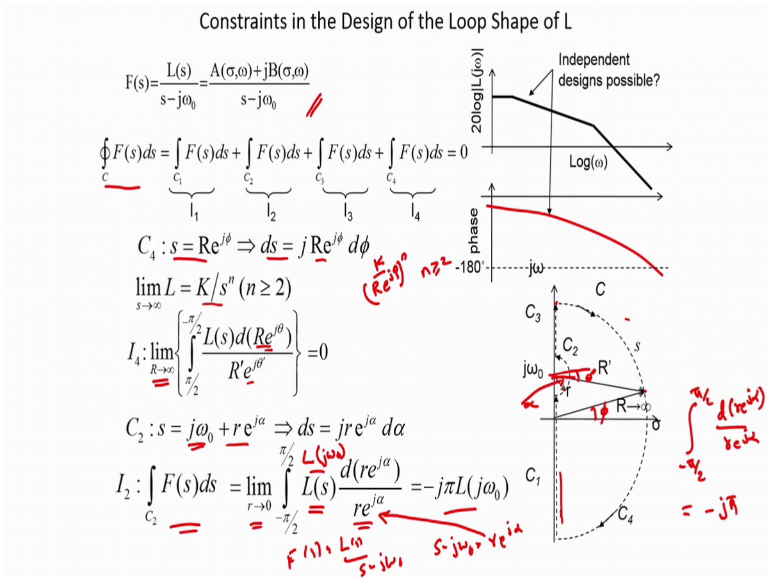

To do that we would be applying Cauchy's theorem to a function F of s that has been given here, F of s is equal to L of s by s minus j omega naught, where L of s is our loop gain and this is form A plus jB. In order to apply Cauchy's theorem to this particular function, we need to make sure that is function does not have singularities within the region where we applying the theorem. Now, if you want to adopt the same D shaped contour that we took in the previous clip, then we would have a problem. Because, the point s is equal to j omega naught would be a point on the D shaped contour and hence it would become difficult for us apply Cauchy's theorem.

In order to avoid this uncomfortable situation, what is done is we shall distort the D shaped contour that we saw in the that you have used in the past. In order to avoid the point s is equals to j omega naught, which is the point at which the singularity of this function F of s is located. So, to do that we chose a curve that is shown by this doted contour here.

So, most of it is essential the D shaped contour that we have always considered when we talked about stability and when we derive the relationship for conservation of sensitivity dirt and so on. But, there is a small kink that is introduced centered at the point j omega naught and of radius small r.

So, this small semi circular arc just avoids the point s is equals to j omega naught and hence excludes it from within the boundary of the contour C. So, the rest of the contour

C is a region where the function F of s does not have any singularities by assumption because, the loop gain L is assumed to be a minimum phase loop gain and all it is poles and 0 are assumed to be located on the left half of the complex plane. So, to apply Cauchy's theorem we shall know divide up this contour into 4 segments. The first is the segment C 1 which is essentially the imaginary axis, all the way from minus infinity to a point that is very close to the point s is equals to j omega naught.

The second is the contour C 2 which is a tiny semicircular arc of the radius small r which just avoids this point s is equals to j omega naught. The third is segments C 3 which extends from s is equals to j omega naught, all the way to plus infinity. And, the fourth is the familiar D shape contour C 4, whose radius capital R is tending to infinity and help us to encompass the entire of the right half of the complex plane. So, if you were to apply Cauchy's theorem to this particular contour, we would have the contour integral over C of F of s would be equal to 0, which would mean that the some of the count integrals along C 1 C 2 C 3 and C 4 would be equal to 0.

Now, we shall evaluate each of these integral separately and finally, simplify the resulting algebraic expression to ultimately arrive at a relationship between the real and imaginary parts of the loop gain. Since, a lot of algebra is involved to here I am stating at the very outset that the goal of all the algebra simplifications and solving of integrals that we would do in subsequent one or two slides, would be to ultimately obtain the relationship between the real and imaginary parts of the loop gain. So, having stated that, let us now plunge into solving these integrals and undertaking necessary algebra for the same.

Let us first call is integrals I 1 I 2 I 3 and I 4 with I 1 corresponding to the integral along the curve C 1 I 2 along the curve C 2 and so on and so forth. And, let us first try to solve the integral I 4. The integral I 4 is along the curve C 4 and along the curve C 4 we have that our complex number s would be of the form s is equal to Re power j phi, where R capital R is the radius of this big D shaped contour which is tending to infinity. And phi is the angle that a complex number on this D shaped contour makes with a real axis and I am indicating here the angle phi.

