Control System Design
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Lecture – 03
Homogeneous linear time invariant ordinary differential equations

Hello in the previous clip, we discussed what linear time invariant systems are and we converged on systems with a particular structure for the input output relationship which will which we will be looking at for the rest of this course. So, on this slide here I have indicated the structure.

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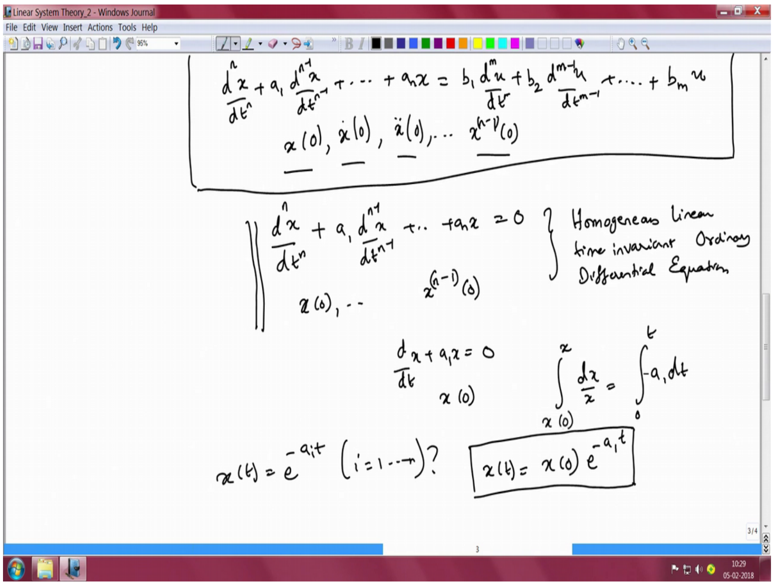

So, differential equation that we assume that the systems we are interested in would obey is of this kind. So, nth derivative of x with respect to t plus a 1 times nth n minus 1 derivative of with respect to t and so on plus a and x is equal to b 1 times mth derivative of u with respect to t and so on and so forth plus b m u. So, where u is the input and x is the output. So, it is not enough for us to just write down the differential equation that the system obeys, we also have to specify its initial conditions.

So, since our system is assumed to obey an nth order differential equation we would have n initial conditions namely x of 0 x dot of 0 x double dot of 0 and up to x n minus 1 of 0. Now, as control engineers we discussed that since we are interested to get systems to do over bidding. It is very worthwhile for us to understand how to obtain the response x of t for any specified input u of t.

Now if you look at this initial if you look at this differential equation, you discover firstly, that the number n, the index n is any arbitrary value, it could be an arbitrarily high order differential equation and so is m. So, m can also be a very high arbitrarily high value and you can therefore have a large number of initial conditions as well. Also you notice that u of t can be any particular input, I mean let us probably we can at best say it can be continuous or piecewise continuous inputs, but otherwise there isn't any particular restriction on u of t.

So, what you discover therefore, is we are faced with a fairly daunting challenge namely to take down a problem. A to solve a different equation whose order is can be anything and to which the input can be anything and the n initial conditions can also be any particular values that one might choose to assign.

But what is quite elegant and attractive about linear system theory is that despite the intimidating appearance of this fairly large order differential equation, its possible for us with a combination of intuition and analogies and so on. To take down this big problem, but we have to do it in a step by step manner and we cannot do it in one shot. So, for the moment let us assume that we have forgotten most of our you know, undergraduate mathematics. And we just want to use whatever we have learned in our pre university in our 11th and 12th standards to solve this differential equation.

And once again I want to underscore the elegance of linear system theory, in that just this amount of knowledge is enough for us to solve a differential equation of this level of sophistication; where n is arbitrary, m is arbitrary, u is arbitrary and all the initial conditions are arbitrary.

So, in the first step we shall right away acknowledge our difficulties with dealing with all these arbitrary values for n m and so on and so forth. Let us try to simplify the problem and see if after simplifying it adequately we can take it down, we can solve the problem. Then once we succeed at that game then we shall see if it is possible through combination of guesswork and intuition and so on.

To progressively increase the complexity until we are in a position to solve this fairly big differential equation. So, the first simplification that I would do is to set all the input the u and all its derivatives to 0. In other words the right hand side of the equation we shall set to 0. In which case we would get n equation, a differential equation of this kind d nth derivative of x with respect to time plus n minus 1th derivative of x with respect to time and so on and so forth, plus a n of x equal to 0.

