Control System Design
Prof. G.R. Jayanth
Department of Instrumentation and Applied Physics
Indian Institute of Science, Bangalore

Lecture – 32
Issues connected with 2-degree of freedom control design using root-locus

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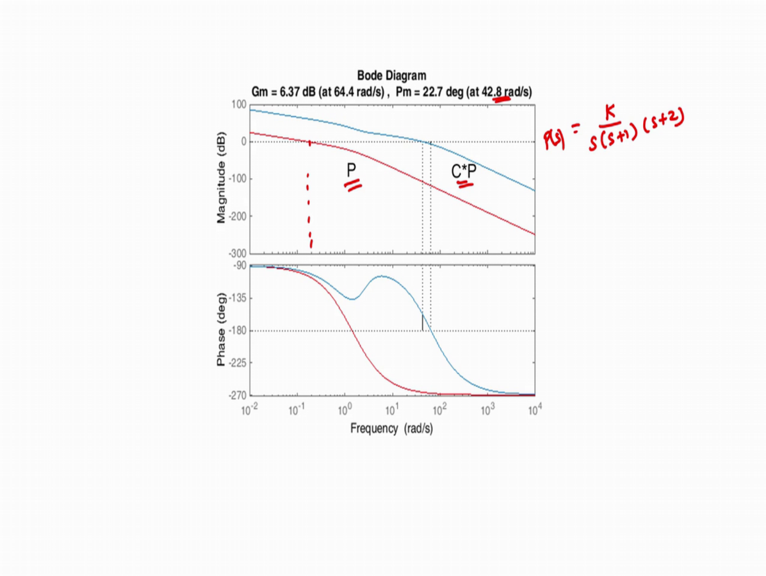

Hello. In the previous clips, we have seen how to perform 2-degree of freedom control design in order to achieve robustness for the closed loop systems response to variation implant model. We have seen that it can be employed to achieve a high degree of insensitivity to variation in the plants gain, to variation in the plant pole and indeed to variation in multiple parameters of the plant including it is gain and perhaps all its poles if that happens to be the case.

So, in this clip what we would look at are some of the notes in connection with 2-degree of freedom control and some of the issues associated with 2-degree of freedom control and we shall see how these can be addressed. So, the first point that I want to address is this apparently magical manner in which the transient response of a 2-degree of freedom control system becomes insensitive to variation implant parameters.

So, what is it that is really going on behind the scenes that helps us to achieve such a high degree of insensitivity to variation in the plants parameters. To understand this question what I have done here is I have plotted the Bode plot of the plant alone which has been shown as the red trace in this graph here, while the blue trace shows the Bode plot of the feedback controller times the plant.

Now, if you look at the gain cross over frequency of the plant alone, you see that it is quite small. It is close to about 0.2 radians per second whereas, if you look at the gain crossover frequency of the plant times the controller it is exceedingly large, it is about 42.8 radians per second. Now, 42.8 radians per second as a number is significantly higher and a location: where we wanted our dominant closed loop dynamics to lie which was at minus 1.5 plus minus 2j.

Hence what we see from a Bode plots of the feedback part of the 2-degree of freedom control system alone is that it is not really the feedback part of the control system that gives us a desired transient response. The desired transient response has dominated by the poles placed at minus 1.5 plus minus 2j is actually obtained from the pre filter of the overall control system.

So, our 2-degree of freedom control system can be thought of as a cascade of two subsystems; one is the pre filter and the second is the feedback control system. And, we see from the Bode plot of the feedback control system that it is not the feedback system that gives us the desired transient response because, the feedback system has a bandwidth that is much higher than what has been dictated to us as a desired dominant closed loop pole positions.

So, the desired dominant closed loop pole positions are guaranteed to us primarily by the pre filter. Then what is the role of the feedback control system? The role of the feedback control system is to deal with the uncertainty associated with the plant. It ensures that the high loop gain that the feedback control system achieves in open loop ensures very small variation in the response of the feedback part of the control system to changes in the plant parameters within the bandwidth that is of important to us. Namely up to a corner frequency as dictated by the dominant dynamics of our feedback control of our overall control system namely minus 1.5 plus minus 2j.

