Control System Design
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Lecture – 04
In-homogeneous linear time invariant ordinary differential equations

So, in the previous clip, we saw how we could solve the problem of you know the homogeneous linear time invariant ordinary differential equations. And we shall now see whether we can apply the same tricks, and the perhaps a couple more to take down the more complicated problem the one that we are really interested in; namely the in-homogeneous differential equation, where you also have inputs on the right hand side. So, to recap what we are talking about let me write out the differential equation that we are out to solve.

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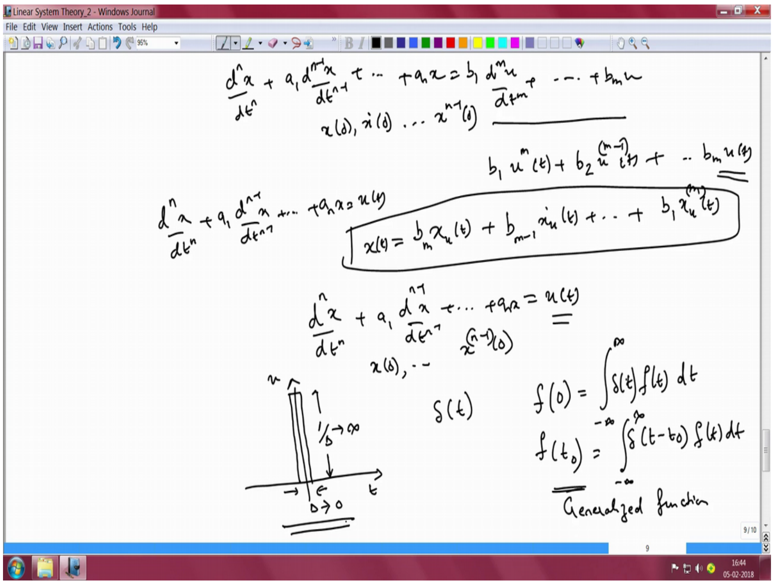

It is nth derivative of x with respect to time plus a 1 times nth derivative of n minus 1th derivative of x with respect to time and so on and so forth plus a n x is equal to b 1 times mth derivative of u with respect to time and so on, up to b m u, and with these initial conditions x dot of 0 and so on, x n minus 1 of 0.

So, in the previous clip, we saw how this problem can be tackled, when the right hand side it has been set equal to 0, but having done that that is not really our final target. So, we want to also see how we can solve the problem, when the right hand side is not equal to 0. And whether, we can draw any lessons from what we have gone through already for the homogeneous case.

Now, if you stare at this equation for a minute, you will discover that it is not very easy to make inroads into this differential equation again, because u is firstly arbitrary I can apply any arbitrary input. And the value of m is also m could be any integer. So, we have arbitrariness both in the value of m as well as in u, which makes it difficult for us to directly apply the lessons that we learned in the case of the homogenous differential equation for the non-homogenous case.

But, I can certainly simplify the problem a little bit by exploiting the fact that we are dealing with linear systems. For example, on the right hand side, I have m terms b 1 u mth derivative of u plus b 2 m minus 1th derivative of u and so on, up to b m u of t. What I discover is that I do not really have to solve the differential equation for this entire sum of inputs because we are dealing with linear systems.

So, for instance, I can just solve the differential equation for the case 1, I have only u of t on the right hand side. Let me say x u of t is the solution, when I have only u of t on the right hand side, in other words if my differential equation were to look something like this u of t. Suppose, the solution to that differential equation is x u of t then if I were to how b m times u of t on the right hand side, the solution will be b m times x q of t. Now, if I have b m minus 1 times u dot of t as another extra input, then the solution to that would be b m minus 1 times x u dot of t and so on and so forth, so that my final solution will be b 1 times x u to the power m of t (Refer Time: 04:09) x u to the power n x u differentiate it m times, mth derivative of x u with respect to time.

