Control System Design

Prof. G.R. Jayanth
Department of Instrumentation and Applied Physics
Indian Institute of Science, Bangalore

Lecture – 29
2 – Degree of freedom robust control design for plants with uncertain gain (Part 2/2)

So, the next step, the second step in the design is to determine the location of the far off poles namely P 4 and P 5. Before we do that, let us first redraw the root locus for the system that we have we are in the process of designing.

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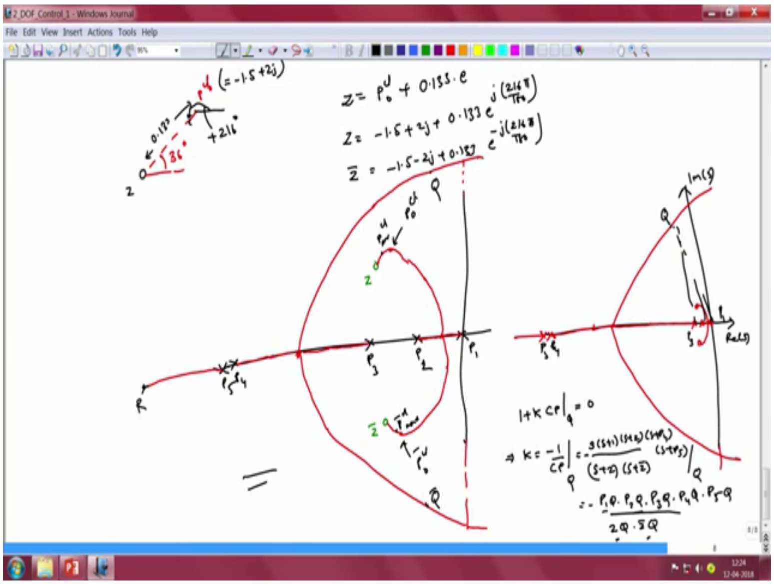

So, we have marked out the 3 plant poles P 1, P 2, P 3 which are at 0, minus 1 and minus 2 respectively and we wanted the dominant dynamics to be located at minus 1.5 plus minus 2 j and we have now discovered where we have to place our 0, Z and Z bar.

So, we have discovered the distance and the orientation of Z and Z bar with respect to the point P cl naught which is the point at which we want our closed loop pole to like. So, this is P cl naught, that is P cl naught bar.

Now, we also have two far away poles namely P 4 and P 5 we have not yet fixed the positions of these poles. It turns out that it is not an arbitrary choice and there is some caution that needs to be exercised while choosing the positions of P 4 and P 5. So, the second step in the design is to determine where we can place P 4 and P 5. But, now that we have placed the 0 Z and Z bar in the complex plane let us briefly sketch the approximate root locus for this system. So, we would have one branch of the root locus starting at P 1 and another branch starting at P 2 and both these branches meet at some point on the real axis and then breakaway into the complex plane and these branches will eventually go through the point P cl naught and finally, sink into the 0 Z and the other branch will sink it to the 0 Z bar while passing through the point P cl naught bar.

So, this is going to be the fate of two of the branches of root locus, then you would have one more branch starting from the point P 3 namely at s is equal to minus 2 and heading off to the left of the point P 3.

Now, since we have 2 poles P 4 and P 5 we would have one branch from this combination also heading off towards the pole P 3 and these two branches are going to meet at some location on the real axis and then they are going to have another breakaway and these two branches have no zeroes remaining for them to sink into. And, therefore, they break away and head away in this particular manner and there will be a particular gain at which they cross the imaginary axis. So, if the gain of the overall open loop system is more than this particular value then one pair of poles are going to lie on the right side of the imaginary axis or we would have an unstable system on our hands.

Now, all together we have five poles. So, we need to have five branches to the root locus and hence the fifth branch would start from the point P 5 and head towards its left towards s is equal to minus infinity along the real axis. So, for the closed loop pole on the branch starting from the point P 5, there is no concern associated with stability. However, for the branches that start from P 3 and P 4 and which meet at some point and then head off into towards the imaginary axis there is a real concern as regards the stability of the closed loop system.

In particular, what we need to make sure of is that when the plant gain reaches it is maximum value so, when the plan gain goes to 3 then where would we have the closed loop poles? We would have one pair of closed loop poles on the first pair of branches. So, we would have the closed loop pole at P cl max and P cl max bar. So, this will be where the first pair of poles would be located and by design they would be located in such a manner that it would be well within the prescribed variation of 0.4 units from the point P cl naught.