So, for all points on the contour C 4 we would have the complex number s to be of the form Re power j phi and if we negotiate this contour in the clockwise sense, as we have done in this schematic on the right. Then the complex number s starts on the positive imaginary axis and traverses along this D shaped contour in semicircular arc and finally, ends in the negative imaginary axis. Hence, we would have that angel phi going from plus pi by 2 to minus pi by 2. Since, s is equals to Re power j phi we would have that ds will be equal to j times Re power j phi d phi.

So, here we have the radius R to be a constant and hence you would get this as the expression for ds. The second fact that we can exploit while solving the integral I 4 is to note that since the loop gain is the product of the controller and the plant transfer functions. And each of the transfer function the controller and the plant are both physical system, we need to have the relative degree of each of these transfer functions to be at least equal to 1.

Hence, in the limit that s tends to infinity or equivalently as capital R is tending to infinity, we can easily show that the loop gain L in the limit s tends to infinity would assume the form K by s power n, where the index n is greater than or equal to 2. We would have this index to be greater than or equal to 2 because, L is a product of the controller and the plant. And, each of them has to tend to 0 as K by s or K by s power alpha, where alpha is greater than 1. And, hence the product of the 2 tends to 0 as K by s power n, where n is greater than or equal to 2.

So, if we exploit this fact along with the expression for ds along the contour C 4, we would have the integral I 4 to be limit R tends to infinity integral pi by 2 to minus pi by 2 L of s ds. So, for ds I have substituted d of R e power j phi divided by s minus j omega naught. In the s minus j omega naught represents R prime e power j theta prime, where R prime is the phase R starting from s is equal to j omega naught at ending at the point s and theta prime is the angle that is phase R makes with respect to the real axis. So, this angle here is theta prime.

Now, once again since we have the loop gain L on this contour C 4 to be of the form K by s power n or another words K by R e power j phi to the power n, where n is greater than or equal to 2. Following the exact same argument that we had in the previous clip, we can show easily that in the limit R tends to infinity we would have the integral I 4 going to 0. So, of the 4 integrals that we have to simplify, one of them has a fairly simple expression namely that I 4 is equals to 0, for the particular function F of s that we have chosen. Let us now focus on the integral I 2 and for the integral I 2, you would be computing this integral along the curve C 2 and along the curve C 2, C 2 is essentially this tiny semicircular arc centered at the point s is equals to j omega naught end of radius small r, where the small r is vanishingly small it is tended to 0.

So, any complex number s along this contour C 2 would have the form s is equal to j omega naught plus r e power j alpha, where alpha is the angle made by a complex number s on this contour C 2 with respect to the real axis. So, I am marking the angle alpha in this graph here.

So, as you see if we traverse the entire contour C in the clock vice sense, then the angle alpha changes from minus pi by 2 to pi by 2 as the complex number moves along the contour C 2. So, since s is of this form and omega naught is a constant, we would have ds to be equal to j times small r e power j alpha d alpha. So, once again small r is also a constant so, we do not need to differentiate the variable small r.

So, the integral I 2 therefore, becomes I 2 is equals to integral over C 2 F of s ds and that is equal to limit small r tended to 0 integral from minus pi by 2 to pi by 2 L of s d of r e power j alpha divided by r e power j alpha. So, we get d of r e j alpha because, ds essentially will become equal to d of r e power j alpha. Since, j omega naught is a constant and in the denominator we get r e power j alpha because, in the denominator of the function F of s, we have F of s to be equal to L of s divided by s minus j omega naught. And, if you notice along the curves C 2 we have s to be equal to j omega naught plus r e power j alpha which implies that s minus j omega naught will essentially simply be equal to re power j alpha.

So, that is the reason the because, we have the term s minus omega naught in the denominator of the integrant, we will have the term r e power j alpha appearing in the denominator of the integral. Now, since this contour C 2 is of vanishingly small radius and is situated very close to the point s is equal to j omega naught, for all practical purposes the loop gain L of s does not change within this contour C 2. And therefore, L of s is can be assumed to be a constant along the contour C 2 equal to L of j omega naught.

So, this term L of s can be replaced by L of j omega naught simply because, the radius r is tended to 0. So, we are only considering the point that are extremely close to point s is equals to j omega naught and for all those points the loop gain L of s will be extremely close to L of j omega naught. So, we can replace L of s by L of j omega naught. So, if you do that the L of j omega naught is a constant, and it can be removed out of the integral and all we would be left to evaluate is the integral minus pi by 2 to pi by 2 of d of r e power j alpha divided by r e power j alpha.

And, this can be shown to be equal to minus j times pi. Hence, when we combine that with the term L of j omega naught we would get the expression of I 2 to be equal to minus j times pi times L of j omega naught. So, we are now done solving I 4 founded to be equal to equal to 0. We just solved I 2 and found it to be equal to minus j pi of L of j omega naught. What we are left with are integrals I 1 and I 3, let us take these two integrals up next.

