Control System Design
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Lecture – 35
Robust control design using Quantitative feedback theory (Part 1/2)

Hello, in the previous clip we took a look at how we could use the Nichols plot to undertake control design and the design that we undertook was that of a one degree of freedom control system. But, in that one degree of freedom control system there were multiple performance specifications. There was an input disturbance, there was an output disturbance and each of them had to be rejected by different extents and then there was also at a stability constraint.

And the general approach as we outlined in the previous clip was that we first identified the set of frequencies where we are interested in achieving a certain performance or a certain extent of stability for the closed loop system. And the second step is that at every one of these frequencies we determine the permissible area in the complex plane where the loop gain L can lie in order for it to satisfy a particular performance specification or a stability specification.

So, if we have multiple performance specifications, each specification will allow the loop gain L to lie within a certain region in the complex plane or in other words it will allow the loop gain L to have a certain range of values for fail and magnitude. And the actual region in which the loop gain can lie for it to satisfy simultaneously all the specifications would essentially be the intersection of all these different regions.

Hence, we performed is design in the Nichols plot which as we discussed is in some sense a distorted version of the Nyquist plot and is once again a complex plane of the loop gain L of j omega. So, each of the performance specifications resulted in what was known as a performance bound. And this bound divided the Nichols plot into 2 parts; the part in which the loop gain was allow to lie and the part in which the loop gain was not supposed to lie.

So, at each frequency we obtain the bounds for input disturbance rejection, output disturbance rejection and stability. Then we find the intersection of all these bounds and that tells us the permissible values of the loop gain that we can have at a particular frequency in order for it to simultaneously satisfy all the requirements. We repeat this at all the other frequencies of interest to us from the point of view of performance and stability and obtain similar bounds at other frequencies. And finally, execute the design where we determine that controller structure which enables us to achieve loop gains for at each of the individual frequencies that we have picked that are within the bounds at that have been specified for each of those frequencies.

So, what we shall do in today's clip is to use the Nichols plot to solve the problem of robust tracking. So, if we go back a couple of clips, we discussed the problem of robust tracking and solved it using root locus as the canvas upon which the design was done. What we shall do in this case is take up the same problem, but relax some of the constraints that had to be imposed when we under took the design using root locus; because, the root locus technique demanded such constraints to be placed for it to be easily applied.

So, we shall relax those constraints and with that relax set of constraints root locus will cease to be a intuitive and useful tool for performing design and we shall see how it can be performed using the Nichols plot. So, the problem statement has been indicated on this slide, we have a plant which is uncertain. And, if you recollect this plant is very similar to the plant that we consider when we were undertaking 2 degree of freedom control design using the root locus.

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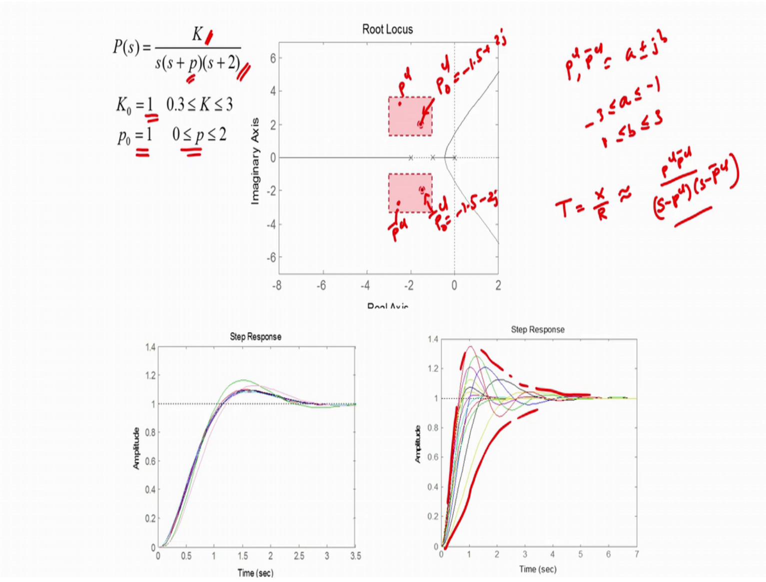

So, the plant has an uncertain gain K and we assume that the nominal value of this gain is K naught equal to 1, but it can vary by a factor of 10 it can vary from 0.3 to 3. Likewise this plant also has one uncertain pole p. Nominally, the pole location is p naught equal to 1, but the pole could lie anywhere between 0 and 2 or in other words the pole in the complex plane the pole could be at anywhere between s is equal to 0 and s is equal to minus 2.