Then, we would have one more pair of poles closed loop poles and they would lie on this branch. So, I shall call that pole as Q and its complex conjugate as Q bar. So, we would have one pair of poles on the branch that emanates from P 4 and the other branch that emanates from P 3 and breaks off towards the imaginary axis and these two closed loop poles are not guaranteed to be stable because it is very likely that when the plant assumes its maximum gain possible it is very likely that the point Q is on the right half of the complex plane and thus our closed loop system might become unstable.

The final closed loop pole will be located to the right of the point P phi. So, the final closed loop pole will be located somewhere here. I shall call that closed loop pole as R and that is something whose stability is not of concern to us; it neither dominates the dynamics nor causes problems with stability. So, that is not of any concern to us. So, what we really need to be worried about now is the fate of the pole at the point closed loop pole at the point Q. So, we need to make sure that when the open loop gain of the system assumes it is maximum value which happens when the plant gain assumes its maximum value the point Q the closed loop pole position Q is utmost on the imaginary axis and never to the right of the imaginary axis. So, that is the second challenge that we have to take up and that is going to be part of the design in the second step.

Now, the root locus that I have drawn here is really not to scale that is because we have assumed the distances of the points P 2, P 3, P 4, P 5 and so on from the origin are all nearly comparable to one another so that they can all be shown on the same graph. However, as we discussed you have chosen to place the points P 4, P 5 very far away from the origin therefore, what would be a more realistic root locus would be something that looks like this I am redrawing the same root locus that I have drawn here, but by paying slightly better attention to the scale the typical distances between the points P 1, P 2, P 3 the point Z and Z bar and the points P 4 and P 5.

So, we would have the real part of S versus the imaginary part of S. Now, since the points P 1, P 2, P 3 and Z and Z bar are very close to the origin I shall indicate them by three cross marks each of which represent the points P 1, P 2 and P 3 and two circles which represent the two zeros Z and Z bar the all these are near the origin. In contrast we would have the point P 4 and P 5 to be very far away from the origin. So, this is probably a more realistic representation of the typical distances of these poles and zeros from the imaginary axis.

So, if we were to redraw the root locus that we have just drawn here we would have one branch starting at P 1 and one branch, but starting at P 2 and they both sink into the 0, Z and Z bar we would have one branch starting at the point P 3 and another branch starting at the point P 4, these two collide somewhere on the real axis and then they break off into the general complex plane and eventually cross the imaginary axis at some particular point and then you have another branch which is on the which starts from the point P 5 and heads towards the negative real axis and the closed loop pole on the branch will not pose any problems to us as far as stability is concerned.

The real problem would be posed by a pole Q on the branch that emanates from the points P 4 and P 3. So, this is the point P 3 this is the point P 4 we are really concerned about the stability of the pole Q. Now, if we were to be located at the point Q then we would have the root locus equation namely 1 plus K times C P at the point Q to be equal to 0 which means that K will be equal to minus 1 by C P at the point Q. So, if we were to once again recollect the geometric interpretation of the term 1 by C p this would essentially be equal to S times S plus 1 times S plus 2 divided by times S plus P 4 times S plus P 5 divided by S plus Z times S plus Z bar and this whole thing gets multiplied with minus 1.

So, that is K and this is evaluated at the point Q. Now, if we recall the geometric interpretation of the complex number S evaluated at the point Q it essentially is the complex number P 1 Q. So, the first term therefore, will be equal to P 1 Q and the second term would be the segment. So, S plus 1 evaluated at the point Q will essentially be equal to P 2 Q and S plus 2 will be P 3 Q and likewise you would have P 4 Q and P 5 Q as the other distances and in the denominator we have S plus Z and S plus Z bar. So, that would be Z Q and Z bar Q. So, these would be the terms.

Now, if you recollect we have chosen to place P 4 and P 5 very far away from the origin and what that implies is that the point Q therefore, is also very far away from the collection of poles P 1, P 2, P 3 and the zeros Z and Z bar. So, since the point Q is very far away from this collection of poles P 1, P 2, P 3 and Z. Z bar what happens is that, the length P 1 Q and P 2 Q and P 3 Q are very nearly identical to one another. The angles subtended by P 1, P 2, P 3 at the point Q will also be very nearly equal to 1 another and. So, will the angles subtended by Z Q and Z bar Q all of these subtend the same angle very nearly at the point Q and all these distances will very nearly be equal to one another.

