Control System Design
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Lecture – 40
Fundamental properties of the loop gain (part 1/2)

Hello over the last few lectures we have seen how we can perform 2 degree of freedom control design using the Nichols plot and with that discussion we bring to an end the different tricks and techniques that we would be discussing in this course, regarding the tools available and the controllers that are available for control of physical systems. What we would be looking that starting from this lecture onwards would be some of the fundamental properties of feedback systems.

As we have occasionally pointed out mathematics is the language of control systems and the control systems that we have designed, therefore how to essentially how to necessarily obey some of the theorems in complex analysis and these theorems in turn impose limitations on what we can do as control engineers.

So, in this clip we would be looking at some of the important limitations that are imposed by these theorems in mathematics which the control systems that we are designing have to necessarily obey. So, we would be discussing 3 such constraints that are imposed on the feedback systems and in particular on the loop gain because, all the benefits and price of feedback is tied to the loop gain and the higher the loop gain the better is our performance in terms of disturbance rejection or robust plucking and so on and so forth.

So, the first theorem is what is known as the Bode sensitivity integral theorem and this theorem attempts to quantify the limitations of feedback control. So, we know that we cannot achieve the benefits of feedback namely very high gain over arbitrarily large frequencies, there will be a frequency at which the plants transfer function will start to reduce and when the controller transfer function also does not assume very large values in that frequency range, then the loop gain will inevitably cross over. So, the loop gain will reduce from a value greater than 0 dB to a value less than 0 dB and beyond that frequency we cannot expect any meaningful performance from our feedback control system.

So, the gain cause of a frequency essentially determines the bandwidth of the feedback control system beyond which we cannot expect any performance improvements that would accrue as a result of employing feedback control. But these same qualitative observations that there is only a finite frequency range within which we can expect benefits of feedback can actually be quantified, so there way we go about quantifying it is by employing Bode sensitivity integrals.

The second limitation that we would discuss has to do with the connection between the magnitude and the phase characteristics of the open loop system in a Bode plot. The question that we would try to ask and subsequently answer is whether it is possible for us to independently design the magnitude characteristic and the phase characteristic. Or equivalently can be independently realize the real parts and the imaginary part of the loop gain L of j omega.

So, let us say we have specified the real part of the loop gain over the entire frequency range omega, do we have any freedom left in specifying the imaginary part of the loop gain or equivalently if you have specified the magnitude characteristics of the loop gain over the entire frequency range is there any freedom left for us to specify the phase characteristics.

Now, if you think a little bit about this particular constraint it is quite important to us as control engineers, because let us say we can independently design the magnitude and the phase characteristics of the loop gain, then we can first design the magnitude characteristics in order for these characteristics to satisfy whatever performance specifications we might have.

So, you might want to achieve fairly high loop gains in some frequency ranges, so that the disturbance specifications or tracking specifications are met in those frequency ranges and having completed that design we can then turn our attention to the phase characteristics and subsequently shape the phase characteristics in such a manner that our final closed loop system would have whatever phase margin we would want it to have.

So, such a decoupled design approach would become possible, if it was actually possible for us to independently design the magnitude and the phase characteristics of the open loop system. But what we shall see in the second part of this clip is that unfortunately the magnitude and the phase characteristics are intimately related.

So, if we specify the magnitude characteristics of the loop gain or equivalently the real part of the loop gain over the entire frequency range, we are essentially also specifying the phase characteristic over the entire frequency range or equivalently the imaginary part of the loop gain over the entire frequency range. And we would have no flexibility left to pick these 2 independently.

So, this is a second fundamental constraint that we control engineers have to contend with and this therefore imposes limitations on what we can accomplish as control engineers. Although it attractive for us to be able to independently design these 2, so that having taken care of performance then we can separately worry about stability this particular theorem states that this is not possible for us to realize.

The third constraint is an extension of what we would have discussed in the second part of this clip, namely the relation between the magnitude and the phase characteristics. So, the second part will reveal that it is not possible for us to independently design the magnitude and the phase characteristics.

The third part which is due to H W Bode indicates that the relationship between the magnitude and the phase characteristics is even more intimate than what would be revealed by the second constraint. So, it is not necessary for us to specify the magnitude characteristic over the entire frequency range for us to determine the phase characteristic, actually it turns out that the phase characteristic is dependent predominantly on the magnitude characteristic in the neighborhood of the frequency at which we are determining the phase.