So, this plant is identical to the plant that we considered when we undertook root locus based design. When there was simultaneous uncertainty in the gain of the plant as well as its pole location. Now, when we undertook root locus based design we insisted that the closed loop pole be located at P cl naught equal to minus 1.5 plus 2 j and P cl naught bar equal to minus 1.5 minus 2 j. This was very dominant closed loop poles were expected to be located for the nominal values of the gain K of the plant and its pole location.

However, when the pole location vary or the gain changed we expected that the closed loop poles should wander by a small amount about its nominal position. And this has been indicated in this graph by this tiny dotted circle that I have drawn here; as the area within which we were with permitting the variation in the closed loop pole.

Now, we had to restrict a variation of the closed loop pole to within such a small area about its nominal position. Primarily because the equations that we wrote when we perform the design using root locus could be simplified only when the permitted variation was much smaller than the typical distances between the other open loop poles and zeros.

What we shall do in this clip; however, is to take a look at a problem that cannot be handled using root locus, but is essentially of similar flavor as what we did earlier. In particular, instead of restricting the variation of the closed loop pole to a very small region near the, you know desired dominant pole location namely at minus 1.5 plus minus 2 j.

We shall allow for variation of the dominant pole P cl anywhere within this 2 big rectangular regions. So, we can have the closed loop pole P cl and P cl bar anywhere within this rectangle. And therefore, the closed loop pole P cl and P cl bar would be of the form a plus or minus jb, where a is the real part of the pole and b is the imaginary part of the pole. And from the limits of this rectangle we see that we can allow for a to be anywhere between minus 3 and minus 1 and likewise b can also be anywhere between 1 and 3.

So, what you see here is that we want to design a feedback control system whose closed loop pole can vary within this rectangle of dimension 2 units along the real axis and 2 units along the imaginary axis. And its nominal closed loop pole position is exactly where we had it before namely minus 1.5 plus minus 2 j. Now, if you notice the dimensions of this rectangle, you note that the size of this rectangle is of comparable magnitude to the typical distances between the other open loop poles of the plant. And this is what makes it difficult for us to employ root locus as the tool for performing design.

The assumption that the distance over which the closed loop pole varies is much smaller than the distances between the open loop poles breaks down in this particular problem specification. And that is what makes it that is what make root locus a tool that cannot be easily employed in this particular case. Now, if our dominant dynamics were allowed to vary within this fairly large rectangle then what we essentially imply is that the closed loop step response could assume any of the trends that have been shown in the graph on the right. So, it could have a range of overshoots and the range of price times and settling times.

So, what we are essentially saying is that as control engineers we are with the closed loop system having such a relatively large spread in its response because, this is acceptable for our particular application. And if we permit for this large a spread in the response here essentially therefore, also indirectly saying that we are with the closed loop pole wandering over a fairly large area in the complex plane. And, in this particular case this area is bounded by the rectangle that we just talked about.

If you think about it this problem is more general and also practically more realistic because, as control engineers we do not always desire for the closed loop response to vary by negligible amounts when the plants model changes. Sometimes, it is to permit some variation in the closed loop response when the plants model changes, but unfortunately the tool of root locus does not permit us to execute design in a transparent and intuitive manner; when such relatively large variations in the closed loop pole position are permitted when the plants parameters change.

So, in comparison to what is now considered an acceptable variation in the closed loop response. I have plotted on the left what was considered an acceptable variation in the closed loop response when we undertook the design using root locus. Now, if you look at the graph on the left we have insisted in this particular case that the variation be extremely small. And the price that we saw that we had to pay when we undertook the design was that our controller the feedback controller ended up having a much larger bandwidth than the overall transmission function as a consequence of having to restrict the variation of the closed loop response due to uncertainty in the plants parameter.

So, that was the price that we needed to pay in the case when we undertook the design using the root locus. So, in this case what we are saying is that if you are not willing to pay such a high price, if you are not willing to invest in controllers with such high bandwidths and correspondingly we are also with permitting a much larger variation in the closed loop response of our overall control system.