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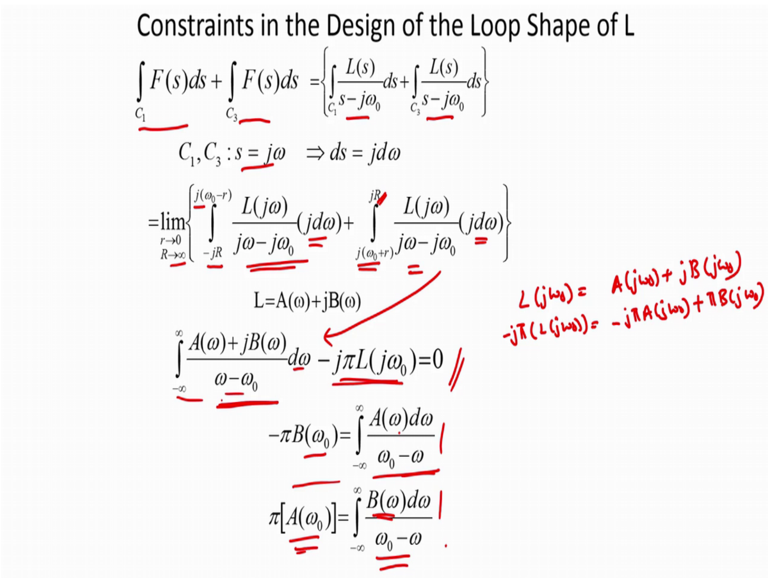

So, I 1 plus I 3 which is what we need to evaluate is essentially, integral along the contour C 1 of F of s plus integral along the contour C 3 of F of s and that essentially given by this expression, because F of s is L of s by s minus j omega naught. Now, if you notice in the previous slide the curve C 1 and C 3 are essentially coincident with the imaginary axis of our complex plane. Therefore, along these curves we would have s to be of the form s is equal to j omega.

So, making that substitution so, that ds internal equal to j times d omega and the curve C 1 starts at minus j infinity and ends at a point it is just below the point s is equal j omega naught. And hence, we would have the limits of the first integral namely the integral over C 1 of L of s by s minus j omega naught ds. To start from minus j times capital R, the capital R is tended to infinity and stop at j times omega naught minus small r which is the extent to which the curve C 1 extends. So, the integrand will be L of j omega by j omega minus j omega naught because, along this curve s is going to be equal to j omega and we have replace ds by jd omega.

So, this first integral within the bracket essentially represents integral over C 1 of F of s ds. The second integral represents the integral over the curve C 3, the curve C 3 starts at point along the imaginary axis, just a little bit above the point s is equals to j omega naught. In fact, the distance of the point at which it starts from the point s is equal to j omega naught will be small r.

So, the point at which it is start will be j times omega naught plus r that is the lower limit of the integral, when we are trying to do the integration along the curve C 3. And it extends all the way to plus j infinity or in other words plus j capital R, where capital R is tending to infinity and the integrand once again is a same L of j omega by j omega minus j omega naught and ds once again gets replaced by jd omega.

So, this is the sum of the integrals I 1 and I 3. Now, we know that L is of the form A plus j B and if we substitute that in this expression we would have that this entire integral, the sum of I 1 and I 3 can be written out as integral from minus infinity to infinity A of omega plus j times B of omega by omega minus omega naught d omega. That is because, the first integrals start with minus j infinity or minus j capital R or R is tended to infinity, stops just a little bit before j omega naught. And the second integral start just a little bit above j omega naught and go all the way to plus j infinity or plus j capital R, where R is tended to infinity.

So, if you combine these two since the distance at which the first integral ends from the point s is equal to j omega naught and where the second integral begins from the points s is equal to j omega naught is small r, which is tended to 0. We can combine these two integrals and write out the integral that we have shown here. The limits will go from minus infinity to plus infinity A of omega plus j B of omega divided by omega minus omega naught d omega. This is essentially simplified version of the integral that we have written out here.

And, from the previous slide we know that the other integral namely the integral I 2 will be equal to minus j pi times L of j omega naught and that is going to be equal to 0. Now, we know that L of j omega naught will essentially be equal to A of j omega naught plus j times B of j omega naught.

So, the integral here which I have shown which I have underlining now, has a real part and an imaginary part and the expression minus j pi L of j omega naught also has a real part and an imaginary part. So, if you compute minus j pi times L of j omega naught, we would have here to be equal to minus j pi A of j omega naught plus pi B of j omega naught.

Hence, if we equate the real part of the integral plus the real part of this term minus j pi L of j omega naught to be equal to 0 and imaginary parts correspondingly to be equal to 0, we end up with these two expressions here. So, we would have minus pi times B of omega naught is equal to integral minus infinity to infinity A of omega d omega by omega naught minus omega.