So, here we have three poles near the origin and two zeros once again located close to the origin and very far away from the point Q. So, therefore, if I am sitting at the point Q this collection of three poles and two zeros will essentially look like a single pole to me because the effect of two of these zeros cancels the effect of two of the poles namely P 1 and P 2. So, effect gets cancelled both in magnitude and phase by the terms at Q and Z bar Q. So, for all practical purposes I can replace the three poles and two zeros near the origin with just a single lumped pole.

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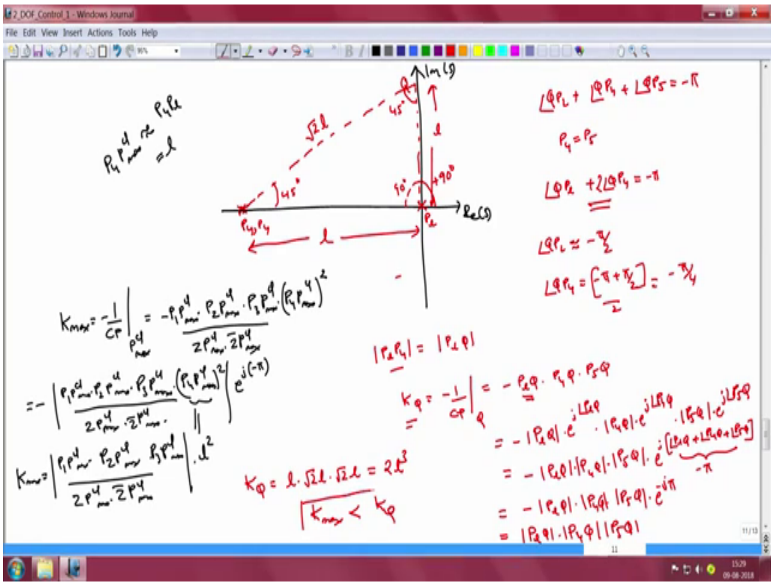

So, in this schematic here I have now replaced the three poles and two zeros which were near the origin with a simple single lumped pole which is how this combination would appear for the point Q which is on the branch of the root locus that is very far away from this combination of poles and zeros.

Now, we want to make sure that when our plants gain assumes its maximum value possible the point Q does not lie on the right half of the complex plane. At most it should lie on the imaginary axis although strictly speaking even this is not really desirable it should in general lies slightly to the left of the imaginary axis, so that we still have a stable closed loop system on our hands or in other words a system all of whose closed loop poles are on the left half of the complex plane.

So, we can get our point Q to be arbitrarily close to the imaginary axis, but not really cross the imaginary axis when the overall gain of the open loop system assumes its maximum value. So, in this situation when the overall gain is at it is maximum value let us assume that the point Q is located near the imaginary axis arbitrarily close to the imaginary axis. Since the point Q is a point on the root locus we should have the magnitude criterion and the phase criterion of the root locus to both be valid at this particular location.

So, let us first apply the phase criterion what it indicates is that the phase, the angle subtended by the poles P l which is the lumped pole at the origin the effective pole which is the combination of the three poles and the two zeros that were located close to one another. So, this one effective pole and the two far away poles P 4 and P 5 put together should subtend an angle of minus 180 degrees at the point Q since the point Q is a point on the root locus.

So, what this indicates is that the angle of Q P l plus the angle of Q P 4 plus the angle of Q P 5 should be equal to minus pi or minus 180 degrees. Now, we have not yet decided where to locate the points P 4 and P 5. In fact, we do not know what distance they need to be from the origin for the moment; let us assume that these two are coincident poles in other words the location of P 4 is the same as the location of P 5. Since we have two coincident poles our problem now boils down to determining the distance of these closed loop poles which I have indicated by the symbol l from the origin. If we can determine it is distance then we can determine where to locate the closed loop poles and what their exact numerical values will be.

So, since we have assumed them to be coincident we would have the angle of Q P 4 to be the same as the angle of 2 Q P 5. So, we would have angle of Q P l plus the angle of Q P 4 times 2 should be equal to minus pi. Now, what is the angle of Q P L since P l is a lumped pole near the origin and the point Q is a point that is very close to the origin we can note that the angle subtended by this complex number Q P l with respect to the real axis is actually equal to 90 degrees. So, this angle is 90 degrees.