So, even though we may have a certain magnitude characteristic and other frequencies, it is the magnitude characteristic in the vicinity of the frequency where we are looking at the phase of the open loop system that predominantly determines the actual value of the phase and this contribution is due to H W Bode and the consequence of this contribution is that one can have a more simplified Bode plot wherein one does not even need to separately plot the phase characteristics of the plot. If one plots the magnitude characteristics then from the magnitude characteristics one can actually already extract the approximate phase characteristics without separately plotting it for the open loop system.

So, it is these 3 particular limitations that we would spend the remaining time that we have in this clip looking at, all of these 3 limitations arise from theorems in complex analysis and therefore it will only prudent for us to refresh our memory regarding some of the important definitions in complex analysis and subsequently we will be using one important theorem called Cauchy's theorem. And we will be applying it repeatedly for different functions to extract the different limitations that we talked about just now.

So, the first property that we would be employing in doing the analysis in this clip is what is known as analyticity of a complex function.

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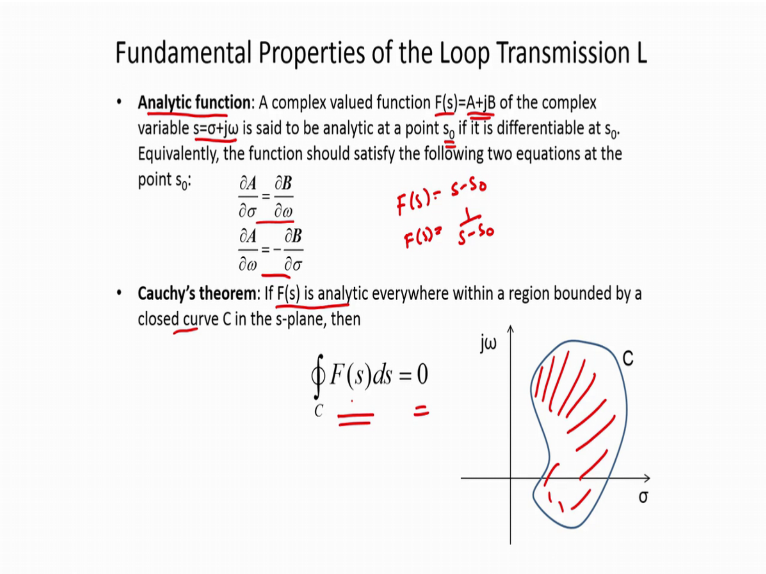

So, the definition has been indicated in this slide here, so an analytic function is a function F of s which is differentiable everywhere. So, if a function F of s is differentiable at a point s naught in the complex plane, then it is have said to be analytic at the point s naught. Equivalently if we write this function in the form F of s is equal to A plus jB, where the complex number s is given by s is equal to sigma plus j omega then for a function to be analytic at a particular point.

Then we should have these 2 equations to be simultaneously satisfied delta A by delta sigma should be equal to delta B by delta omega and delta A by delta omega should be equal to minus of delta B by delta sigma.

So, an analytic function is something that we have already seen when we discussed the proof for the Nyquist stability theory and it is essentially function that is infinitely differentiable and one simple example of an analytic function is F of s is equal to s minus s naught, such a function is analytic everywhere in the complex plane. Where as a function of the kind f of s is equal to 1 by s minus s naught is analytic everywhere in the complex plane except at the point s naught, simply because the function blows up at the point s naught it is value would be infinity and therefore it cannot be differentiated at the point s is equal to s naught.

So, the notion of an analytic function is quite a straightforward one and the next theorem that would be using repeatedly in the course of this clip is what is known as Cauchy's theorem. And the Cauchy's theorem is applicable to analytic functions and the statement has been indicated here. So, if a function F of s is analytic everywhere within a certain region bounded by a closed curve C and such a region has been shown here. So, if the function F of s is analytic everywhere inside this shaded area of the, that is bounded by the curve C.

Then the contour integral of F of s along the curve C which is given by the contour integral F of s ds is going to be equal to 0. So, this is Cauchy's theorem and we shall not prove Cauchy's theorem in this clip, but we shall just assume take it for granted and apply it to different functions of interest to us as control engineers and see what kind of limitations will this particular theorem impose on the loop gain or functions of the loop gain of the open loop system. So, in order to quantify the limitations of feedback we shall applying this theorem to the function F of s is equal to ln of capital S.