While, pi times A of omega naught will be equal to integral minus infinity to infinity B of omega d omega by omega naught minus omega. So, we got these two expressions by separately equating the real part and imaginary part of the equation that we have here to 0.

Now, this equation tells an important story, what is B of omega naught? B of omega naught essentially represents the imaginary part of the complex number L of j omega at the frequency omega naught. What is A omega? A omega represents the real part of the loop gain and let us say that has been specified over the entire frequency range, namely omega going from 0 to infinity then, A of omega is known very well.

So, we see that if you specified A of omega we are automatically therefore, specifying the numerical value of B of omega naught via this first equation. Conversely, if we are specifying the imaginary part of the loop gain which is B of omega over the entire frequency range, omega going from 0 to infinity.

Then the second equation revels that this specified B of omega over the entire frequency range can be used in combination with this particular integral on the right hand side to compute A of omega naught or other words the real part of the loop gain at any particular frequency omega naught. The trouble with the integrals on the right hand side; however, is that the frequency where is from minus infinity to infinity rather than from 0 to infinity. So, we can simplified this expression by taking one additional step through which we can replace a limits of the integral from minus infinity to infinity by 0 to infinity. So, let us undertaking this next step for a sake of this last step of simplification.

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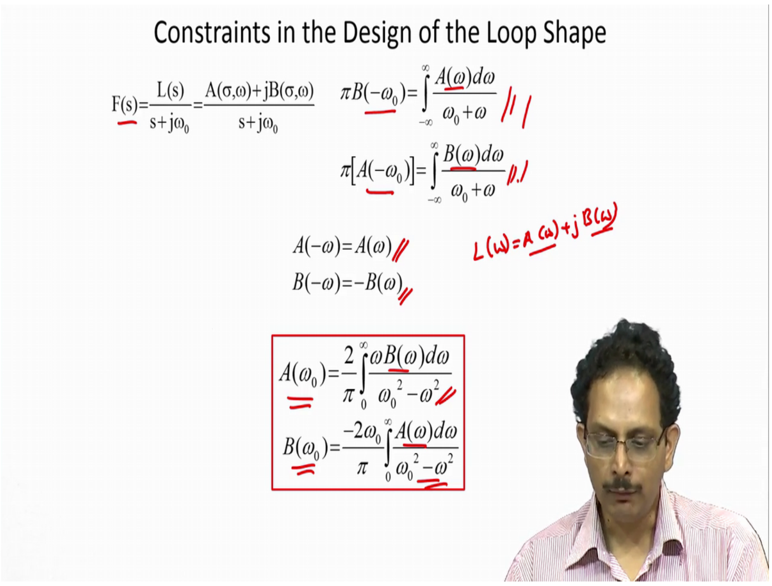

To do this we apply the exacts same step that we have undertaken so far to the function F of s is equal to L of s by s plus j omega naught, which is basically A plus j B by s plus j omega naught. Please recall that two slides back we choose to apply Cauchy’s theorem to the function F of s is equals to L by s minus j omega naught, here we are choosing to apply to L is F is equal to L by s plus j omega naught.

Now, in order to apply Cauchy’s theorem to this particular function we have to introduce a tiny semicircular kink to the contour C at the location s is equal to minus j omega naught, in order to avoid the singularity of this function at the particular point. Otherwise, if you follow the exact same step as we did in the previous two slides, then we would get a very similar expression that relates B of minus omega naught to A of omega and A of minus omega naught to B of omega.

So, for negative frequencies omega the real part and imaginary parts of the loop gain are related as given by these two equations. Now, there is one last step that allow us to replace the limits of these integrals in a manner that we have to plan to do; namely to get the lower limit of both this integrals is to be 0 rather than minus infinity. To do that, we notice that since the loop gain represents the transfer function of some physical system. We would have the real part of the loop gain be an even function of frequency or in other words you would have A of minus omega to be equal to A of omega. And the imaginary part of the loop gain would be an odd function of frequency or in other words B of minus omega would be equal to minus B of omega.

So, by exploiting the fact that if L of s, if L of omega is equal to A of omega plus j times B of omega, we would have A of omega as a even function of frequency and B of omega as odd function of frequency. We can use the expression we have in this slide along with expression that we had in the previous slide combine them, take the linear combination suitably and show that A of omega naught can be written out as 2 by pi times integral 0 to infinity omega times B of omega d omega by omega naught square minus omega square. And B of omega naught is equal to minus 2 omega naught by pi times integral 0 to infinity A of omega d omega by omega naught square minus omega square.