Then how do we undertake systematically? The design of the feedback controller and the pre filter that will allow us to get a control system whose closed loop response varies within the particular bounds that have been indicated by this new set of step responses within this particular bound and no larger. So, you know be with this amount of variation, but we are not with any amount of variation. We want the closed loop poles to still be restricted within a certain area. It is just that the area that we are considering in this example is much larger and the area that we consider when we undertook the design using root locus.

So, if we are willing to permit the closed loop poles to lie within the rectangles that we have shown there then the overall transmission function T that relates the output to the reference R. So, T is given by X by R can approximately be written as T is equal to P cl P cl bar divided by s minus P cl times s minus P cl bar. This is the approximate transfer function for the overall transmission function simply because by definition P cl and P cl bar are the dominant poles of our closed loop system. So, for all practical purposes our closed loop system looks like a second order system, we whose poles are at the points P cl and P cl bar.

In practice; however, I want to underscore the fact that the overall transmission function will have other terms in addition to P cl and P cl bar and the number of poles that it would have an number of zeros it would have is dependent on the order of the plant as well as the order of the controller. But, what we are claiming here is that whatever order we might have for the controller or for the plant it has a pair of dominant poles. And therefore, it can be approximated as a second order system whose poles are at P cl and P cl bar.

The exact locations of P cl and P cl bar; however, can vary that is because whenever we have plant whose gain and pole locations are different from the nominal gain and pole locations then the closed loop pole P cl can lie at a different location. And what we want to do is to design the feedback controller and the pre filter such that for any particular combination of the plants gain K or its open loop pole position P, our dominant closed loop pole P cl and P cl bar lie within the 2 rectangles that we have indicated. Now, if we return to this transmission function T if you want to draw the Bode plot of this transmission function, then what we are essentially claiming is that we are willing to permit.

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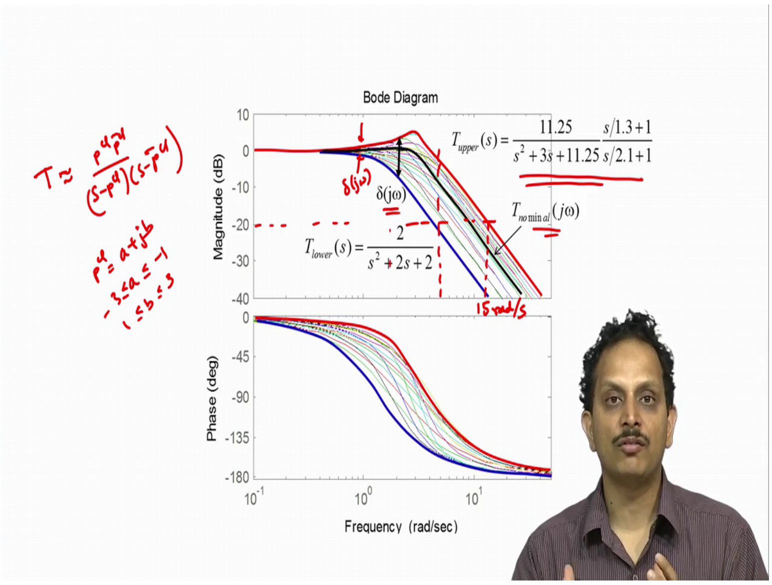

The Bode plot of the overall transmission function to be any one of the curves in this bundled here; so, for each particular value of P cl and its corresponding value of P cl bar that are within the rectangle. We have computed the transmission function T which is approximately equal to P cl times P cl bar divided by s minus P cl times s minus P cl bar. And for this transfer function I have drawn the bode plot.

Now, P cl and P cl bar can be numbers that would be of the form P cl is equal to a plus jb. And we allow for the parameters a to vary between its particular limits namely minus 3 and minus 1 and we allow for b to vary between its particular limits namely 1 and 3 and when we do when we take a number of values for a between its particular limits and correspondingly a number of values of b within its particular limits.

And for each of these particular combinations of a and b we compute a transmission function and then draw the Bode plot of that transmission function. We get the set of curves that I have shown here. It is worth noting that the actual transmission function is not going to exactly look like this because there are going to be other terms that are contributed by the other closed loop poles of the overall system. In this case; however, you are plotting the bode plots only for the approximate model of the transmission function.

Now, if you look at this transmission function what we can identify is that there is an upper limit that we are willing to permit for the variation in the transmission function and there is also a lower limit which we are willing to permit. The lower limit has been indicated by the blue curve and the upper limit has been indicated by the red curve.