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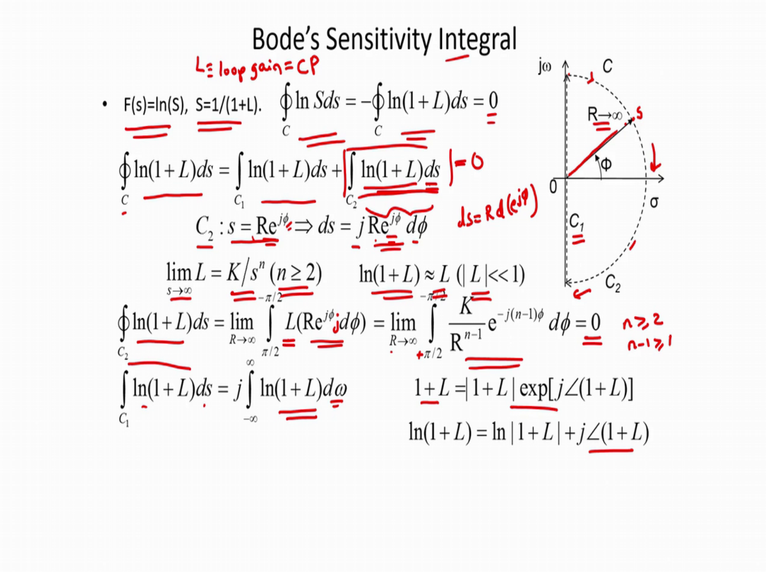

Where the function capital S is given by capital S is equal to 1 by 1 plus L, where L is the loop gain of the system. So, L corresponds to the loop gain and is given by L is equal to the product of the controller transfer function and the plant transfer function. So, we shall apply Cauchy's theorem to the function f of s is equal to ln of S where s is equal to 1 by 1 plus L, if you recollect the function capital S is also called the sensitivity function.

So, if our loop gain L has all it is poles and zeros on the left half of the complex plane, then there are no singularities in the function ln of capital S in the right half of the complex plane, because the singularities will occur only at the location of the poles and 0's of the term 1 by 1 plus L. So, the poles of the term 1 by 1 plus L are essentially the poles of the closed loop system and the 0 of 1 by 1 plus L are essentially the open loop poles of our system.

If our closed loop system is stable then all the closed loop poles will be on the left half of the complex plane and if our plant and the controller both have their poles on the left half of the complex plane, then the 0's of capital S a sensitivity function will also be on the left half of the complex plane. Therefore, there is no point on the right half of the complex plane at which the function F of s equal to ln of capital S is going to go to infinity is going to process singularities.

Hence if we define the curve C along which we would be in evaluating the contour integral to encompass the entire of the right half of the complex plane or in other words the curve C is essentially this D shaped contour whose flat side coincides with the imaginary axis and whose bulging D shaped contour has a radius R that is tended to infinity and hence encompasses the entire of the right half of the complex plane. If this is the contour of the region within which we are going to be applying Cauchy's theorem.

We note that in this region the function F of s is going to be analytic for the reasons that we discussed just a few minutes back and hence we would have that the contour integral of ln of S ds over the contour C which is essentially going to be equal to minus of the contour integral of ln of 1 plus L d s and that is going to be equal to 0. Because F of s equal to ln of s is an analytic function everywhere within the contour C that we have defined here.

Now, this is going to be the starting point for us to derive the fundamental limitation as far as feedback control is concerned and such a limitation is revealed by what is known as the Bode's sensitivity integral and that is what we would be getting to at the end of the derivation. So, starting from this point where we have applied Cauchy's theorem to this particular function to the point where we derive the Bodes sensitivity integral and it is implications and interpretation to us as control engineers.

There is quite a bit of algebra we shall patiently go through that algebra and I request the viewer's indulgence while we patiently simplify the expression that we have got here for the particular function that we have assumed, namely capital S sensitivity function equal to 1 by 1 plus L.

So, starting from this equation here namely that the contour integral of lawn of 1 plus L ds is equal to 0, we can then break up the contour C which is this entire D shaped contour into 2 smaller contours one, is the part C 1 which represents essentially the entire of the imaginary axis and the D shaped part C 2 which encompasses the entire of the complex plane on the right hand side.

So, the contour C can be looked at as a union of these 2 curves C 1 and C 2 and hence we can write the contour integral ln of 1 plus L ds over the curve C is equal to the integral over the curve C 1 of the same function ln of 1 plus L plus the integral over the curve of C 2 of the function ln of 1 plus here.