And we once again have the n initial conditions x n minus 1 derivative of 0. Now an equation where the input is 0 is called a homogeneous linear time invariant ordinary differential equation. So, it is linear because we are dealing with linear operators here time invariant because the coefficients a 1 to a n are assumed to be constants and ordinary differential equations because we don't have any partial derivatives.

Now, the question is can we solved this differential equation at least. Now if you stare at it for a little while you discover that once again we are although we have simplified the problem somewhat it is still not simple enough for us to readily guess the solution for this differential equation.

That is because n is still an arbitrary number arbitrary it can it can even be it can be arbitrarily large and one is not it is not very clear, how starting from our knowledge of pre university calculus one can solve this differential equation. So, let us try to simplify it even further and see if in its most simple form we can solve this differential equation. So, the most simple form of course, is if we remove all the derivatives in which case it just reduces to a n of x a n times x equal to 0 which is a trivial case I shall not look into that.

But I shall look into the next most simple form for this differential equation namely d by d t of x plus a 1 times x equal to 0. And this has one initial condition namely x of 0 can we solve this differential equation at least and indeed. It turns out that it is not too difficult to solve this is something that can be handled by a student in his eleventh standard or his or her eleventh standard or twelfth standard and how do we do it. So, we integrate we cross multiply and have d x by x integral of d x by x is equal to integral of minus a 1 d t. And the limits of time are 0 to t and the initial conditions go from x of 0 to some initial to some value x. And one can easily show that the solution after simplification of this integral is going to look something like this x of t is equal to x of 0 e to the power minus a 1 times t.

So, in its most simple form we have managed to solve the differential equation, but we are still very far away from what we really are interested to do. Namely to solve this fairly big intimidating differential equation, but we have taken the first baby steps in the right direction. So, having solved it for the case when n is equal to 1 where in you have d x by d t plus a 1 x equal to 0.

The question is can we now do it for slightly higher values of n here is where we can take help from the solution that we obtained for the case that n was equal to 1. In this case we noticed that the solution was of the form x of t is equal to e to the power minus a 1 t. And what is a 1 a 1 was the coefficient of one of the terms in the differential equation. So, one can therefore, wonder whether a general solution to the homogeneous problem namely this one here can have a solution of the form x of t is equal to minus e power a i t.

Where I could be a 1 to a n is that possible is that possible to verify this one only need to substitute this particular form of solution into the differential equation. And one will quickly discover that this very straightforward extension of the special case of n equal to 1 is not going to work for higher values of n. So, x of t equal to minus e power minus a i t where i is a 2 a 3 a 4 a 5 and so on and so forth, is not a solution to the differential equation solution to the homogeneous linear time invariant differential equation. So, what is the way forward do we give up our hope all altogether actually not we can give it one other shot instead of looking for solutions of the form e power minus a i t.

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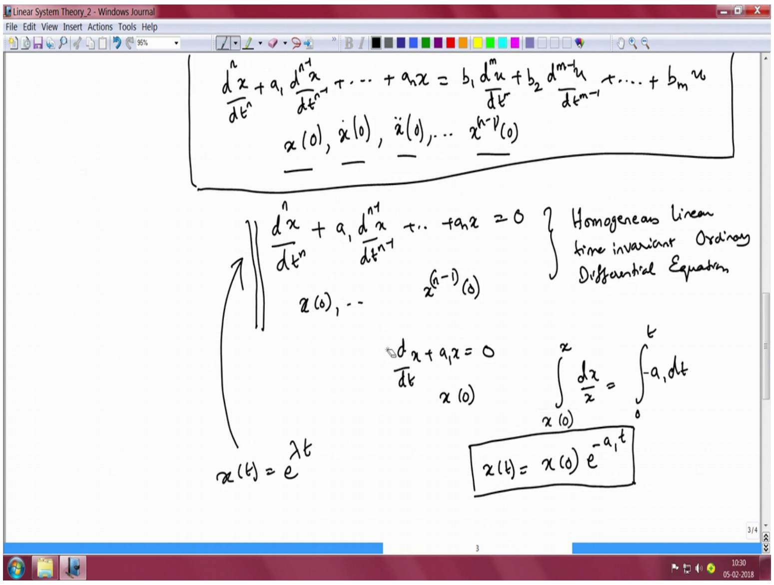

Is it possible at all that for some other value of coefficient lambda x of t equal to e power lambda t could be a possible solution? So, lambda clearly we discovered cannot be a 1 or a 1 to a n, but is there any other lambda for which this particular form of solution. Which we have obtained which we have seen in the case of n is equal to 1, could possibly also be a solution for the more general case of n for any arbitrary value of n? To verify this particular hypothesis, all we need to do is substitute this particular x of t into this differential equation here. If we do that what we get is the following.