Furthermore, there is also the nominal transmission function which we would get when the dominant closed loop pole P cl is located at its nominal position namely P cl naught, namely minus 1.5 plus minus 2 j. And that nominal Bode plot has also been plotted along with the upper limit and the lower limit.

Now, there is this nominal transfer function and there is this spread in transfer function namely delta of j omega that we are willing to permit. Now, when we undertake our control design we have to make sure that the spread in our transmission function does not exceed delta of j omega at each of the frequency omega. And this in turn will guarantee that our dominant closed loop poles will lie within the rectangle that we have defined in the previous slide.

Now, we can obtain the exact numerical expression for the upper limit which is indicated by the rectrace with some trial and error. But, also by making note of the fact that there will be a particular closed loop pole location for which the corresponding Bode plot will define the upper limit of the variation of the closed loop pole. And likewise we can also obtain the expression for the lower limit of the permitted variation. Starting from the fact that there will be a certain closed loop pole location P cl for which the Bode plot would define the lower limit of the permissible spread in variation.

However, sometimes it so, happens that within a certain frequency range there will be one particular closed loop pole for which the Bode plot will define the upper limit. And in a different frequency range there will be another location for the closed loop pole which will define the upper limit for the spread in variation. Hence, this transmission function T upper has been obtained with some trial and error. By starting with the guideline that I just talked about, but modifying it little bit in order to incorporate the fact that in different frequency ranges, different closed loop pole positions can define the upper limit for the permissible variation in the transmission function.

So, we have this nominal transmission function that we need to design firstly. And secondly, we also need to restrict the variation of the transmission function from its nominal characteristic by the amount delta of j omega at each frequency omega. So, at very low frequencies, the permitted variation of delta of j omega is quite small. And as frequency increases we are willing to admit slightly larger variation in delta of j omega.

Now, we have 2 controllers namely a feedback controller and the pre filter. The question is which controller should we use to do what? Do we use the feedback controller to design the nominal characteristics of the control system? And do we use the pre filter to restrict the variation in the transmission function? Or vice versa do we use the feedback controller to restrict the variation in the transmission function? And do we use the pre filter to achieve the desired nominal response for the overall transmission function? This is the question that we first need to answer before we undertake the design of the 2 controllers.

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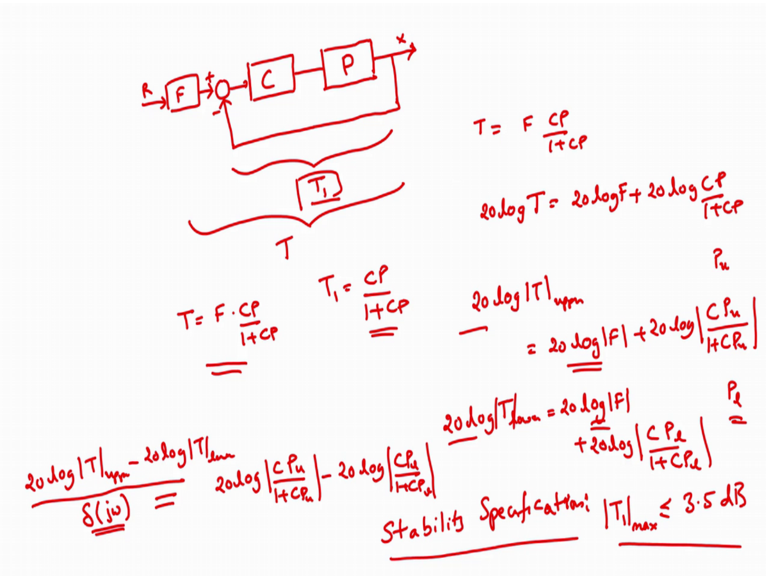

To answer this question I have indicated the 2 degree of freedom control system here. And I have defined the overall transmission function which is T and that is given by F times CP by 1 plus CP. And I have defined T 1 as the transmission function for just the feedback part of the overall system and that is simply given by T 1 is equal to CP by 1 plus CP.