Now, we have to separately simplify the 2 integrals, namely the integral over the curve C 1 ln of 1 plus L ds plus and the integral along the curve C 2 ln of 1 plus L ds. First we shall take up the second integral namely integral over the curve C 2 of ln of 1 plus L ds, let us first evaluate that, subsequently let us get to the first integral namely integral along the curve C1 of the function ln of 1 plus L.

So, if you take the second integral you note that along the curve C 2 our complex number would be of the form s is equal to Re power j phi, where R is the radius of the D shaped contour and phi represents the orientation of the complex number s with respect to the origin on this D shaped contour. So, in along the curve C 2 our complex number s will be at a constant distance from the center of the complex plane given by the radius capital R, but it is angular position will change and that is captured by the change in the phase phi of the complex number s.

In particular if the d shaped contour starts somewhere near the positive imaginary axis and ends somewhere on the negative imaginary axis, we would note that the angle phi would change from a value of plus pi by 2 to minus pi by 2. Now since s is given by Re power j phi along the contour C 2 we would have that the term ds is going to be given by j times Re power j phi d phi. This is because the term R is a constant, so ds will be equal to R times d of e power j phi and one can easily verify that the differential of e power j phi is given by j times e power j phi times d phi. So, that is how we obtained the expression for the term ds in the second integral which we are now taking and considering for simplification purposes.

There are a couple of other simplifications that we can make for the function ln of 1 plus L on the contour C 2, since the radius of this contour is R capital R which is tendered to infinity and hence this contour encompasses the entire of the right half of the complex plane. We note that the values of s that we would have on this contour are going to be significantly greater than all the poles and zeros of our loop gain L.

Now, when s is tended to infinity or equivalently when capital R is tended to infinity we note that our loop gain L, which is essentially a cascade of the controller transfer function in the plant transfer function will reduce to the term K by s power n. Since the controller is a transfer function whose relative degree at least has to be 1 because, it is a physically realizable transfer function it has to be strictly proper and hence the new denominator polynomial of the controller has to be at least it has to have a degree it is at least 1 unit greater than it is numerator transfer function. And likewise since plant is also a physical system and it is it has to be strictly proper and therefore the denominator polynomial of the plant should have a degree that is at least 1 unit greater than it is numerator polynomial.

We would note that in the limit s tends to infinity we can write the loop gain L which is a product of the plant and controller transfer functions in the form K by s power n, where n is the sum of the relative degrees of the plant and the controller respectively. Since each of these subsystems the plant and a controller each have a relative degree of at least 1, we note that the term n should be greater than or equal to 2.

So, the loop gain at very large frequencies is going to be a very small number firstly and secondly it would be decaying as function of s in this particular manner in the form K by s power n. This can be easily verified by taking any particular specific example of a transfer function that represents the loop gain and tending s to infinity. We will note that the effect of all the poles and zeros will go away and we would have L going to 0 in this particular manner as yes is tended to infinity.

So, because L is going to be a very small number at very high frequencies because, L rolls off the planned poles and 0's result in reducing trend for L at high frequencies we can make one more approximation. And it is that ln of 1 plus L is approximately going to be equal to L in the limit that the magnitude of L is very small. Since we are now looking at complex numbers on the contour C 2 where which are very far away from the origin, which are at a distance of capital R from the origin and our capital R extended to infinity.

We note that the magnitude of L is going to be very small or those high frequencies and in those frequency ranges we can approximate ln of 1 plus L to be approximately equal to L itself, this we can obtain from the Taylor series expansion of logarithm of one plus x. Where we ignore the higher order terms on account of the fact that the term L is very small in magnitude with this particular approximation, we can write the integral over C to ln of 1 plus L ds in this particular form.

So, we can have, we can replace the term 1 plus L by the term L which is what has been done here and a term ds by j times R times e power j phi d phi there is a term j missing here and I shall add that here and we note that L would be of the form K by s power n and s would be of the form or e power j phi on the contour C 2 and hence we would have this integral to be given by limit of R tends to infinity. The angle phi going from pi by 2 to minus pi by 2 because we are traversing this contour starting from some point on the positive imaginary axis and finally going to some point on the negative imaginary axis the angle phi changes from plus pi by 2 to minus pi by 2 as shown here.