However, since we have a pole near the origin the contribution of this pole is going to be the negative of this angle. So, Q P l is therefore, going to be equal angle of Q P l is therefore, going to be approximately equal to minus pi by 2 radian or in other words in degrees it will be minus 90 degrees. This is because the point P l is situated close to the origin and we are considering the case when the point Q is very close to the imaginary axis, but not to the right of the imaginary axis when the overall loop gain of our system has assumed its maximum value.

So, if Q P l is approximately minus pi by 2 then this equation here indicates that the angle of Q P 4 should be equal to minus pi plus pi by 2 divided by 2 and which essentially comes down to minus pi by 4. What this indicates that the coincident poles P 4 and P 5 should both subtend an angle of 45 degrees or minus pi by 4 radian at the point Q. So, this angle should be 45 degrees.

So, since we have this angle here to be 90 degrees and this angle is 45 degrees. So, in this triangle that has been formed by the point P l the point P 4 and the point Q we would have the other angle to also be 45 degrees which implies that this triangle is a right angled isosceles triangle and that in turn implies that the line P l P 4 should be equal in magnitude to the line P l Q. So, this also therefore, should be of the same magnitude l as the distance of the point P 4 from the origin.

All though the schematic that I have shown here does not indicate that these two lengths are the same our calculation reveals that because of the particular distribution of the poles in this particular system at the location Q the points P 4 and P 5 through each subtend an angle of minus 45 degrees.

Now, that we have determined the angle that the far away poles subtend at the location where the root locus crosses over from the left half plane to the right half plane, let us now apply the magnitude criterion of root locus at the point Q. Since Q is a point on the root locus we would have K q to be equal to minus 1 by C p evaluated at the point Q and since we have one lumped pole near the origin and two poles P 4 and P 5 at some distance away from the origin this would simply be equal to minus of P l Q times P 4 Q times P 5 Q.

And, this in turn is given by minus of magnitude of P l Q times e to the power j angle of Pl Q times magnitude of P 4 Q times e to the power j angle of P 4 Q times magnitude of

P 5 Q times e to the power j angle of P 5 Q which can be simplified to be equal to minus of the magnitude of P l Q times the magnitude of P 4 Q times the magnitude of P 5 Q times e to the power j of the angle of P l Q plus the angle of P 4 Q plus the angle of P 5 Q. So, this is going to be the expression, but since the point Q is a point on the root locus we note that the sum of the three angles P l Q, P 4 Q and P 5 Q should be equal to minus 180 degrees or in other words minus pi radians.

So, if you substitute that they would get this to be equal to minus magnitude of P l Q times magnitude of P 4 Q times magnitude of P 5 Q times e to the power minus j pi and we know that e to the power minus j pi is minus 1 and there is a minus one here at the start of this expression and the 2 cancel one another. So, that we get K q to be equal to simply the magnitude of P l Q times the magnitude of P 4 Q times the magnitude of P 5 Q. Now, what is the magnitude of P 4 Q and P 5 Q? Since P 4 and P 5 are coincident poles the distance of the point Q from the point P 4 or P 5 would be equal to root 2 times l and that is because the triangle P 4 Q P l is a right angle isosceles triangle and the sides of the triangle are both equal to l. So, the hypotenuse will therefore, be equal to root 2 times l.

So, if we exploit that fact we would get that K q will therefore, be equal to the magnitude of P l Q which is going to be equal to l times the magnitude of P 4 Q which is going to be equal to root 2 times l times a magnitude of P 5 Q which is again going to be equal to root 2 times l and hence take Q is going to be equal to 2 l cube.

Now, this gain K q should be greater than the maximum gain that the open loop system would achieve on account of the uncertainty that exists in the gain of the plant itself. Now, when the plant gain reaches it is maximum value the closed loop pole position P cl will be at the point P c l max and hence the gain K max of the overall open loop system when the plants gain is at its maximum value is given by K max is equal to minus 1 by CP evaluated at the point P cl max and what is that equal to that is equal to minus P 1 P cl max times P 2 P cl max times P 3 P cl max times P 4 P cl max times P 5 P cl max, but since P 4 and P 5 are coincident poles we would just have that to be equal to P 4 P cl max the square divided by Z P cl max times Z bar P cl max. So, this is going to be the expression.

Now, once again we can make a further simplification of this expression by noting that since the point P cl max is a point on the root locus we would have the sum of the phases of the complex numbers P 1 P cl max P 2 P cl max P 3 P cl max P 4 P cl max P 5 P cl max and the difference of this sum from the phases of the complex numbers at P cl max and Z bar P cl max that net algebraic phase contributed by all these different open loop poles and zeros at the point P cl max will add up to minus 180 degrees.