Now, the question is do we design the controller C to restrict the variation delta of j omega? Or do we design the controller C to achieve the desired nominal response for the overall transmission function? To answer this question let us first write out that since T is equal to F times CP by 1 plus CP. We would have 20 log T to be equal to 20 log F plus

20 log CP by 1 plus CP. Now, for a particular combination of the parameters of the plant, our transmission function will assume its value T upper which is the maximum permissible variation in the transmission function. And I shall call that particular plant model as P upper or P u, P subscript u for short.

So, for this particular plant model we would have 20 log magnitude of T upper to be equal to 20 log magnitude of F plus 20 log magnitude of C times P upper divided by 1 plus C times P upper. So, this is the equation that we would have that would relate the transmission function T upper 2 a particular structure of the plank that gives us the transmission function T upper.

Likewise, there will be another combination of parameters of the plant, namely its uncertain gain and its pole location that will result in the overall transmission function; assuming the value T lower at a particular frequency omega. Hence, if we were to call that particular plant which results in the transmission function being equal to T lower at some particular frequency omega P l, P subscript l. Then we would have that 20 log T lower log magnitude of T lower to be equal to 20 log magnitude of F plus 20 log magnitude of C times P lower divided by 1 plus C times P lower.

Hence, if we subtract the 2 equations that I am highlighting [under/underlining] underlining here you will note that 20 log magnitude of T upper minus 20 log magnitude of T lower is going to be equal to. We would have the terms 20 log F getting cancelled from both these equations. And hence you would how this to be equal to 20 log magnitude of C times P upper divided by 1 plus C times P upper minus 20 log magnitude of C times P lower divided by 1 plus C times P lower.

And on the left hand side we note that the term 20 log magnitude of T upper minus 20 log magnitude of T lower essentially is a permissible spread delta of j omega at the frequency omega. So, if we restrict the spread to within delta of j omega at all frequencies omega we essentially ensure that our close loop pole does not vary outside the boundaries of the rectangle that we have drawn 2 slides back.

So, we have this equation that delta of j omega is equal to 20 log C times P upper by 1 plus C times P upper minus 20 log C times P log magnitude of C times P lower by magnitude of 1 plus C times P lower. Now, this equation tells us all that we need to know in order to determine which controller we use for what purpose.

If we note this equation we see that the pre filter F is not represented at all in this equation and what that essentially tells us is that no matter what we do to the pre filter, we cannot use it to control the spread delta of j omega of our transmission function when the plants parameters change. Therefore, we have no choice, but to use the feedback controller for restricting the variation in delta of j omega and the pre filter for achieving the overall nominal performance T nominal.

So, now, that this analysis has revealed to us the division of labor that needs to be undertaken between the feedback controller and the pre filter namely, at the feedback controller is intended to restrict the spread in variation of the transmission function at each frequency. And the pre filter is employed in order to achieve the nominal response of the transmission function at each frequency.

We shall now proceed with performing the design using the Nichols plot. Now, if you recollect from the previous slide the first step in undertaking design using the recalls plot is to select a set of frequencies that are of interest to us from the point of view of performance. So, if we look at the transmission function of the overall system which I drew in the previous slide, what you see is that the transmission function has a bandwidth that is close to around 4 radians per second.

The upper transmission function has a bandwidth that is close to around 4 radians per second and the lower transmission function has a bandwidth that is even lesser that is on the order of a little bit more than 1 radian per second. So, what we are essentially claiming therefore, when we have decided to choose a bandwidth of around 4 radians per second for our overall transmission function is that our reference signals frequency content will not exceed 4 radians per second.

If our reference signal had frequency content beyond 4 radians per second, then we had to choose our dominant pole locations for our overall closed loop system in such a manner that the transmission function of the overall system assumes a magnitude close to unity within the entire frequency spectrum of the reference signal.

So, what is therefore, clear from this discussion is that we will whatever reference it is that we might choose to track, the frequency content of the preference is decidedly going to be less than 4 radians per second. Hence, we are not obligated to restrict the variation in the transmission function beyond 4 radians per second. Because, we will not have any references whose frequency content is beyond this particular frequency for us to track.

On basis of this consideration, we shall choose an upper limit to the frequency up to which we restrict the variation in the transmission function. Although this frequency could have been the frequency at which that upper transmission function or the upper limit T upper falls down to minus 3 dB of each DC value, in the analysis that we undertake during this discussion. We shall assume that the frequency up to which we shall restrict the variation of the transmission function would be the frequency at which the transmission function T upper falls to minus 20 dB. And this particular frequency it so, happens is around 15 radians per second.