So, therefore one does not really need to include all the inputs on the right hand side. One can just look at one of the inputs alone, and then obtain the response, which I have called x u of t. And then differentiate the input m times, and combine these derivatives in the right proportion namely b 1 times x u mth derivative of x u with respect to time plus b 2 times m minus 1th derivative of x u with respect to time and so on and so forth. All the way to b m times x u of x u of t, and that is going to give me x of t.

So, the problem that I really need to tackle is to obtain the solution for this differential equation plus a 1 times n minus 1 of x with respect to time plus so on plus a n of x is equal to u of t. If I can take this down, then I know that I can take the more complicated case also down, now how do we do this. Once again I want to remind you of the initial conditions, we have n initial conditions x n minus 1 nth minus 1 derivative of x with respect to time at time t equal to 0. These are nth initial conditions how do we solve this problem, can we in any way draw inspiration from our attempt to solve the homogeneous problem well. If you look at it for some time, you discover that it is not really that easy, because u of t can be any general input, and it is not 0, it is only when u of t is equal to 0, that we have the homogeneous case.

So, now we next ask ourselves, if we can find an input u of t l is rather close that brings the system, rather close to the system, that we had for the homogenous case. And it is in this context, that we stumble upon the impulse input as a good candidate to have. What is an impulse input, you might have come across an impulse function in your undergraduate mathematics as a function of magnitude tending to infinity, for a time that is tending to 0.

So, if I have the width of this function delta to be tending to 0, and it is height to be tending to infinity, in such a manner that the height times the width is a constant equal to 1. Then this function we have called, you might have been exposed to as a delta function, this is also called as a Dirac-delta function. However, this is a very restrictive definition for a function of this kind, a more general definition is a rather indirect one.

So, you call a function as a delta function, which represent, which we represent by using the symbol delta of t. As a function that gives us the value of any other function f at time t equal to 0, when we perform this particular operation. When we weight down f of t with delta of t, and integrate that over time, then we should get f of 0 or more generally, if we multiply f of t with delta of t minus t naught, and integrate it over time from minus infinity to infinity, then that should give a give me f of t naught.

So, you see here that there is no direct definition for a delta function. It is an indirect definition in that a delta function is one, which when weighted with the integrand f of t, and integrated between the limits minus infinity and infinity gives me this particular function f of t 0. Hence, a delta function is also called a generalized function or a distribution. So, any function, which will multiply with f of t satisfies this particular equation here is a candidate delta function. Of course, you can easily verify that the Dirac-delta function that we have drawn here easily satisfies this particular relationship, but there are also other functions that satisfy this relationship.

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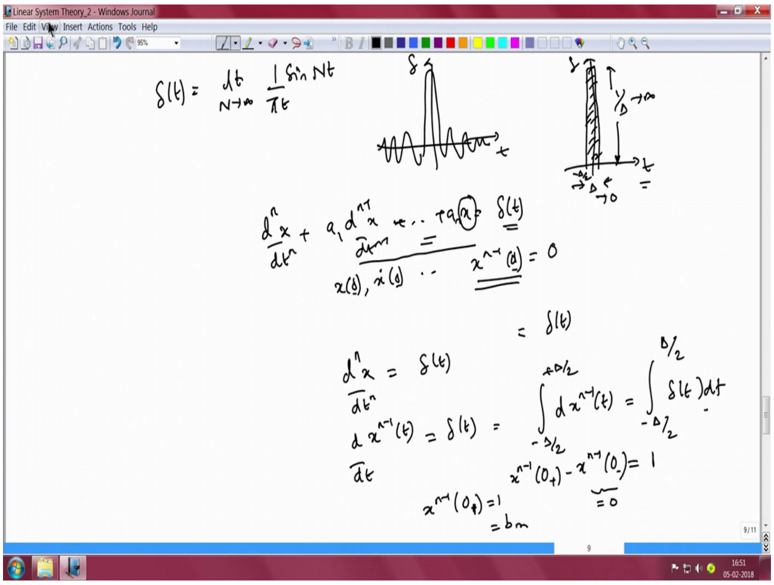