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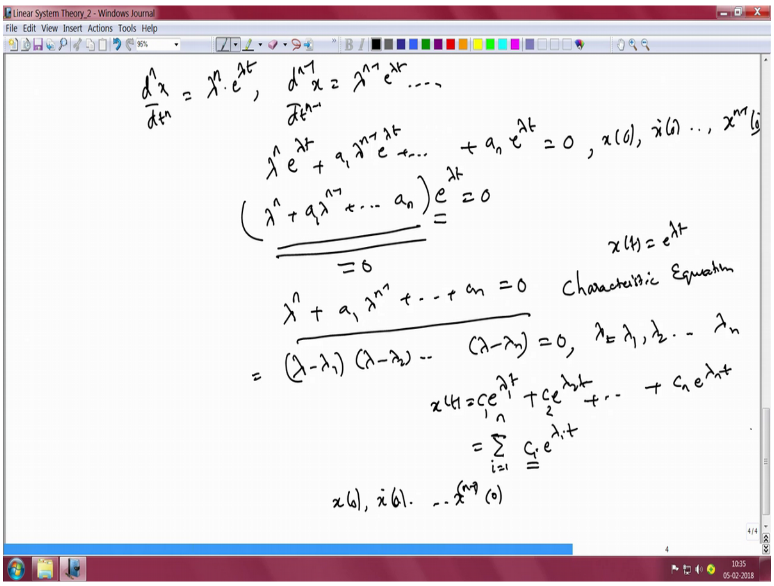

So, nth derivative of x with respect to t would be lambda power n times e power lambda t n minus 1th derivative of x with respect to t will be lambda power n minus 1 e power lambda t and so on and so forth. So, if I were to substitute it in the original homogeneous differential equation. If a solution of the form x of t equal to lambda e power lambda t exists then what should be satisfied is lambda power n e power lambda t plus a 1 times lambda power n minus 1 e power lambda t and so on and so forth plus a n e power lambda t should be equal to 0.

And of course, we would have the n initial conditions x of 0 x dot of 0 and so on so forth up to x n minus 1 of 0. Now for this to be true if I remove e power lambda t out lambda power n plus a 1 lambda power n minus 1 and so on plus a n times e power lambda t should be equal to 0. One can easily see that e power lambda t cannot be 0, for all t indeed for any t any finite value of t. So, therefore, we should have this other term being equal to 0.

So, I shall write that out separately here lambda n plus a 1 lambda n minus 1 plus a n should be equal to 0. So, if x of t equal to e power lambda t is a solution then the lambda should in turn satisfy this particular equation. And this equation is called the characteristic equation, characteristic equation for the particular homogeneous differential equation. Now how many solutions do we have? How many lambdas do we have that solve that satisfy this particular algebraic equation? We know that if you have an nth degree algebraic equation, then we would have n roots. In other words we can factorize this as lambda minus lambda 1 times lambda minus lambda 2 and up to lambda minus lambda n and that should be equal to 0. So, we would therefore, have n roots lambda equal to lambda 1 lambda 2 all the way to lambda n.

So, there are n specific values of lambda for which x of t equal to e power lambda t is a solution. Now since we are dealing with linear systems, if e power lambda 1 t is a solution and e power lambda 2 t is a solution, then we can easily show that owing to superposition and scaling e power lambda 1 t plus e power lambda 2 t is also a solution.

How do we confirm this? We just substitute this in the homogeneous differential equation and verify that it is a solution is an acceptable solution. Indeed, we can also show that c 1 times e power lambda 1 t plus c 2 times e power lambda 2 t is also a solution a c 1 and c 2 are 2 constants. More generally we have n roots lambda 1 to lambda n. So, c 1 e power lambda 1 t plus c 2 e power lambda 2 t and so on and so forth, up to c n e power lambda n t is a general solution to the homogeneous linear time invariant differential equation.