Hence, we see that the role of making sure that the transient response is dictated by the poles at minus 1.5 plus minus 2j is taken up by the pre filter while the role of ensuring minimal variation in the overall transmission function due to variation in the plant parameters within the frequency range dictated by the dominant dynamics of the overall system is taken care of by the feedback control part of the overall system.

So, the 2-degree of freedom controller has a finely divided distribution of roles and responsibilities between the pre-filter and the feedback controller. The feedback controller is intended to deal with uncertainties associated with the plant while, the pre filter is designed to give you the desired dominant dynamics or the overall transmission function relating the output to the reference.

This is the first point that I wish to make. The second point has to do with the controller structure that we obtained in the course of the designs that we undertook over the last three clips.

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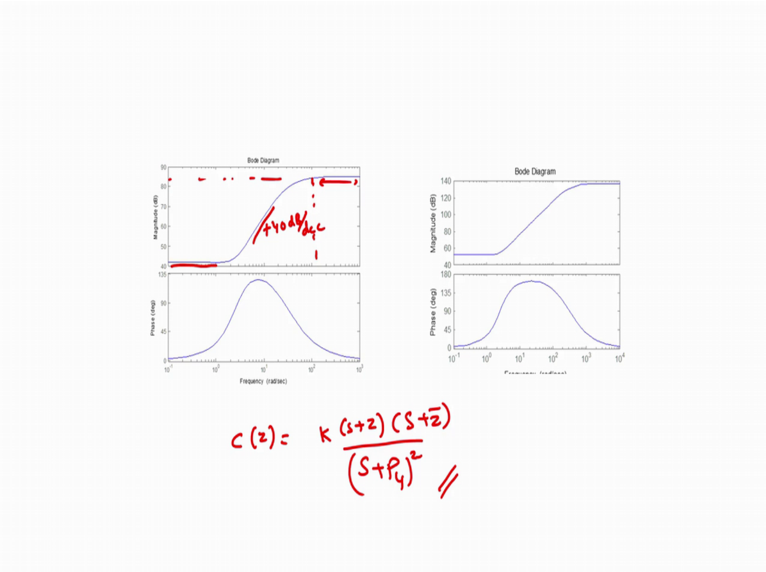

So, if we look at the controller structure in all cases the controller had a structure of the kind that has been shown here. It was of the form K times S plus Z times S plus Z bar divided by S plus P 4 a square because we are assumed that the controller has coincident poles we have the term S plus P 4 the square appearing in the denominator. And, we went over the elaborate design step necessary to determine the locations of the Z of the zeros Z and Z bar which are intended to restrict the variation in the closed loop pole due to changes in the plant parameters.

And, the gain K of the feedback controller ensures that the for the nominal value of the plant parameters the closed loop poles are located exactly where we want them to be located namely at P cl naught and P cl naught bar. Now, if one were to draw the Bode plot of a controller that has this particular structure we note that when we substitute S is equal to j omega and look at the magnitude of the controllers transfer function for frequencies that are below the corner frequencies of both the 0, as well as the poles of the controller then the magnitude response will be flat as seen from the plot here.

Now, when the when we approach the corner frequency of the 0 Z or Z bar then beyond this corner frequency there will be a 40 decibels per decade rise in the magnitude characteristics because, we have two zeros contributing to the rise in the magnitude characteristics. So, this is the region where we have plus 40 decibels per decade rise in the magnitude characteristics.

But, then at some far away location we have the pole two poles P 4 and P 5 which are coincident and at that particular location you would have a minus 40 decibels per decade characteristic that gets added on to the plus 40 decibels per decade characteristic of the two zeros. And, hence beyond that frequency you would once again have a flat characteristic since these two contributions cancel each other out. So, this is once again going to be about 0 dB per decade slope.