For instance, if I have if I were to define delta of t, as limit n tends to infinity 1 by pi times t sine N t. If I graph it, it will look like a sinusoidal signal with decaying amplitude. And the frequency of the sinusoidal tended to infinity. I can show that this also represents a delta function. So, there is no unique definition for a delta function it is indirect. But, for our case for our engineering applications the function that we looked at in the previous slide is namely a rectangle of unit area. And of height that is tended to infinity and of width that is tended to 0 is an adequate definition. So, height is 1 by delta, and that is tended to infinity, and the width is delta that is tended to 0, so that the area is 1 unit. So, this is an adequate definition for a delta function.

Now, let us apply a delta input on the right hand side of our system. Our system was x is equal to u and u I have set as delta of t. Now, why is this problem rather close to the homogeneous case, you see that in the case of a delta function. The input exists only for a various brief amount of time in the neighborhood of time t equal to 0. Before, that there was no input, after that no there is no input, which means that except for that small duration of time in the neighborhood of time t equal to 0.

Our system behave looks like a homogeneous system, where you have no input on the right hand side, it is only in the neighborhood of time t equal to 0 is there any input. And all we need to look at is how the input in the neighborhood of time t equal to 0 changes our response. So, once again I want to remind you, that I have n initial conditions.

However, I already know how to solve the problem, when I have any initial conditions and no input on the right hand side.

So, let us now simplify the problem by assuming that initially all the initial conditions are 0. In other words before application of the input in other words at time t equal to minus delta by 2 or I shall also call it as 0 minus ok, at time t equal to 0 minus all its initial conditions are 0. And then, there is this storm namely this huge input of infinite nearly infinite magnitude that that is applied on the system for a very brief infinite assembly small duration of time. And then, that changes the initial conditions in some way and then once again, you have the homogenous case, because the input has gone to 0 on the right hand side of this equation you would have 0. And you would have to solve the problem for the new set of initial conditions that have that come about because of application of this delta input.

Now, how do we find that new set of initial conditions? We find it by matching the singularities on the right hand side, and on the left hand side of the differential equation. We notice that on the right hand side, we have delta of t. And therefore, we should have a matching delta of t on the left hand side. Now, the question is where can this impulse input reside on the left hand side, can it be in the term x.

If it is in the term x, you see that you also have x dot, x double dot and so on and so forth, on the left hand side, which means that if x of t is proportional to delta, then we would have x dot proportional to delta dot and so on and so forth. But, we do not have terms related to delta dot, delta double dot, up to delta to the power n on the right hand side. So, clearly delta the delta input cannot reside in x of t, how can it reside in x dot of t. Similarly if you have delta function in x dot in other words, if I were to set x dot equal to some constant times delta of t, you will discover that x double dot will have delta dot, and so on and so forth, which are not represented on the right hand side.

So, you can continue this argument, until you discover that none of the lower order derivatives of x can contain the delta function. Because, if they do then there will be a higher derivative of the delta function on the left hand side, which will not be balanced by a corresponding term on the right hand side. Therefore, we would have to have the nth derivative of x alone to be a delta function.

Now, what this implies is that the nth derivative of x is essentially the derivative of n minus 1th derivative of x with respect to time, and that is equal to a delta function. Now, if I were to integrate this expression between the limits minus delta by 2 to plus delta by 2, then I would have between the time limits of minus delta by 2 to plus delta by 2. Then, I would have this to be delta of t d t between the limits minus delta by 2 to plus delta by 2; now, between the limits minus delta by 2 to plus delta by 2.

We see that integral delta of t d t represents the area, under this curve. And this, we know by definition is equal to 1, because the height of this delta function is 1 by delta. And it is width is delta. So, what we discover therefore is that on the left hand side, what I would get is the n minus 1th derivative of x at time t equal to 0 plus minus n minus 1th derivative of x, at time t equal to 0 minus is equal to 1. Now, we know from our initial conditions that the n minus 1th derivative of x, at time t equal to 0 minus was equal to 0, which implies that the n minus 1th derivative of x with respect to time, at time t equal to 0 plus has to be equal to 1.