So, although we could have stopped with restricting the spread in the transmission function only up to the frequency at which the upper transmission function T upper falls to minus 3 dB. In the interest of coming up with a somewhat conservative design we shall choose to restrict the variation delta of j omega up to the frequency at which the transmission function T upper assumes a value that is greater than minus 20 dB, and in this case it happens to be around 15 radian per second.

So, as part of the first step in the Nichols plot base design we have to pick a set of frequencies between 0 radians per second and 15 radians per second within which we try to restrict the variation of the transmission function and also achieve the desired nominal response for the transmission function.

So, this is the first step, but as control engineers we are not interested purely in performance alone. We are also concerned about stability; however, the specifications that have been given to us do not talk anything about the stability or the feedback part of the control system. Or in other words the specifications do not tell us the extent of stability that we desire for the transmission function T 1.

So, in keeping with the strategy that we would adopt in the Nichols plot to specify the stability requirement namely to specify the maximum permissible value for the magnitude of the feedback transmission function T 1. We shall add this particular stability specification namely, that a magnitude of T 1 max should be less than or equal to 3.5 dB.

So, this was specification that was not provided to us at the outset we were just told what the dominant dynamics of the overall control system was for that does not reveal to us what the stability specifications for the transmission function T 1 is. And hence we had added this extra specification to complete the problem statement. So, we want our transcription function T 1 to assume values less than 3.5 dB in the in the interest of stability.

So, when our transmission function T1 assumes a value of around 3.5 dB which corresponds to about 1.5 in a linear scale, the corresponding phase margin can be shown to be close to around 40 degrees which is typically considered an acceptable phase margin when we are undertaking control system design. So, having now decided which controller we would use for what purpose and also having defined the problem completely, let us now get started with undertaking a control design.

So, the first step in undertaking the control design is to determine the permissible bounds for performance as well as for stability at each of the frequencies of interest to us. And we have already decided that the frequencies of interest to us from the point of view of performance go from 0 radians per second to 15 radians per second. So, let us take one particular frequency within this range.

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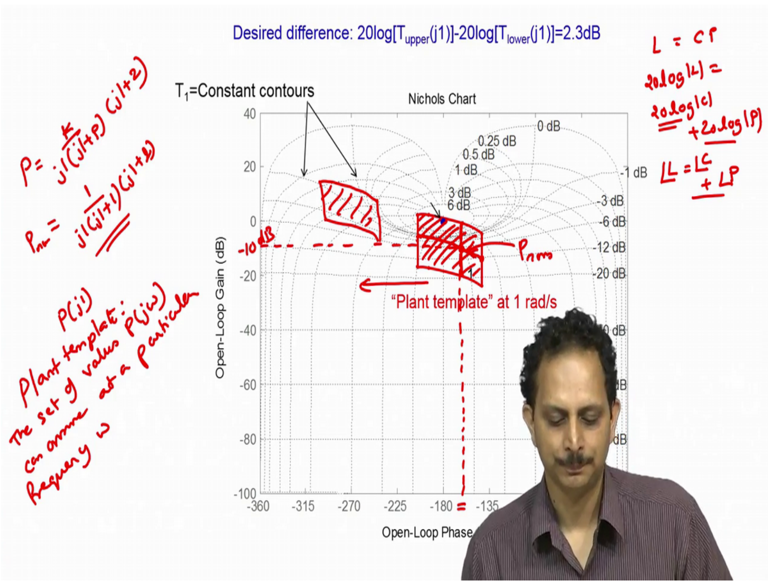

And, in this particular case I have taken 1 radian per second, now at 1 radian per second I would have a plant transfer function P to be essentially equal to K by j 1 times j 1 plus P times j 1 plus 2. Now, we know that the plant gain K and the plant pole location P are uncertain.

Now, if we pick a particular combination of the plant gain K and P, for instance if we choose the gain K to be its nominal value K naught. And the plant pole location to be its nominal location namely at s is equal to minus 1 then the plant transfer function which is the nominal transfer function of the plant is given by this particular expression.

We note that the value of K naught is equal to 1. So, the nominal plant transfer function, at 1 radian per second is given by this particular expression. Now, we note that this results in the plant having a certain phase lag which is determined by the phase of the complex number in the denominator and a certain magnitude.