I shall write this in a more compact form as x of t is equal to sigma i going from 1 to n c i e power lambda i t. Now, we have n initial conditions; x of 0 x dot of 0 all the way up to x n minus 1 derivative or at the time t equal to 0. So, the coefficient c i which are at the moment unknowns can be fixed with our knowledge of the n initial conditions x of 0 to x n minus 1 of 0.

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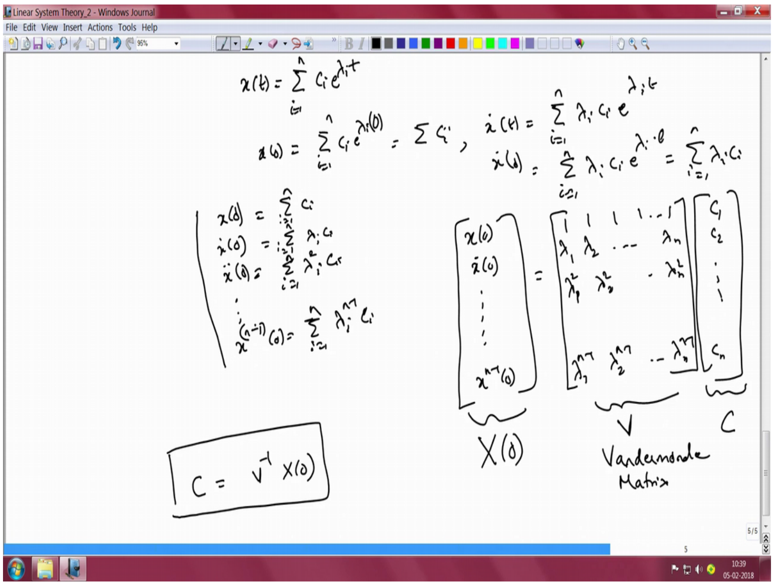

So, let us do that next. So x of t is equal to sigma i going from 1 to n c i e power lambda i t. So, x of 0 therefore, is equal to sigma i going from 1 to n c i e power lambda i times 0 which is basically sigma c i. Likewise, x dot of t is equal to sigma i going from 1 to n lambda i c i e power lambda i t. Therefore, x dot of 0 is equal to sigma i going from 1 to n lambda i c i e power lambda i times 0 which is equal to sigma lambda i c i; i going from 1 to n and so on and so forth.

So, we can assemble all the initial conditions and the corresponding coefficients here x of 0 is equal to sigma c i x dot of 0 is equal to sigma lambda i c i lambda i c i x double dot of 0 is equal to sigma lambdas square i c i. In all cases i goes from 1 to n, all the way down to x n minus 1 derivative of 0 that will be equal to sigma lambda i power n minus 1 c i; i going from 1 to n.

Now, we have n specified values on the left hand side x of 0 x dot of 0 and so on. These are the n specified values and c i c 1 to c n are the n unknowns and we have n algebraic equations. So, one can easily solve this equation in general assuming that we don't have any repeated roots and lambda 1 to lambda n are distinct. So, to cast this in a slightly better form, let me write this out in using vectors and matrices.

So, I shall on the left hand side write down the vector of initial conditions x of 0 x dot of 0 and so on and so forth, up to x n minus 1 of 0. And I shall call that capital X of 0 which is the vector of initial conditions. And on the right hand side, I have the vector of unknowns c 1 to c n c 2 all the way to c n. And these are related by some coefficients which are functions of the roots of the characteristic equation.

So, the first row you can show is all equal to 1 the second row is lambda 1 lambda 2 and so on, lambda n the third row is lambda 1 square lambda 2 square and so on lambda n square. The last row is lambda 1 power n minus 1 lambda 2 power n minus 1 all the way to lambda n power n minus 1. So, I shall call the vector of unknowns as capital C and I shall call the matrix of lambdas as v. Now there is a special name for this matrix v it is called the vandermonde matrix.

Now, we can obtain each of the elements of the vector c by inverting the vandermonde matrix and multiplying it with the vector of initial conditions capital X of 0. And this solves the first problem that we assigned ourselves namely to obtain the solution to the homogeneous, linear, time invariant ordinary differential equation. Now let us illustrate the steps that we undertook by means of a numerical example.