So, now we see the effect of a delta function. Immediately, after the delta function has been applied, the delta function changes the initial condition of the system. All the other initial conditions remain unchanged. It changes the initial condition of only the n minus 1th derivative of x by 1 unit. Of course, if I had b m times delta of t, then it would have changed the initial condition by b m, just by matching my singularities I can get that. So, now we have obtained the response of the system to an impulse input. And this, response is called the impulse response. Let us now obtain the impulse response for a particular numerical example.

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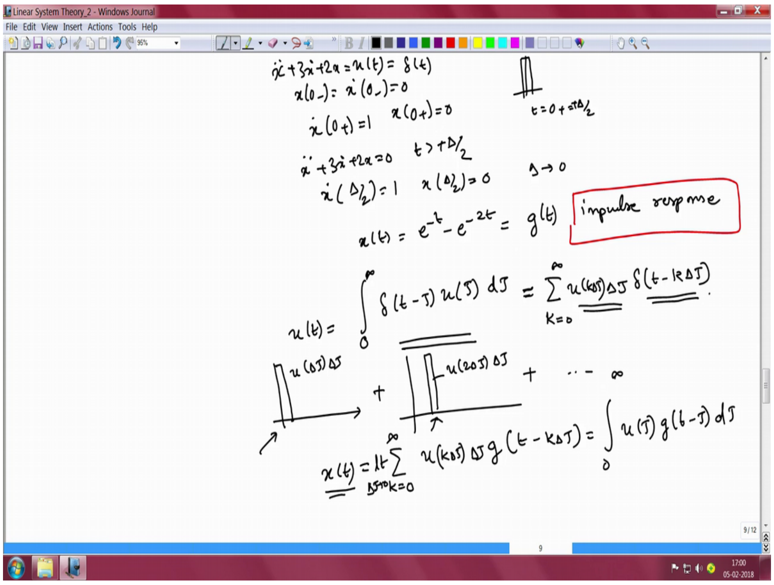

So, let us take once again the differential equation that we considered x double dot plus 3 x dot plus 2 x equal to 0 is equal to u of t. And I am setting x of 0 minus equal to x dot of 0 minus equal to 0 to obtain the impulse response; I set the right hand side u of t equal to delta of t. And I matching the singularities, what I would discover is that x dot of 0 plus would be equal to 1. And x of 0 plus will be equal to z.

Now, at time t equal to 0 plus I have these new initial conditions. And my delta of t is such a function that it is ceases to exist beyond time t equal to 0 plus or t equal to 0 plus is the same as t equal to plus delta by 2. Therefore, we have a homogeneous differential equation, namely x double dot plus 3 x dot plus 2 x equal to 0, for t greater than plus delta by 2. And with the initial conditions x dot of 0 plus or in other words x dot of delta by 2 is equal to 1. And x of delta by 2 is equal to 0, where delta is a very small quantity.

Now, what is the solution to this differential equation, we can apply the same tricks. Now, it is a homogeneous differential equation with a certain given set of initial conditions. This is something that we have already seen. And we can solve it, and get the solution to be x of t equal to e power minus t minus e power minus 2 t. And since this is a response to a specific kind of input namely, an impulse input. This is given a specific symbol g of t and it is called the impulse response, impulse response.

So, now we have made some progress in that we have managed to solve the in homogeneous differential equation not for any general input u of t, but for 1 specific input namely a delta function. How do we go from here? We notice that our delta function by definition allows us to connect the input u applied at a time t to its previous history, by definition. Therefore, u of t is equal to delta of t minus tau u of tau d tau, where the limits of integration are from 0 to are from minus infinity to plus infinity. This is just a definition of the delta function.