What is a little worrisome about this characteristic is that for frequencies beyond P 4 we note that the magnitude of the controller transfer function has to be quite high namely around 80 decibels per decade. This may be because the example we considered was a peculiar one where we assume very large variation in plant parameters and we imposed a restriction that the closed loop pole should very by a very small amount in response to variation in the plants parameters. And, this is what has resulted in such a high magnitude for the controllers gain for frequencies beyond P 4.

What is more worrisome though is that this gain has to be maintained for all frequencies starting from P 4 and all the way up. So, all the way up to infinite frequency we need to have the magnitude characteristic of the controller b equal to 85 or so, decibels and this is an impractical proposition there is no physical controller that can give us infinite gain bandwidth product which is what is demanded by a controller of this particular structure.

So, how do we address this issue and how do we achieve a controller which has a finite gain bandwidth product this is the next question that I want to address. To address this question let us return to the steps that we had undertaken in order to obtain the structure of this controller and determine the locations of the poles P 4 and P 5.

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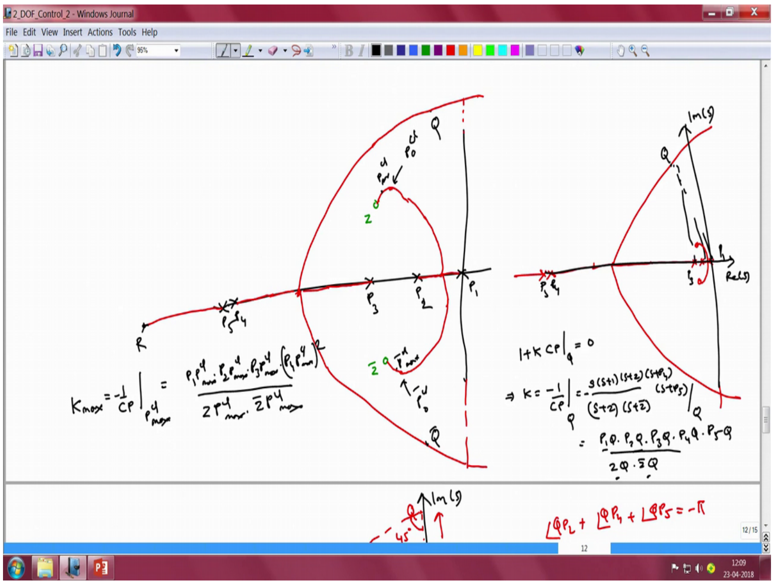

So, in order to determine the locations P 4 and P 5 of the controller we first noted that the root locus of the overall system would have branches, but crossed over from the left half of the complex plane to the right half of the complex plane on account of the two poles P 4 and P 5. And, we considered a point Q on this branch and we demanded that the gain at which the root locus process over or the gain at which the point Q will cross over from the left half of the complex plane to the right half of the complex plane should be greater than the maximum gain that the overall open loop system would assume when we have variation in plants parameters.

So, this was what was captured by the equations that we wrote three clips back we determine the maximum value of gain which is K max in terms of the distances P 1 P cl max, P 2 P cl max, P 3 P cl max, Z P cl max and Z bar P cl max all these terms for known the only term that does not known as P 4 P cl max the square and that was what had to be determined and that had to be less than K Q and K Q was essentially P 1 Q P 2 Q P 3 Q divided by Z q times Z bar Q times P 4 Q times P 5 Q and these two terms if there if P 4 and if P 4 and P 5 are coincident poles these two terms will simply be equal to P 4 Q the square.

Now, we noted that since the points P 4 and P 5 for assume to be very far away from the origin then in a plot where we indicate the locations of the three of the plants poles and the two of the controller zeros these three plant poles and two controller zeros that are near the origin will appear as though they just constitute a single lumped pole at the origin. In other words the big picture of the root locus gets simplified to something that looks like this.