So, we can write therefore, K max as minus of the magnitude of P 1 P cl max times P 2 P cl max times P 3 P cl max times P 4 P cl max the square divided by Z P cl max times Z bar P cl max times e to the power j times the sum of all the phases which as in the case of the point Q will end up adding up to minus pi radians.

So, exactly as in the previous derivation that we undertook a few minutes back e to the power minus j pi is minus 1 and that minus 1 multiplies with the term minus 1 at the start of this expression and a 2 multiply to give us plus 1. Therefore, K max will be simply equal to the term within this modulus sign namely the magnitude of P 1 P cl max times P 2 P cl max times P 3 P cl max times P 4 P cl max the square divided by Z P cl max times Z bar is P cl max.

Now, in this expression since we know the point P cl max and we know the open loop pole positions P 1, P 2 and P 3 the 3 terms P 1 P cl max P 2 P cl max and P 3 P cl max are known to us. So, we can determine their magnitude without any trouble. Likewise the term Z P cl max is also a term whose magnitude is known to us because we have already fixed the location Z of the controller 0 and we also know where the closed loop pole P cl max would be located with respect to the point Z and likewise Z bar P cl max also is known because we have already fixed the locations of both P cl max and Z bar.

So, all these terms are known the only term that is not known is P 4 P cl max the square. Now, what is that term if we come back to the figure that we have here we note that P 4 P cl max refers to the distance from the point P 4 to the point P cl max and since the point P cl max is very close to the origin the distance from the point P 4 to the point P cl max is for all practical purposes equal to the distance of the point P 4 from the point from the origin itself. Hence for all practical purposes we would have therefore, that P 4 P cl max is approximately equal to P 4 P l which is the lumped pole at the origin. So, and this distance from this figure here is equal to l.

So, therefore, we would have that K max is equal to the magnitude of P 1 P cl max times P 2 P cl max times P 3 P cl max divided by Z P cl max times Z bar P cl max times l square because the term P 4 P cl max is equal to l square. So, in order for our closed loop system to be stable the gain at which the root locus crosses over from the left half of the complex plane to the right half of the complex plane should exceed the maximum gain that our open loop system would assume on account of plant parameter variations. So, in other words what we require therefore, is that the gain K max should be less than the gain K Q, this is what is required of us.

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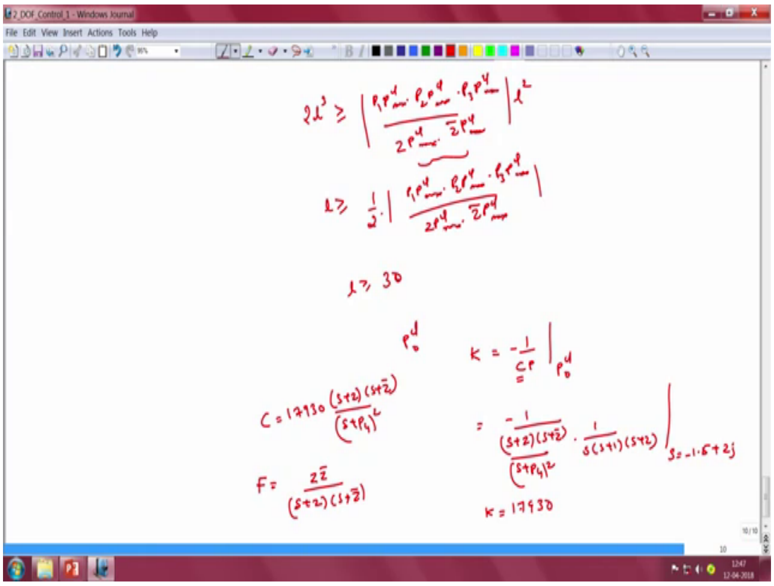

Now, let us write out the two terms once again. The gain K q is of course, given by 2 l cube and gain K q should be greater than K max in order for our closed loop system to be stable. So, 2 l cube should be greater than or equal to the magnitude of P 1 P cl max times P 2 P cl max times P 3 P cl max divided by Z P cl max times Z bar P cl max times l square and what this indicates therefore, is that the distance l which is the minimum distance at which the far away controller poles need to be placed away from the origin is given by l should be greater than or equal to 1 by 2 times the magnitude of this term above namely magnitude of P 1 P cl max times P 2 P cl max times P 3 P cl max divided by Z P cl max times Z bar P cl max.