So, this expression is directly the result of using the two equations that we have in this slide along with two similar equation that we had in the previous slide. Combining them appropriately the exact steps that allow us to get to the final expression from these two expressions has been given in the lecture notes. But, what is important is what this two equations tell us. What they tell us is that there is absolutely no freedom available to us in picking the imaginary part of a loop gain, which is essentially going to be B of omega naught. If we have specified its real part namely, A of omega over entire frequency range.

So, A of omega is specified everywhere, then all we can do is compute this integral that we have on the right hand side to get B of omega naught. Likewise, if we have specified the imaginary part of the complex number namely, B of omega over the entire frequency range then we have no flexibility what so ever, left in determining the real part of this complex number at any particular frequency omega naught.

So, determine its real part we just have to compute the first integral that is shown in this equation. So, we would get A of omega naught to simply be equal to 2 by pi times integral from 0 to infinity omega times B by omega naught square minus omega square d omega and that gives us A of omega naught. So, this is a real big constraint on us control engineers because, it tells us that we cannot independently determine the real and imaginary parts of a loop gain or equivalently the magnitude and the phase of the loop gain.

So, if we specify one over the entire frequency range thus, other gets automatically specified every single frequency omega naught and that is given by the expressions that is shown here. But, the relationship between the magnitude and the phase is far more intimate then what is apparent from these two equations.

The intimacy of the relationship between the magnitude and a phase was revealed by H. W. Bode who took this expressions further, made some simplifications reorganization to reveal how closely the phase characteristics get determined by the magnitude characteristics in the neighborhood of a frequency omega naught. So, what we shall do next is follow along H. W. Bode's footsteps and starting from these two expressions, we shall see how the magnitude and the phase characteristics in a Bode plot are related.

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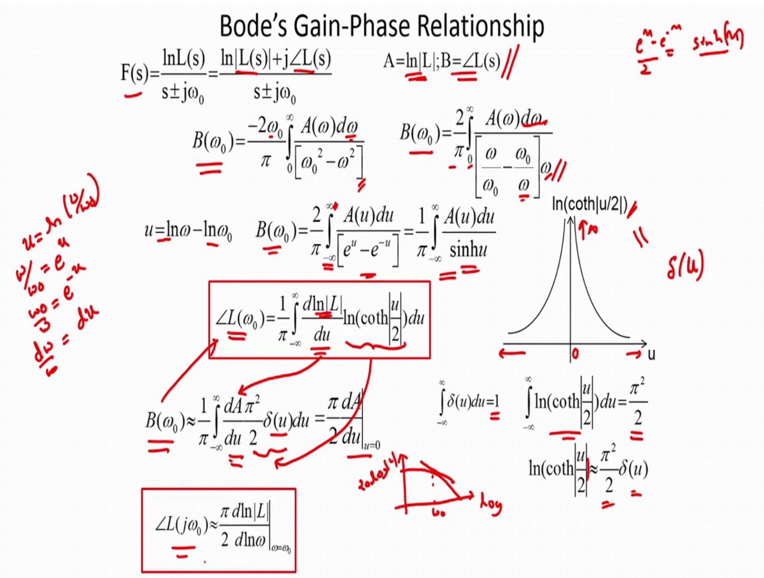

To do that let us first chose to apply Cauchy's theorem, this time to the function F of s is equal to ln of L of s divided by s plus or minus j omega naught or in other words ln since ln L of s given by ln of magnitude of L of s plus j times angle of L of s. You would have F of s to be equal to ln of magnitude of L plus j times angle of L divided by s plus or minus j omega naught. So, we first apply to the case when we have s plus j omega naught in the denominator and subsequently to the case where we have s minus j omega not in the denominator.

And, exactly as we did in the previous slides and then add the two, combine the two and we would get the expression B of omega naught, where B is the angle of L and A is the ln of magnitude of L to be equal to minus 2 omega naught by pi integral 0 to infinity A of omega d omega by omega naught square by minus omega square. So, this expression we borrowed from the previous slide, but we are applying it to the function F of s equal to A plus j B. But, this time A does not represent the real part of L and B does not represent the imaginary part of L.

Instead, A represents ln of the magnitude of L and B represents the phase of L or the angle of L that is because, we have chosen our starting point to be that we would apply Cauchy's theorem to the function F of s is equal to ln of L divided by s plus or minus j omega naught instead of directly L divided by s plus or minus j omega naught. Now, that we have this particular expression let us rearrange it a little bit.

So, the term omega naught in the numerator was taken to the denominator so, that and the term minus 1 also observed by the integrant. So, that we would have B of omega naught to be equal to 2 by pi times integral 0 to infinity A of omega d omega which is same as what we had in the previous expression divided by omega by omega naught minus omega naught by omega times omega.

So, by simple manipulation and rearrangement you can quickly see that the expression that we have here on the right side is identical in every sense to expression that we have decided on the left hand side. It is just a rearrangement and this arrangement has been made to help us take the next step, in the next step what you would do is define a new variable u.