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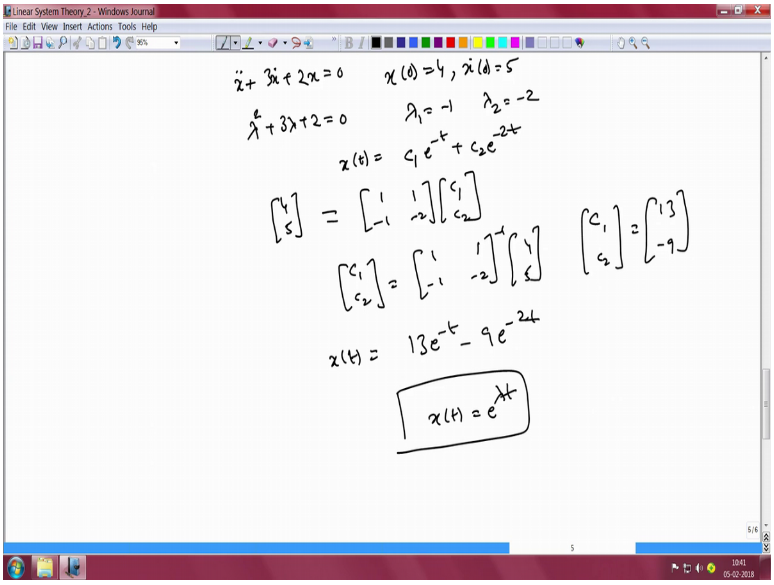

Let us take a differential equation of the form x double dot plus 3 x dot plus 2 x equal to 0, with the initial condition x of 0 equal to 4 x dot of 0 equal to 5. Now, we first write down the characteristic equation for this system which is lambda square plus 3 lambda plus 2 equal to 0. And there are 2 roots namely lambda equal to lambda 1 equal to minus 1 and lambda 2 equal to minus 2.

So, the general solution to this equation is of the form x of t is equal to c 1 e power minus t plus c 2 e power minus 2 t. To fix the values of c 1 and c 2, we employ the two initial conditions. So, if we use the language of Vandermonde matrices then we would have 4 comma 5, which is the vector of initial conditions is equal to 1 1 minus 1 minus 2 times c 1 c 2.

And therefore, c 1 c 2 is equal to this matrix, Vandermonde matrix minus 1 minus 2 inverse times 4 comma 5. And we find that from this c 1 c 2 vector will be equal to 13 minus 9, in other words x of t will be equal to 13 times e power minus t minus 9 times e power minus 2 t. There is just one additional aspect that I want to discuss in relation to our approach to solving the homogeneous linear time invariant ordinary differential equation.

The key point, the key step in solving this problem was our recognition of the fact that x of t equal to e power lambda t is a possible solution to this equation. Then we went forward and we found that there are n lambda such that satisfy this equation and then we could work out the entire solutions starting from that point.

Now, you might ask if x of t equal to lambda t is the only possible form for the solution. Is it possible that we have perhaps since we anyway depended on intuition and guesswork to guess this form for the solution? Is it possible that there exist other solutions, whose form we cannot guess that still satisfy this differential equation and still obey the same initial conditions, is that a possibility? What we shall try to show is that that is not possible and the solution is indeed unique. To do this we shall adopt what is known as proof by contradiction.

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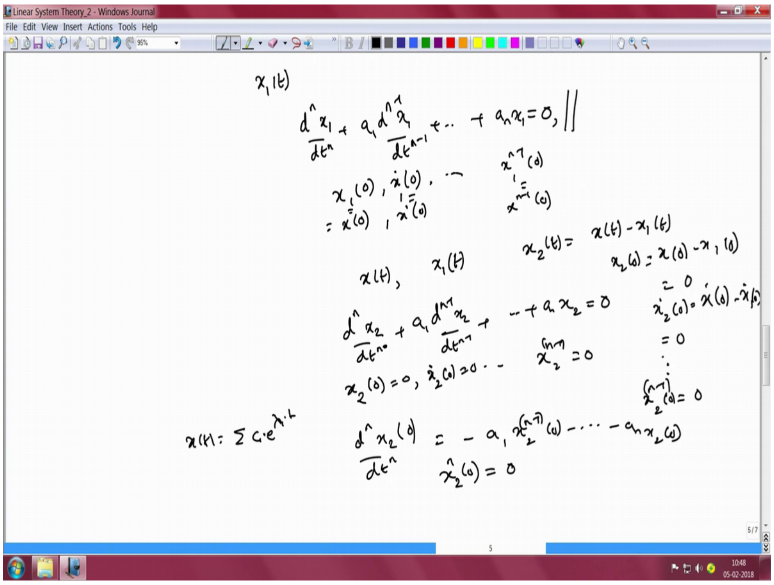