So, the expression for ln of 1 plus L ds on the contour C 2 is going to be given by this particular expression and here we note that n is greater than or equal to 2 and hence therefore n minus 1 is going to be greater than or equal to 1. Since we have the term R to the power n minus 1 in the denominator of the integrand and we are tending the radius R to infinity, we would have that this particular integral namely integral over C 2 ln of 1 plus L ds is going to be equal to 0.

So, the second integral here namely this particular integral has therefore shown to be equal to 0, as a consequence of the simplifying steps that we undertook just now. Let us now focus on the first integral namely integral over C 1 ln of 1 plus L ds and we note that on the curve C 1 we would have that number s to be essentially equal to j omega, because the curve C 1 essentially coincides with the imaginary axis with the consequence that we would have ds to be equal to j times d omega. And I have substituted the same in this expression and we have obtained ln of 1 plus L ds to be equal to j times integral minus infinity to infinity ln of 1 plus L d omega.

So this simplification is obtained from this realization. Now, we note that 1 plus L the complex number 1 plus L is essentially given by the magnitude of 1 plus L times exponential of j times the face of 1 plus L, this is by definition the complex number 1 plus L, it is given by the magnitude times e to the power j times the phase.

Hence we would have the ln of 1 plus L or locked logarithm to the base e of the function 1 plus L to be equal to ln of the magnitude of 1 plus L plus j times the angle of 1 plus L. We get this expression by simply taking the logarithm of this particular expression, so with this particular expression we plug it into the integral that we have here. and we obtain that ln of 1 plus L ds over the contour C 1 is equal to j times minus infinity to infinity ln of magnitude of 1 plus L d omega plus j times integral minus infinity to infinity the angle of 1 plus L b omega.

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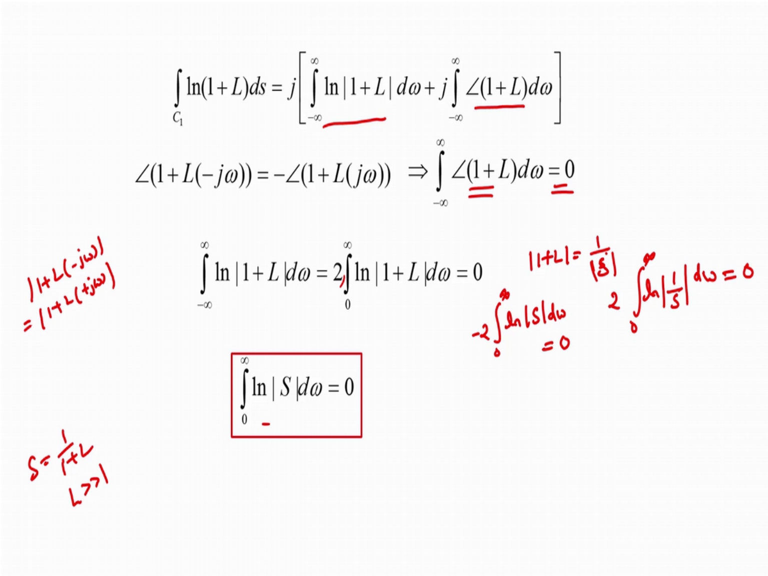

So, it would be the sum of these 2 integrals, let us evaluate these 2 integrals separately and to do that let us first consider the second integral. So, in the second integral the integrant happens to be the angle of 1 plus L and this is the in the limits of integration happen to be minus infinity and infinity. Now it so happens that the angle of 1 plus L is an odd function of omega or in other words the angle of 1 plus L of minus j omega is going to be equal to the negative of the angle of 1 plus L of j omega.

Now since the limits of integration of minus infinity to infinity, so we are going we are allowing omega to assume all values both along the negative imaginary axis as well as the positive imaginary axis and we have the integrand to be a function that is an odd function of omega. We would have this particular integral here namely integral from minus infinity to infinity of the angle of 1 plus L d omega to be equal to 0, simply because, the angle this function angle of 1 plus L is an odd function of omega.

What that leaves us with is just the first integrant integration alone and that is given by the term minus infinity to infinity ln of magnitude of 1 plus L d omega and since magnitude of 1 plus L, the function magnitude of 1 plus L is an even function of omega. We would have that the magnitude of 1 plus L of minus j omega to be equal to the magnitude of 1 plus L of plus j omega. And as a consequence of this fact we would have that integral minus infinity to infinity ln of magnitude of 1 plus L d omega to essentially be equal to 2 times the integral 0 to infinity of the same terms.