However, I can choose to apply my input to the system at any specific time. And therefore, for reasons of convenience, I shall choose to apply it at time t equal to 0, so that my lower limit of the integral becomes 0. Now, this is an integral. And an integral to me, as an engineer is a glorified sum ok. It is an infinite sum. I can, therefore write it approximately. As I can break it up, I can look at d tau as sum elemental increment of time delta tau. And write the input as u of k times delta tau. And the impulse input as delta of t minus k times delta tau.

So, in the limit that k tends to infinity. This sum essentially converges to this particular integral. So, as an engineer, therefore I can view this integral as a sum of different inputs u applied at different instants in time, as decided by this delta function. Now, as an engineer, I can consider the input u of t, as a train of impulses of magnitude u of k times delta tau, times delta tau.

Now, let me just graph this particular expression. So, if I for the case, when k equal to 1, I would have a delta function at center at delta tau. And of height and of area given by u times delta tau, times delta tau. And plus for the case, when k equal to 2, I would have a delta function centered at 2 delta tau, of area equal to u of 2 delta tau, times delta tau and so on and so forth.

So, the summation of all these inputs is the actual input u of t. Now, we see that we are dealing with a linear system. And since, we are dealing with a linear system; the response to a summation of inputs is a summation of the response to individual inputs. The response x of t is equal to the summation of the responses to each of the individual inputs. So, k equal to 0 to infinity. The response to one input delta of t minus k times delta tau is the impulse response, center at k times delta tau, which is g of t minus k times delta tau. And the magnitude of the response is decided by the magnitude of the input namely u of k times delta tau, times delta tau.

Therefore, my response x of t can be written as a summation of the responses to each of these individual inputs u of delta tau, times delta tau an impulse function of that magnitude applied, at time t equal delta tau plus an impulse function, at time t equal to 2 delta tau of magnitude u of 2 times, delta tau times delta tau, and so on and so forth. And that is what is represented by this equation. And in the limit delta tau tends to 0. This would reduce to the integral 0 to infinity u of tau g of t minus tau t tau. And thus, we have succeeded in obtaining the response to any general input u of t.

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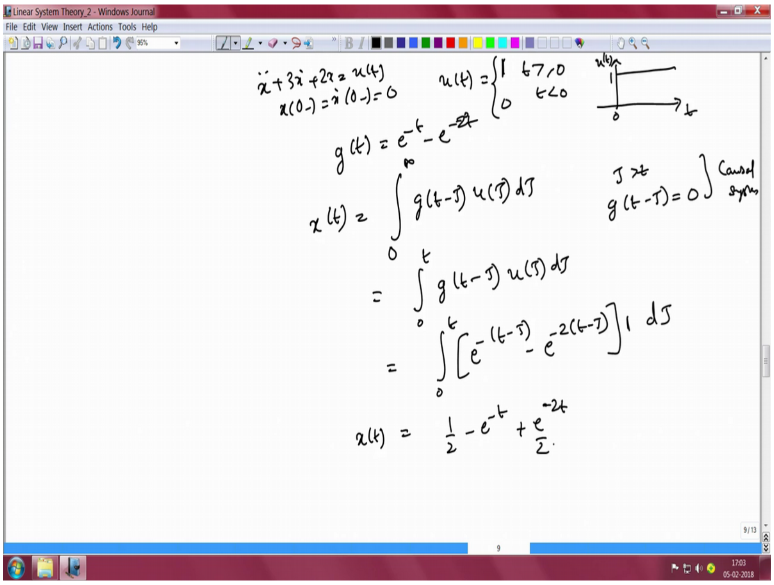

Let us now, illustrate this by means of a numerical example. For a same system, it will be considered before namely x double dot plus 3 x dot plus 2 x is equal to u of t. x of zero minus is equal to x dot of zero minus equal to zero. Let us try to obtain the response to the input u of t is equal to 1 for t greater than or equal to 0, and 0 for t less than 0. So, this is called the Heaviside step.