So, we have called u as ln omega minus ln omega naught and we shall right out this integral in terms of u. So, the movement you define u to be equal to ln omega minus ln omega naught, you notice that u is equal to ln of omega by omega naught or in other words omega by omega naught will be equal to e to the power u. Likewise omega naught by omega will be equal e to the power minus u.

Finally, we would also have d omega by omega to be equal to du since, ln of omega naught is a constant and hence, d of ln of omega naught will be equal to 0. So, by noting that the term d omega by omega in the right most integral here is equal to du and omega by omega naught is equal to e power u. And omega naught by omega is equal to e power minus u, we can right that d of omega naught is equal to 2 by pi integral minus infinity to infinity A of u du by e power u minus e power minus u. Kindly, note that the limits of integration have also changed as a consequence of the change of variables.

Earlier we have omega and omega was going from 0 to infinity, now when we right out the same integral in terms of u which is ln of omega. We would ln of 0 to be minus infinity and hence the lower limit of this integral becomes minus infinity. And, ln of plus infinity still remains infinity, hence the upper limit still remains plus infinity.

We would therefore, have B of omega naught to be equal to 2 by pi integral minus infinity to infinity A of u du by e power u minus e power minus u. Now, we notice that e power u minus e power minus u divided by 2 is essentially equal to sin hyperbolic of u.

So, by noting this particular definition of sin hyperbolic of a variable we can write out the same integral here, as 1 by pi times integral minus infinity to infinity A of u du divided by sin hyperbolic of u. Next, what we do is under take integration by parts to simplify this particular integral. This as not been done by me in this particular slide, but it is in the lecture notes and you can refer to it at some later point. What we can show is that it is integral here, the right most integral that are currently under lining can essentially be simplified by using integration by parts to this particular expression here.

Namely, that B of omega naught which is essentially equal to angle of L is equal to 1 by pi times d A by du, where A is ln of magnitude of L noise] times ln of cotangent hyperbolic of magnitude of u by 2 du. So, this is the relationship between the angle of the loop gain and the magnitude of the loop gain. So, the angle of the loop gain is equal to 1 by pi times its integral of some slope of the loop gain namely d ln of magnitude of L divided by du, waited by this function ln of cotangent hyperbolic of magnitude of u by 2 du.

Now, if you examine this relationship it is not easy to tell exactly what it is about this so called simplification that sheds any further light on the relationship between the magnitude and the phase characteristics of the Bode plot or the angle and the magnitude of the loop gain in a Bode plot. That is because; we do not know exactly how this particular function ln of cotangent hyperbolic magnitude of u by 2 looks like the moment we plot this function thing will start to become immediately a bit more clear. So, if one word to plot this function ln of cotangent hyperbolic magnitude of u by 2 it looks something like this.

So, as u tends to minus infinity it tends to 0 as u tends to minus infinity once again it tends to 0 and at u equal to 0 the function tends to infinity. So, this function is of infinite magnitude in the neighborhood of u equal to 0 and it tends to 0 as you move further away and as u is tended to infinity. Now, we might wonder what function does this particular function resemble?

A moments thought might lead us to suspect that perhaps this function ln of cotangent hyperbolic of magnitude of u by 2 looks somewhat like a delta function, namely delta of u. Because, if I want to draw a delta function, a delta function also has infinite magnitude in the neighborhood of u equal to 0 and quickly goes to 0 outside of u equal to 0. So, you would imagine therefore, that perhaps ln cotangent hyperbolic u by 2 is actually a delta function, but the bad news is this function is not exactly a delta function.

So, a delta functions definition was provided to a provided in the very beginning of this course and if you apply the definition we will discover that this particular function not really a delta function. But, it can be approximated as a delta function simply because, it weighs the integrand the namely d of ln of magnitude of L by du; significantly more in the neighborhood of u equal to 0 compare to other location or other location of u.

There is a second problem in equating or approximating ln of cotangent hyperbolic of magnitude of u by 2 with a delta function. That is because, when we compute the area under the curve for a delta function or when we compute integral minus infinity to infinity delta of u du by definition we should get that area to be equal to 1.

However, when we compute the integral ln of cotangent hyperbolic of magnitude of u by 2 du, we do not get that area to be equal to 1; instead you get it to be equal to pi square by 2. Hence, this term ln cotangent hyperbolic u by 2 is not only not a delta function, but also that it is the area under the curve is not 1 it is actually equal to pi square by 2. However, in order to simplify analysis and given the fact that it raise the integrant significantly more around u equal to 0, we can approximate ln cotangent hyperbolic u by 2 as pi square by 2 times delta of u approximately.