So, let us assume for the moment that there does exist another solution x 1 of t which satisfies the same differential equation, in other words nth derivative of x 1 with respect to t plus a 1 times nth derivative of x 1 with respect to n minus 1 th derivative of x 1 with respect to t and so on up to a n of x is equal to 0.

And the initial conditions of x 1 namely x 1 of 0 x 1 dot of 0 and so on are all equal to the original initial conditions x of 0 x dot of 0 and so on and so forth. So, x 1 of 0 is assumed to be equal to x of 0 x 1 dot of 0 is assumed to be equal to x dot of 0 and x n minus 1 derivative n minus 1 derivative of x 1 at time t equal to 0 is assumed to be equal to x n minus 1 of 0. Let us assume that there is such a solution x of t which we have not discovered.

Which has a different form compared to what we have just found out; namely, x of t equal to sigma e power lambda i t, and that is also a solution to this differential equation. Now suppose such a solution were to exist we shall show in the course of this discussion that x 1 of t cannot be different from x of t it has to be equal to x of t. Suppose that were to exist, then I know that since I am dealing with a linear time invariant differential equation. If x of t is a solution and x 1 of t is also a solution, then I can conclude that x 2 of t which is a difference between x of t and x 1 of t is also a solution to this differential equation, because superposition is valid in case of linear systems. Now if x 2 of t is a solution to this differential equation what does it mean? It means that nth derivative of x 2 with respect to time plus a 1 times n minus 1th derivative of x 2 with respect to time and so on and so forth plus a n times x 2 is equal to 0.

Now, what are the initial conditions for x 2? We have defined x 2 of t as x of t minus x 1 of t which means that x 2 of 0 is equal to x of 0 minus x 1 of 0 and we have assumed that x and x 1 have the same initial conditions. So, x 2 of 0 is equal to 0; similarly, you could show that x 2 dot of 0 is equal to x dot of 0 minus x 1 dot of 0. And both these 2 are also assumed to be the same we are assuming that x n x 1 have the same initial conditions.

So, that is also 0 and all the way down to n minus 1th derivative of x 2 and that is at time t equal to 0 is also equal to 0. So, if x 2 is a solution then x 2 obeys the same differential equation with this new set of initial conditions x 2 of 0 equal to 0, x 2 dot of 0 is equal to 0 and so on and so forth up to n minus 1th derivative of x 2 which are all 0.

So that is the equation that x 2 obeys. Now what can we say about x 2?If it obeys this differential equation and all its n minus 1 derivatives at time t equal to 0 are 0. Here I want to clarify that we are focusing on x 1 and x 2 which are both differentiable at all times t. So, there are they are not only differentiable, but are infinitely differentiable.

So, in other words if you look at the original solution namely x of t equal to sigma c i e power lambda i t, I could take the derivative of this any number of times I wanted, for all time t greater than 0. So, I am assuming that if there exists another solution x 1 then even that solution is infinitely differentiable at all times t. Why am I making this assumption? It is because there might be other solutions which do not which are not differentiable or so which are either not continuous or differentiable or something like that, but these are solutions that are not physically to be found.

So, therefore, we shall confine ourselves to x 2 and x 1 which are not only continuous, but also differentiable and indeed infinitely differentiable. So, we know that the first n minus 1 derivatives of x 2 at time t equal to 0 or 0. Which means that if I were to plug it into the original differential equation nth derivative of x 2 at time t equal to 0 would be equal to minus a 1 times n minus 1th derivative of x 2 at time t equal to 0 and so on and so forth up to minus a n times x 2 or time t equal to 0. And I know that all the terms on the right hand side are 0. So, therefore, the n minus nth derivative of x 2 at time t equal to 0 is also equal to 0, now I go forward with this particular argument.