So, it changes, its value from 0 to 1, at time t equal to 0. To do this, we first have to obtain the impulse response of the system, which we did a few minutes back. You, know that the impulse response g of t is e power minus t minus e power minus 2 t. So, the response to the general input in this case u of t equal to 1, when t greater than or equal to 0. And 0, when t is less than equal to 0 is given by, x of t equal to integral 0 to infinity g of t minus tau u of tau d tau.

Here, I want to make one more modification to the limits of this integral. We notice that, we are dealing with physical systems. And our physical systems cannot respond to inputs that have not been applied yet. They can only respond to inputs that have been applied in the past or at the instant of application. And that in turn means that, when tau is greater than t g of t minus tau has to be equal to 0. So, such a system is called a causal system. And all physical systems are causal, which means that x of t has to be evaluate only till time t. And beyond time t g of t minus tau will go to 0.

And we do not have to worry about the magnitude of the integral. So, this is the integral that we have to evaluate. And this we shall do by first substituting the value of g of t. So, g of t minus tau e power minus t minus tau minus e power minus 2 times t minus tau. And we multiplied with u of tau, which is a constant of magnitude 1. And we integrate this between the limits 0 to t. If we do the integration, what we will discover is that this integral is equal to half minus e power minus t minus plus e power minus 2 t by 2.

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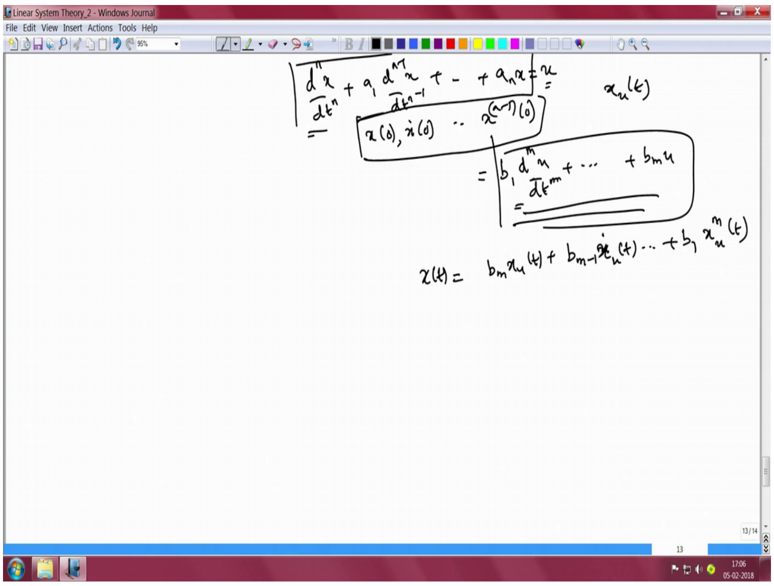

So, to summarize we have obtained the solution to the differential equation nth derivative of x with respect to time plus a 1 times n minus 1th derivative of x with respect to time so on and so forth plus a n x is equal to u; with the n initial conditions x of 0, x dot of 0, x n minus 1 derivative of 0. Of course, when we solved this equation, we assume that the initial conditions were 0. But, if you have some non-zero initial conditions, the response of the system would be the sum of the responses to the initial conditions plus the response to that particular input u.

Now, if you want to obtain the solution to the more general case, where on the right hand side you have instead of u, you have b 1 mth derivative of u with respect to time and so on and so forth up to b m u. All you need to do is having obtained the solution x u of t, that we did just some minutes back, we can represent the response to the general input given by this expression as x of t equal to b m x u of t plus b m minus 1 x dot u of t and so on, up to b 1 x m u of t.

What I hope you have appreciated from the material that we have discussed over the last 2 or 3 clips is that we have managed to solve the differential equation, where the degree of the differential equation with respect to x is n. And the degree with respect to u is m, where both n and m are arbitrary, and u is any arbitrary input. And we have managed to do it by first starting with the homogeneous differential equation, looking at how the solution appears for the most simple case namely, when n is equal to 1. And then from that using a combination of intuition and some special inputs such as the delta function, we have managed to solve the most general case namely when you have this term being equal to this term and for some n set of initial conditions.