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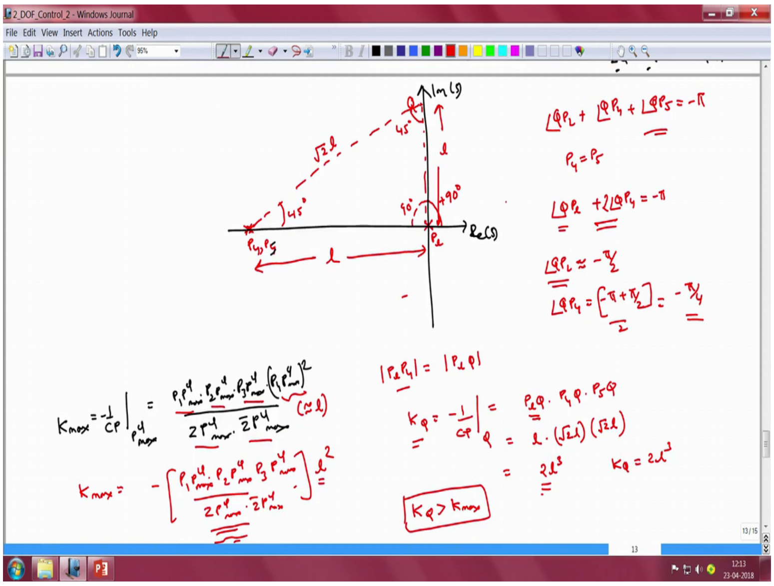

We have one lumped pole at the origin that is the effective contribution of the three plant poles and the two controller zeroes near the origin and then we have the we have the far away controller poles P 4 and P 5 which together result in one of the branches of the root locus namely the branch on which the point Q lies to cross over from the left half to the right half of the complex plane.

In order to determine the location of the point Q we first apply the angle criterion of the root locus and that is what has been done here. At the point Q the sum of the angle subtended by all the open loop poles and zeros has to be equal to minus pi and we have three open loop poles here one effective pole near the origin which I have called as a P l and the two poles P 4 and P 5 and the three angles together should add up to minus pi.

Since P 4 and P 5 are coincident poles we get the angle subtended by the first pole near origin which is the angle Q P l plus 2 times the angle Q P 4 should be equal to minus pi.

And, since the point P l is situated close to the origin and we are trying to determine the point Q at which the root locus crosses over from the left half to the right half of the complex plane we noted that the angle Q P l is minus pi by 2. And, that allowed us to determine that the angle subtended by each of these poles P 4 and P 5 at the point Q has to be equal to minus pi by 4 or equivalently 45 degrees.

And, with that simplified picture available to us we could determine the distance of the point Q from the origin to be equal to the distance of the poles P 4 and P 5 from the origin they both had to be equal to l and we could then express K max some constant times l square because the term P 4 P cl max represents the distance of the point P 4 from the point P cl max which is situated very close to the origin. So, for all practical purposes we could assume that the distance of the point P 4 from P cl maxes nearly equal to the distance of the point P 4 from the origin itself and hence we could replace the term P 4 P cl max with the term ls l and we would get l square in the expression for K max.

And, likewise in the expression for K Q we had K Q to be equal to P l Q times P 4 Q times P 5 Q, P 4 and P 5 are coincident poles. So, those distances are the same from the fact that this is a right angle isosceles triangle we noted that P 4 Q and P 5 Q are both equal to root 2 l, where l is a distance of P 4 from the origin and hence and P l Q was equal to l and hence K Q was equal 2 l cube. So, K Q was equal to 2 l cube and K max was given by this constant times l square by setting K Q to be greater than K max we can work out what the value of l had to be. Now, these were the steps that we have adopted in order to obtain the position of the poles P 4 and P 5 and complete the controller design.

Now, suppose we want a controller with a fine finite gain bandwidth product we note that we cannot have the denominator polynomial of the controller be exactly of the same degree as a numerator polynomial of the controller. This is precisely what landed us in the trouble that we just talked about a few minutes back we ended up with a controller with infinite gain bandwidth product.

So, we have to have a controller that is strictly proper or in other words the number of poles of the controller should exceed the number of zeros. So, since we have to have two zeros in order to restrict the variation of the closed loop dynamics of the overall system we need to have at least three poles for our controller. The question now becomes how does one determine the location of these three poles. To address this question we can follow the exact same steps as what we have done so far.