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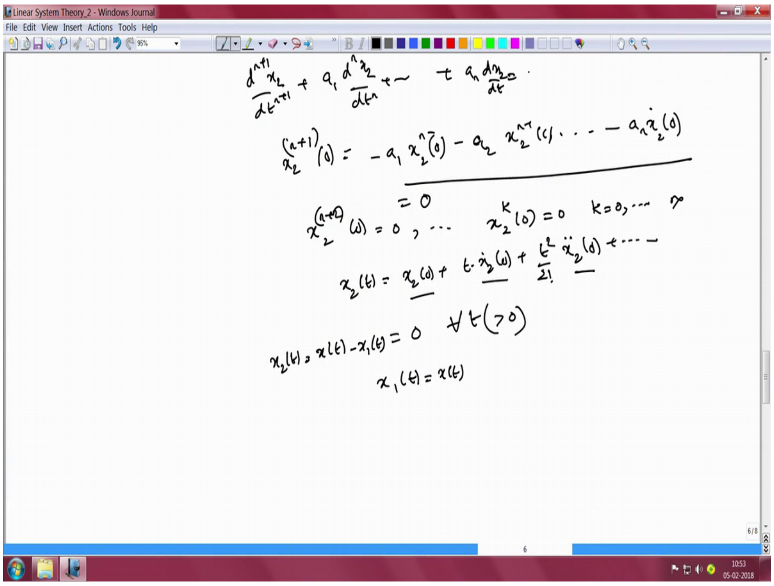

If I were a differential equation once more I would get n plus 1th derivative of x 2 with respect to t plus a 1 times nth derivative of x 2 with respect to time and so on plus a n times d x by d t equal to 0. So, I have I have just differentiated the original differential equation.

Now, if I evaluate n plus 1th derivative of x 2 n plus 1 at time t equal to 0. I would get that to be equal to minus a 1 times the nth derivative of x 2 at time t equal to 0 minus a 2 times n minus 1th derivative of x 2 or time t equal to 0 and so on and. so forth minus a n times x 2 dot at time t equal to 0. Once again we see that all the terms on the right hand side are 0.

So, I would get that the nth n plus 1th derivative of x 2 is also equal to 0. I can continue this argument further differentiate this equation once more and show that the n plus 2th derivative of x 2 at time t equal to 0 is also 0. And in general all the derivatives of x 2 at time t equal to 0 or 0 where k goes from 0 to infinity ok. So, now, since we are dealing with infinitely differentiable functions we have x 2 all of whose derivatives at time t equal to 0 or 0.

So, if you were to take a Taylor Series expansion of x 2 about the time t equal to 0, then I would have x 2 of t is equal to x 2 of 0 plus t times x 2 dot of 0 plus t square by 2 factorial times x 2 double dot of 0 and so on and so forth.

And we notice that each of these coefficients x 2 dot of 0 x 2 double dot of 0 and so on and so forth, are all 0 including x 2 of 0 which means that x 2 of t is 0 for all time t greater than 0. What is x 2 of t by definition it is x of t minus x 1 of t. And that is equal to 0 for all time t and what can be therefore, say about the solution x 1 we can therefore, conclude at x 1 of t has to be equal to x of t. In other words what this implies is that the solution for the homogeneous differential equation is unique and this is of great practical relevance because the technique that we adopted to find the solution for this homogeneous equation was a rather adhoc one. We started with a first order differential equation obtained the solution to it and guessed hope that it would be a possible solution thus the formula solution would be similar for higher values of n as well and indeed the turned out to be the case.

And subsequently we were able to solve the entire problem of solve of for the case of homogeneous linear time invariant differential equations. So, it was a rather ad hoc approach, but that allowed us to solve the equation. Now we were confronted with this question of whether there could be other such solutions which we couldn’t have guessed through our intuition and guesswork and so on.

But we were what we discovered is that such a solution cannot exist and this is used quite often when you are dealing with linear systems in that people try to guess the form of solution. And see if it works and if it does work this uniqueness property associated with the solution guarantees to them that if you have a solution that satisfies the differential equation and the boundary conditions then that is the only solution for the differential equation. In the next clip, we shall take our first baby steps in solving the slightly more sophisticated case where you have inputs on the right hand side, namely on the right hand side we shall not have 0, but we shall have some input let us say p 1 times u of t and see if we can solve that differential equation.