So, if you were to indicate the magnitude of the nominal plant and the phase of the nominal plant at 1 radian per second what we would get is this particular location P nominal in the Nichols plot. So, at 1 radian per second the plant has a phase somewhere between minus 180 and minus 135 degrees. The exact numerical value can be computed from the expression for P nominal at 1 radian per second. And it has a open loop gain namely the magnitude of the plant is going to be somewhere between 0 and minus 20 dB and it appears to be close to minus 10 dB.

So, this is the magnitude and the phase of the nominal plant at 1 radian per second. However, the plant is an uncertain plant and what that essentially implies is that it can assume different values of gain and different and its pole location can also assume a range of values. Now, if the plants gain increases then we note that the value of the plant transfer function P of j 1 we increase without changing the phase of the plant transfer function. Likewise, if the value of the gain K reduces in the phase of this plant transfer function once again will not change, but it is magnitude will reduce.

So, in the first case it will increase in the another transform here and the second case it will reduce. The phase of the plan transform function will remain the same which is given by the by this number here and the magnitude; however, will reduce. Now, if the pole location were to change, but the gain where to be its nominal value. So, if the gain was equal to 1, but the pole location could assume values between 0 and 2 then we can show that the plant transfer function will change its phase and magnitude along a certain curve in the Nichols plot.

So, when the plant pole location is at s is equal to 0 and the gain is 1, the plant transfer function will assume this particular magnitude and phase combination. And as the plant pole location p goes from 0 to 1, the and the gain remains at 1, then the plants gain will travel along this curve and come to the nominal location of the pole and gain combination and when the plants pole goes from 1 to 2, then the plants gain and phase at 1 radian per second moves along this particular curve.

Now, if the plants gain and pole locations change independently, then the complex number P of j 1 essentially assumes values within this particular closed curve. And this closed curve has a special name it is called the plant template at the frequency 1 radian per second. So, by definition the plant template is the set of I shall write it down here, plant template is the set of values P of j omega can assume, but a particular frequency omega.

So, because an uncertainty in the gain and the pole location of the plant the plant transfer function P of j 1 is not a single number, but is actually a set of numbers that we obtain by plugging in the different combinations for the gain K and the pole location P within the limits that have been specified to us. So, if we plug-in these particular parameters and compute P of j 1 the set of magnitudes and the phase values of P of j 1 essentially define a certain area in the Nichols plot as the area within which P of j 1 will lie and this area is called the plant template.

Now, the attraction with doing control design using the Nichols plot comes from the fact that this plan template does not change either its orientation or its size when it is multiplied with a controller. Because, in the undertaking feedback control design we are essentially trying to determine the loop gain L, it is going to be equal to C times P. Now, we would have 20 log of magnitude of L to be equal to 20 log of magnitude of C plus 20 log of magnitude of P. Likewise, the phase of L will be equal to the phase of C plus the phase of P.

Now, unlike the plant transfer function there is no uncertainty associated with the controller transfer function. So, once we fix the controller, it will provide a certain specific magnitude at a particular frequency and a certain specific phase lag or need at that particular frequency.

So, from these equations what we can conclude is that the effect of undertaking control design which essentially involves coming up with an appropriate loop K is the effect of multiplying the plant transfer function with the controller is to displace the plan template either to the left or to the right depending on whether the controller provides a phase lag or a phase need. Or to move it up and down depending on whether the controller amplifies the gain of the overall open loop system or attenuate the gain of the overall open loop system.

So, in the course of our design the set of values that our loop gain L can assume would essentially be depicted by a closed curve whose shape would be exactly identical to the shape of the plant template that we obtain with the controller C set equal to 1. The only difference is that this set of values for the loop gain will be at a different angular position and will be at a different gain in with respect to the plant template. But, the size and the shape of this closed region would be exactly identical to that of the plant template itself.

And this is the attraction of a Nichole's plot based design because, if you go back to our discussion on the challenges introduced by an uncertain plant when we are trying to represent performance specifications in the Nyquist plot instead of the Nichols plot. We noticed that when we multiply the Nichols plot of the plan transfer function with the controller then the area within which the plant transfer function could assumed values in the Nyquist plot not only got rotated, but also got scaled up.

So, unlike what happened in the Nyquist plot, in a Nichols plot the same area neither gets scaled up nor gets rotated. It will only get translated in the Nichols plot when we multiply the plant transfer function with a controller.