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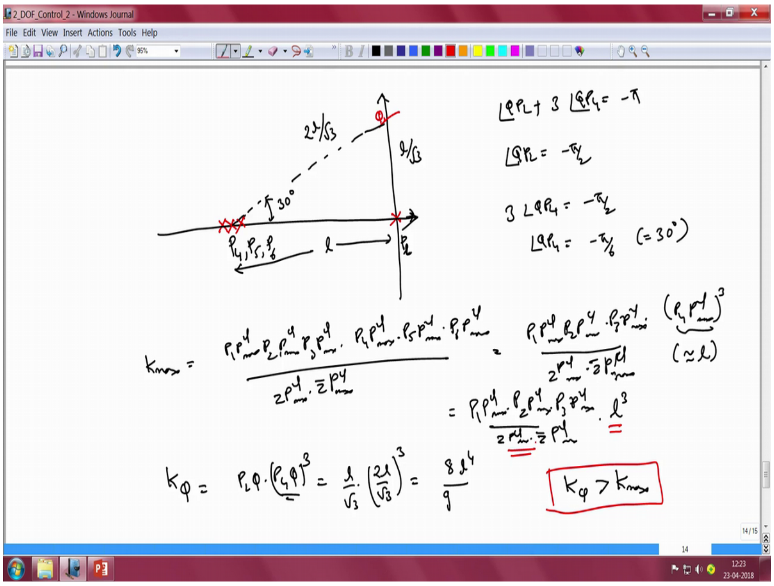

So, if we assume that we have three poles P 4, P 5 and P 6 and Q represents the location at which the root locus cross over into the from the left half of the complex plane to the right half of the complex plane and P l represents a lumped pole at the origin. Then to determine the location of Q we can once again start by applying the angle criteriol. So, the angle of Q P l plus 3 times the angle of Q P 4 should be equal to minus pi this is under the assumption that the points P 4, P 5 and P 6 are going to be coincident or in other words there all be going to be sitting at the same location in the complex plane.

Now, once again we note that the angle Q P l is going to be equal to minus pi by 2. Hence we would have 3 times angle Q P P 4 to be equal to minus pi by 2 or other words angle Q P 4 to be equal to minus pi by 6 or equivalently it is going to be equal to 30 degrees.

So, unlike the previous case where we had two poles for the controller if we choose to go with three poles for the controller the angle subtended by these three poles assuming that their coincident is going to be equal to 30 degrees at the point Q instead of 45 degrees as it happened in the previous case. So, if I want to connect these two poles which are assumed to be coincident to the point Q then the angle formed by this line with respect to the real axis is going to be equal to 30 degrees.

Now, what we need to determine is a distance of each of these poles from the origin. So, what we need to determine is the distance l which has been shown here now from this triangle we note that P l Q is going to be equal to l by root 3 because this is a triangle a right angle triangle with one of the angles being 30 degrees and the hypotenuse is going to be equal to 2 l by root 3.

So, we would have K max to be equal to P 1 P cl max P 2 P cl max P 3 P cl max times P 4 P cl max times P 5 P cl max times P 6 P cl max divided by Z P cl max times Z bar P cl max. And, since we have chosen P 4 P 5 and P 6 to be coincident we would have this to be equal to P 1 P cl max times P 2 P cl max times P 3 P cl max times P 4 P cl max the cube divided by Z P cl max times Z bar P cl max.

Now, we noted that the distance P 4 P cl max represents a distance from the point P 4 to your point that is very close to the origin situated near the dominant poles minus 1.5 plus minus 2 j. Hence for all practical purposes this distance from the point P 4 to the point P cl max is approximately going to be equal to the distance of the point P 4 from the origin itself. So, this distance going to be approximately equal to l. Hence, we would have K max to be equal to P 1 P cl max times P 2 P cl max times P 3 P cl max divided by Z P cl max times Z bar P cl max times l cube that is going to be the value of K max.