So, that the area under the curve for the term on the right hand side namely, pi square by 2 times delta of u will also be equal to pi square by 2. And this term will end up varying the integrant significantly more near u equal to 0. So, by approximating this function ln of cotangent hyperbolic magnitude of u by 2, as approximately being equal to pi square by 2 times delta of u, we would have the relationship between the phase of L and the magnitude of L to approximately be given by B of omega naught is approximately equal to 1 by pi d A by du pi square by 2 delta of u du.

Where, I have replaced ln of cotangent hyperbolic u by 2 with pi square by 2 times delta of u. And B of omega naught essentially represents the angle L as based on the definition that we have laid out at the beginning of the slide here. And, d of ln of magnitude of L by du is essentially d A by du because, by definition A is ln of magnitude of L. Hence, d A by du essentially represents this term here.

So, the term L of omega naught has been reproduced in this equation, the term d L d of ln of magnitude L by du has been reproduced again here. Just that we have renamed the same variable as A and the term ln of cotangent hyperbolic magnitude of u by 2 has been approximated has pi square by 2 times delta of u.

So, when we simplify this particular integral by noting that we have a delta function within the integral and we remove the term pi square by 2 out of the integration. We would get that d of omega naught is going to be equal to pi by 2 times d A by du evaluated at u equal to 0. This we get because, of the presence of the term delta of u inside the integration.

Now, if you once again substitute what A the variable capital A stands for and what the variable capital B stands for, we would get this very interesting expression. Namely, that the angle of j omega naught which is essentially B of omega naught is approximately equal to pi by 2 times d of dA by du essentially represents d of ln of magnitude of L.

Because, by definition A is equal to ln of magnitude of L by du and u represents ln of omega. Hence, we would have the angle of L at j omega naught to be approximately equal to pi by 2 times d of ln of magnitude of L by df ln of omega at the frequency omega equal to omega naught. Now, since we have drawn the Bode plots for quite some time in this lecture series and perhaps also before this, we immediately recognize that in a Bode plot the y axis is proportional to ln of magnitude of L.

It is actually equal 20 times log to the base 10 of magnitude of L which is proportional to ln of magnitude of L and the x axis is essentially log to the base 10 of omega which is essentially again proportional to ln of omega. So, what this expression tells us therefore is that the angle of L at any particular frequency omega naught is simply equal to approximately the slope of the magnitude characteristic in a Bode a plot at the frequency omega naught.

So, if we have the Bode a plot where we plot log of omega versus 20 log of magnitude of L t, let us say have certain Bode a plot for our open loop gain. At any particular frequency omega naught, if we compute the slope of this curve then the slope of that curve is proportional to the phase of the loop gain at that particular frequency. And what is in tar indicates to us is that it is not necessary for us to separately draw the magnitude and the phase characteristics of the loop gain.

If you are given the magnitude characteristics you can automatically derive the phase characteristics by using this approximate expression and that is the power of this relationship between the magnitude and phase that has been derived by H. W. Bode. So, is this really true? To check, if this is really true, we can apply this expression to a few common transfer functions that we have come across. And we tend to come across in practice and see whether this relationship between the magnitude and phase are really correct or not.

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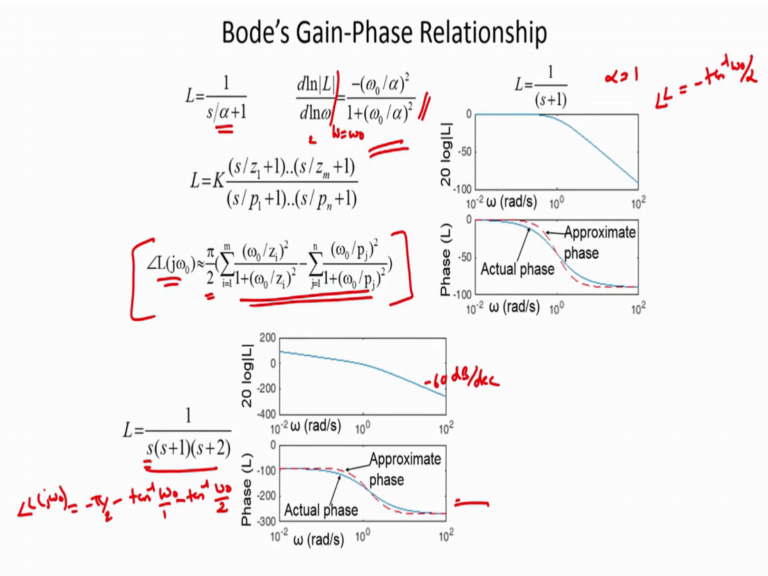

So, I have chosen to first apply it to the loop gain L equal to 1 by s pi alpha plus 1. This is a first order system and for this system we can compute the magnitude of L at any frequency omega. And then take it is logarithm and take the derivative of log of magnitude of L with respect to log of omega and show that after some algebraic simplification, it becomes equal to minus omega naught by alpha the whole square divided by 1 plus omega naught by alpha the whole square.