So, the limits of integration can be changed from minus infinity to infinity to 0 to infinity by multiplying this first integral by a factor of 2 and we note that this entire integral has to now be equal to 0 as a consequence of Cauchy's theorem. Now, if we note that the magnitude of 1 plus L is going to be equal to 1 by the magnitude of s capital S by definition, where capital S represents a sensitivity function, we would have therefore that 2 times integral of 0 to infinity lawn of magnitude of 1 by S d omega to be equal to 0. Or in other words minus 2 times integral 0 to infinity lawn of magnitude of S d omega to be equal to 0.

So, if you were to cast this in better form we would get this particular equation, namely that integral 0 to infinity lawn of magnitude of S d omega is equal to 0. So, this is called the bode sensitivity integral and what this indicates is that the integration of the logarithm of the sensitivity function, namely log of magnitude of S over the entire frequency range is always going to be equal to 0. No matter which particular controller we pick and therefore which particular form the loop gain has as long as the poles and 0's of the controller as well as the plant or both on the left half of the complex plane.

Now, this particular expression is essentially an expression of conservation, what it indicates is that as control engineers we might wish to reduce the sensitivity in a particular frequency range because, a reduction of sensitivity is essentially synonymous with having a high loop gain.

Because S is equal to 1 by 1 plus L, if our sensitivity is small much less than 0 dB then essentially we are having a loop gain L that is going to be much greater than 1 and as control engineers we wish the loop gain to be much greater than 1 in the frequency range where we are expecting performance. But what this expression indicates is that we cannot achieve the benefits of feedback over the entire frequency range.

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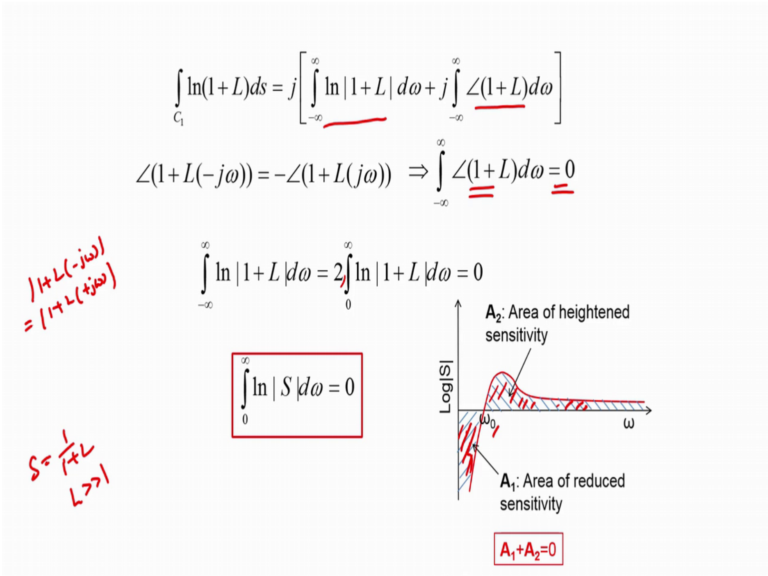

What it indicates is that the area of reduced sensitivity which I have called as a one here is going to be equal to the area of heightened sensitivity. So, if you were to reduce the sensitivity of our overall systems in some frequency range in order to achieve the benefits of feedback. We actually end up increasing the sensitivity in some other frequency range, in such a manner that there is no net change in sensitivity.

So, the areas A 1 plus A 2 add up to 0, so this particular integral is also called as the law of conservation of sensitivity dirt because, function sensitivity capital S is 1 whose magnitude we desire to be as small as possible and hence we call this function ln of magnitude of S as sensitivity dirt. We want to have as little of sensitivity dirt as possible, but what this expression indicates is that sensitivity dirt is can is conserved.

So, if we reduce sensitivity dirt in a certain frequency range as we have done in this range here in the interest of reaping the benefits of feedback, we inevitably end up increasing the sensitivity dirt in a different range which I have highlighted here. In such a manner that the net increase in sensitivity dirt, in the rest of the frequency range will be exactly equal to the amount of sensitivity dirt that we have reduced in one particular frequency range.

In this case this frequency range happens to be up to frequency omega naught. So, this is the first fundamental property associated with feedback systems, although in this clip we do not discuss how we can put this to good use, we shall see later on that we can employ this particular theorem to quickly come up with the estimates of the best possible gain margin and phase margin for a feedback control system.