However, K Q is going to be equal to P l Q times P 4 Q the cube I am taking P 4 Q the cube because we have 3 poles namely P 4, P 5, P 6 which are all coincident and therefore, are at the same distance from the point Q and we note that if this is a right angle triangle with the angle at the point P 4 being equal to 30 degrees then from trigonometry we see that the distance P l Q is going to be equal to l by root 3. So, that is one term and the hypotenuse of the triangle which is going to be equal to the distance P 4 Q is going to be equal to 2 l by root 3 and that is going to get cubed. So, we are going to have a gain at which the point Q crosses over from the left half to the right half of the complex plane to be given by 8 times l power 4 by 9.

Now, in order to make sure that none of the closed loop poles are going to be on the right half of the complex plane when the plant parameter changes by the maximum possible value by which it can change we have to ensure that K Q should be greater than K max the same equation has to be applied and we note that K max has a term l cube times some constant whose numerical value we already know and we note that K Q is given by 8 by 9 l power 4.

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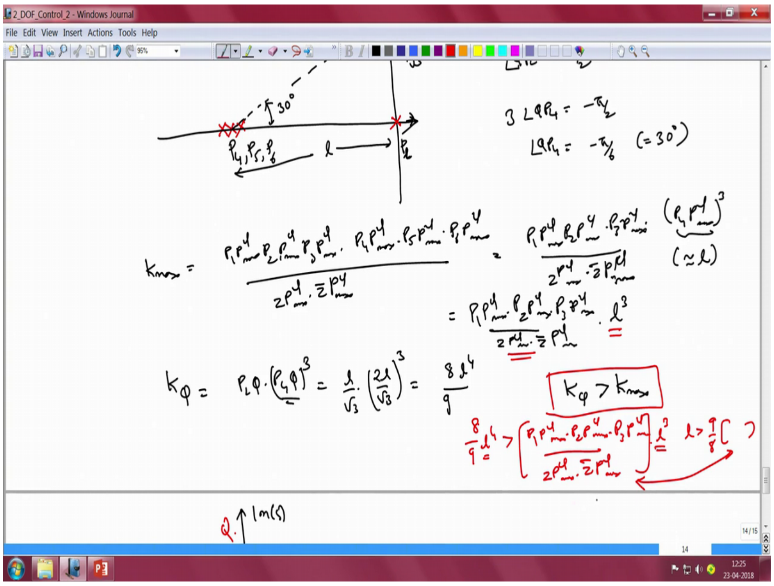

So, if we apply this inequality K Q greater than K max we would have 8 by 9 l power 4 should be greater than P 1 P cl max times P 2 pc l max times P 3 P cl max divided by Z P cl max times Z bar P cl max times l cube and l cube here cancels with l power 4 there. So, that we would get l should be greater than 9 by 8 times this term here the term within the square bracket which I shall not write again in the interest of brevity.

So, by applying this equation we can now obtain the new locations where the points where the controller poles P 4, P 5 and P 6 have to be located and if we do that and finally, compute the controller gain K for this new controller we would end up with a controller whose denominator polynomial is of third degree while a numerator polynomial would be of second degree because of the two controller zeros that we have chosen and hence we would have a controller with a finite gain bandwidth product.

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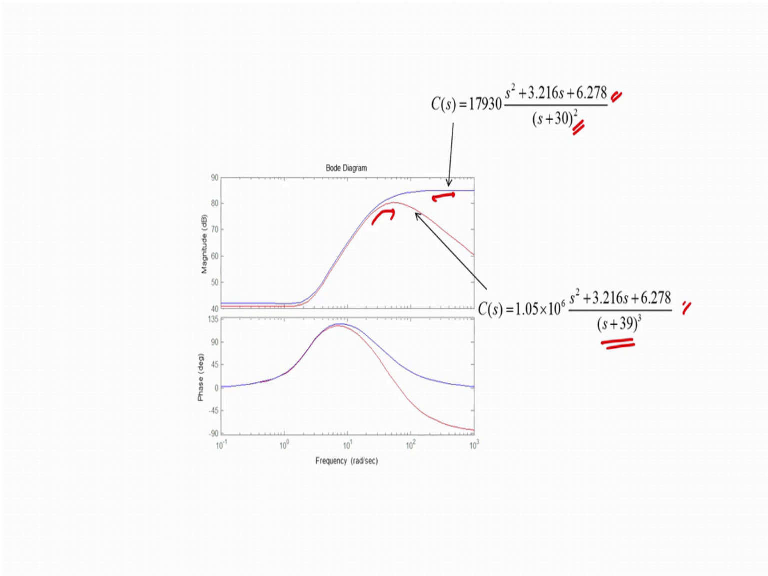