Where, omega naught is the frequency at which we are trying to evaluate the derivative d of ln of magnitude of L by d of ln of omega. So, this is evaluated at omega equal to omega naught. Hence, if Bode is gain phase relationship between true, we would have that the angle of L would be equal to pi by 2 times minus omega naught by alpha the square divided by 1 plus omega naught by alpha the whole square.

So, what I have done on the right hand side is plotted the magnitude characteristics of the loop gain for some particular alpha, in this case I have chosen alpha to be equal to 1. And I have also plotted the actual phase of this loop gain which is quite easy to derive because, the angle of L actually is given by minus of tan inverse of omega naught by alpha at any particular frequency omega naught.

So, the actual phase has been shown in blue colour solid curve here and the approximate phase has been shown by the red dashed curve here, which is obtained by using the approximate gain phase relationship that we derived in the previous slide. Namely, that the angle of L is equal to pi by 2 times approximately equal to pi by 2 times the derivative of ln of magnitude of L with respect to ln of omega.

And, what you see from the graph on the right hand side is that the two are quite close and the maximum difference between the two which you can compute is going to be about 6 degrees, it is not a bad number. And as control designers we can live with the small difference between the actual phase and the approximate phase.

And the benefit you would get from that is that we can just rely on the magnitude characteristics in order to perform our control design. Because, by taking the derivative of the magnitude characteristics we can estimate the approximate phase characteristics of the loop gain; provided this loop gain is the minimum phase loop gain.

And from that work out the other stability specifications such as the phase margin and so on and so forth. Now, if you take a more general loop gain namely, L is equal to K times s by z 1 plus 1 times s by z 2 plus 1 and so on and so forth up to s by z m plus 1. Let us assume we have m 0's z 1 to z m and n poles for our open loop system p 1 to p n. So, that the denominator polynomial can be factorized as s by p 1 plus 1 times s by p 2 plus 1 and so on and so forth up to s by p n plus 1.

We can show with some algebraic manipulation that d of ln of magnitude of L by d of ln of omega is given by this particular expression within the brackets here. Namely, summation i going from 1 to m omega naught pi z i the whole square divided by 1 plus omega naught by z i the whole square minus summation j going from 1 to n omega naught by p j the whole square divided by 1 plus omega naught by p j the whole square.

So, this entire some multiplied with pi by 2 will approximately give us the phase of L at any particular frequency omega naught. To verify whether this is correct or not I have taken once again a specific numerical example L of s is equal to 1 by s times s plus 1 times s plus 2. And we the top graph plots some magnitude characteristics of this loop gain.

Since, we have an integrator initially we have a minus 20 dB per decade roll off, around 1 radian per second which is its first corner frequency the roll of will increase to minus 40 dB per decade. And at the second corner frequency which is at 2 radian per second there will be an additional roll off introduced by the second by the pole at s equals to minus 2.

And so, the ultimate roll off will be minus 60 dB per decade and for this particular magnitude characteristics, we can compute the phase exact phase characteristics because; we know the three terms that make up the loop gain. So, the exact phase any particular frequency omega naught is given by angle of L of j omega naught is equal to minus pi by 2, which arises from the integrator 1 by s that we have.

And then minus tan inverse of omega naught by 1 because, of the pole at s is equal to minus 1 minus tan inverse of omega naught by 2 because of the other pole at s equal to minus 2. So, this is going to be the exact phase. The approximate phase can be computed by using this expression, which I am now highlighting between the brackets.

And the second graph here compares the approximate phase and actual phase. You see that actual phase is shown in blue solid curve and the approximate phase is shown by once again the red dashed curve. And it is evident from this that the two are very nearly identical to one and other. It is only at some particular frequency is there is a difference, but, if you zoom in and look at the difference it is going to be quite small; on the order of a few degrees or at most 10 degrees, 15 degrees and this difference is small enough for us control engineers to live with.

So, the importance of the second half of our discussion in this clip is that not only is the phase characteristic completely determined if we are given the magnitude characteristic of the open loop gain. But, in fact the phase characteristic is determined predominantly by the slope of the magnitude characteristic in a Bode plot, in the neighborhood of the frequency at which we are determining the phase characteristic.

So, although the magnitude characteristic at other frequencies or in particular the slope of the magnitude characteristic at other frequencies also affect the phase at a particular frequency of omega naught, that contribution is rather small. What is dominant is the slope of the magnitude characteristic in a Bode plot or in other words so, slope of the ln of magnitude of L versus ln of a omega at a particular frequency omega naught, is what determines the phase characteristics at the same frequency omega naught.

Thank you.