To illustrate this we have undertaken the design for the case for the first example that we had taken up and we see that in the blue trace is the original controller which had just a pair of poles along with the two zeros that was added and this controller has infinite gain bandwidth product and is therefore, not practically realizable.

In contrast through the slight modification in the design steps that we just undertook we have ended up with a controller whose Bode plot is shown by the red trace here and this controller has three poles and two zeros and this hence a strictly proper controller and we see that beyond the corner frequency of 39 radian per second the magnitude characteristic falls down at minus 20 dB per decade and hence this controller will have a finite gain bandwidth product and hence we would also be practically realizable.

The third and final aspect of root locus based 2-degree of freedom control design that I wish to discuss is concerning the number of lumped poles that we have near the origin. If we return to the original root locus of the overall system we notice that we could exploit the fact that the three of the plant poles and the two controller zeros were very close to the origin compared to the two controller poles P 4 and P 5 in order to lump the effect of these three poles and two zeros as just a single lumped pole at the origin.

If the plant had four poles instead of three poles and we have added our two controllers zeros then the effective number of poles or the lumped poles near the origin would have been two; namely the difference between the number of poles and the number of zeros.

Because, very far away from the origin this collection of n poles and m zeros we simply look like n minus m poles sitting at the same location.

Hence can we allow the number of poles near the origin to be anything other than one can it be 2, can it be 3 or can it be 0. This is the question that we shall try to answer in this last segment.

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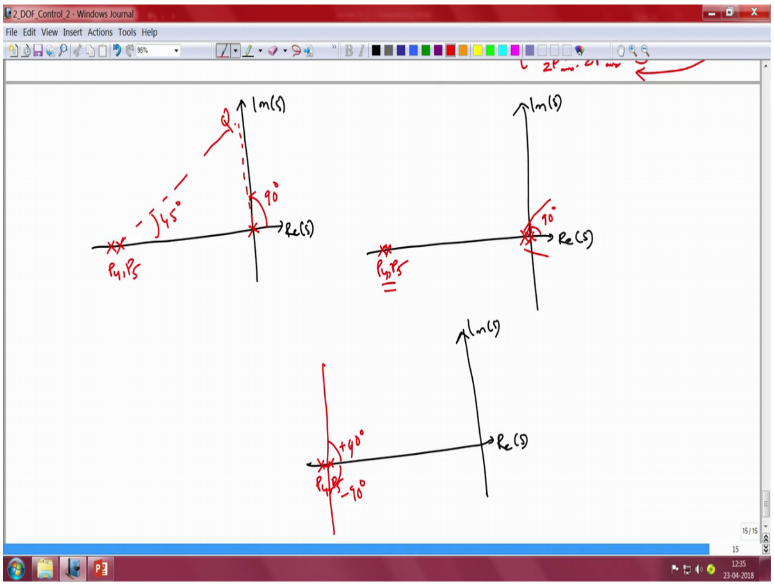

So, in the first example we consider that we had one pole near the origin and because we had one pole near the origin if we consider the point Q on the branch that was about to cross over from the left half to the right half of the complex plane we could conclude that this one lumped pole added subtended an angle of minus 90 degrees at the point Q the negative sign comes because this is a pole write a subtending the angle and not the 0.

And, hence the other poles P 4 and P 5 needed to together subtended an angle of 90 degrees at the point Q and if the points P 4 and P 5 were assumed to be coincident then each of them ended up subtending an angle of 45 degrees at the point Q. But, suppose instead of having a single lamped pole near the origin we had two lumped poles near the origin. I shall redraw the complex plane in order to depict this scenario. So, we have two poles near the origin and we also have above poles P 4 and P 5.

What is going to be the consequence of having two poles two lumped poles near the origin. We would have these two lumped poles if our plant was a fourth order plant and we had just two controller zeros. Then we note that in the vicinity of these two lumped poles each of these poles subtend an angle of 90 minus 90 degrees. And, hence the two together subtend an angle of close to minus 180 degrees. What is indicates is that the root locus will cross the imaginary axis at a point it is situated very close to these two lumped poles.

In other words, I will have no control over the location at which the root locus crosses over from the left half to the right half of the complex plane by appropriately positioning the far away controller poles P 4 and P 5. This is in contrast with the first case where by appropriately locating P 4 and P 5 at a sufficiently far away distance from the origin we could get our closed loop system to be stable even when the plants gain assume its maximum value possible.

However, in the case where we have two lumped poles near the origin since the root locus crosses the imaginary axis at a point that is very close to these two lumped poles because each of these poles contribute to approximately minus 90 degrees phase. And therefore, together the contribute to close to minus 180 degree phase this far away controller poles have very little role in determining the location at which the root locus crosses over. Hence, there is no way in which we can guarantee that our closed loop system is going to be stable by appropriately positioning the points P 4 and P 5.

So, what is the remedy if we are stuck with such a situation? The remedy is to add one more controller 0 near the origin. So, that the net number of poles and zeros near the origin would be 1 instead of 2. This problem also exists if we have the number of poles lumped poles near the origin to be more than 2. So, if it is 3 or 4 or any other number apart from 1, then we would have no control over the location at which the branches of the root locus cross over from the left half to the right half of the complex plane by suitably changing or controlling the positions of the points P 4 and P 5, the far away poles of the controller.

Hence, if we are lucky then for the maximum variation in the plants parameters our control system would be stable, but there is no way whereby we can design the control system to be stable by suitably choosing the structure of the controller. Hence we need to make sure that the number of lumped poles near the origin is 1 or less.

Now, if you look at the next situation which is the opposite of what we have been looking at now namely where we have no poles at all near the origin. Or in other words we have the complex plane and in the complex plane there are no lumped poles near the origin and we just have the two controller poles P 4 and P 5 what will the consequence be. We note that if we have no lumped poles near the origin then it means that all the branches to a root locus near the origin that have originated from the poles near the origin have already sunk into the corresponding zeros near the origin and there are no branches coming out.

So, the only branches will be the branches coming from the points P 4 and P 5. And, if P 4 and P 5 are separated from one another then you would have a branch originating from each of them they collide at a at the midpoint and then take away into the complex plane. The angle at which the asymptotes of these two branches of root locus tend to infinity will be plus minus 90 degrees.

So, what we would see in this case is that we would have no concern about the instability of the overall closed loop system because, of these branches of root locus. That is because these branches of root locus will always subtend an angle of plus 90 degrees or minus 90 degrees with respect to a real axis and hence will never cross over from the left half to the right half of the complex plane.

So, since no issue of instability arises in this scenario; there is no need for us to undertake the elaborate design steps that we undertook in the first case where, we had a single lumped pole near the origin in order to determine the location of the points P 4 and P 5 to ensure that the closed loop system is stable even for the maximum permissible variation in plant parameters.

Hence the last mode in association with 2-degree of freedom control design performed using the root locus is that we have to ensure that the number of lumped poles near the origin or in other words the difference between the number of poles and the number of zeros placed near the origin should be equal to either 1 or 0 in order for our closed loop system to be stable. If we are given a plant whose degree is such that the number of lumped poles is more than 1 than we have to add enough controller zeros near the origin to ensure that the net number of lumped poles near the origin remains either equal to 1 or 0.

With this we conclude our discussion on 2-degree of freedom control design to achieve robustness against planned parameters performed using the tool of root locus.

Thank you.