Now, if you plug in the values for P cl max and P 1 and P 2 and Z and Z bar and P 3 and so on we would get that the distance l should be greater than or equal to 30. So, our poles

P 4 and P 5 should be at least 30 units away from the imaginary axis on the negative real axis of the complex plane. So, the other step that is remaining as far as the feedback control controller is concerned is to determine the gain of that controller. Now, since we know that the point P cl naught is a point on the root locus we can determine the controller gain by once again applying the root locus equation at the point P cl naught. We know that our overall open loop gain K is given by minus 1 by C p evaluated at the point P cl naught.

Since we have now determined a structure of our controller C we know that our controller has two poles and two zeros. So, it is going to be of the form S plus Z times S plus Z bar divided by S plus P 4 the square, that is going to be the structure of our controller and our plant nominally would be of the kind 1 by S times S plus 1 times S plus 2. So, if we evaluate this entire thing at the point S is equal to minus 1.5 plus 2 j then we obtain the controller gain K and with that we would be done with the design of the feedback controller.

If we compute the numerical value we find K to be equal to 17930. Therefore, our feedback controller C is given by 17930 times S plus Z times S plus Z bar divided by S plus P 4 the square. We are now done with the second step of our control design, we have determined in the first step the locations of the zeros in the second step you have determined the location of the far away poles and used both these pieces of information to determine the overall feedback controller of our system what is remaining is for us to determine the structure of the pre filter.

If you recollect the purpose of a pre filter is to cancel the effect of the controller zeros on the overall transmission function relating the reference to the output and hence we would have to choose a pre filter that has the terms S plus Z and S plus Z bar in the denominator. So, we have fixed a denominator of the pre filter by noting that the pre filter is intended to cancel the open loop zeros which are also going to be the closed loop zeros of our feedback system and since we want our pre filter to have unity gain we choose the numerator to be Z Z bar. So, that when we set S is equal to 0 in order to obtain the DC gain of our system we see that the gain of our pre filter would be 1.

And, with this we are done with two degree of freedom control design in order to achieve the specified amount of robustness of the closed loop system to the particular kind of variation we had in the plant parameters.

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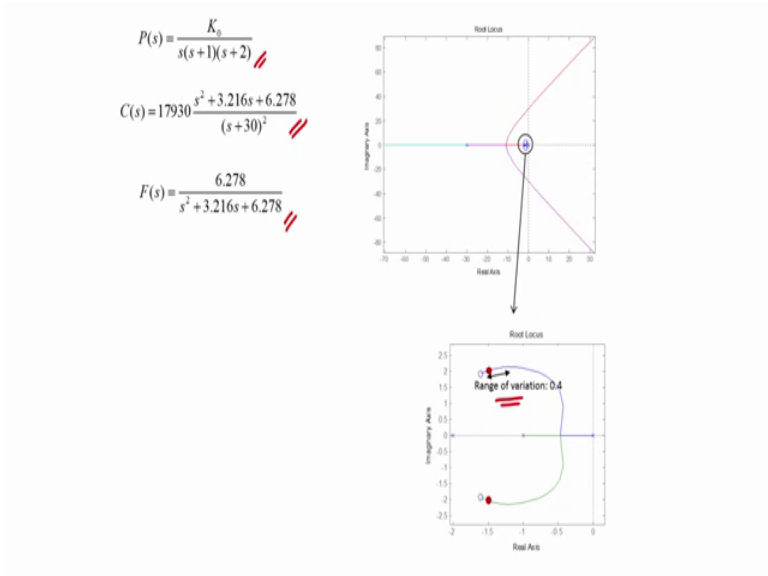

I have shown in this graph the overall root locus of the system. So, we have this plant whose gain K naught is uncertain it can vary by a factor of 10 nominally it is equal to 1, but it can reduce to 0.3 and increase to 3 and in order to make sure that the variation in the closed loop pole, the dominant closed loop pole due to the variation in the plants gain is within 0.4 units we need to have this as our feedback controller and that as our pre filter.

So, if you zoom in near the origin we will be able to look at the root locus in the vicinity of the plant poles P 1, P 2 and P 3 and we see that indeed we have succeeded in placing the closed loop pole at minus 1.5 plus minus 2 j as designed and when we change the open loop gain by a factor of 3 we can show that the range of variation of the closed loop pole is also within 0.4 units.

Now, as a consequence of restricting the variation of the closed loop pole to within 0.4 units we are we will also be able to accomplish a corresponding suppression in the spread in the transient response of our closed loop system when the plants model varies.

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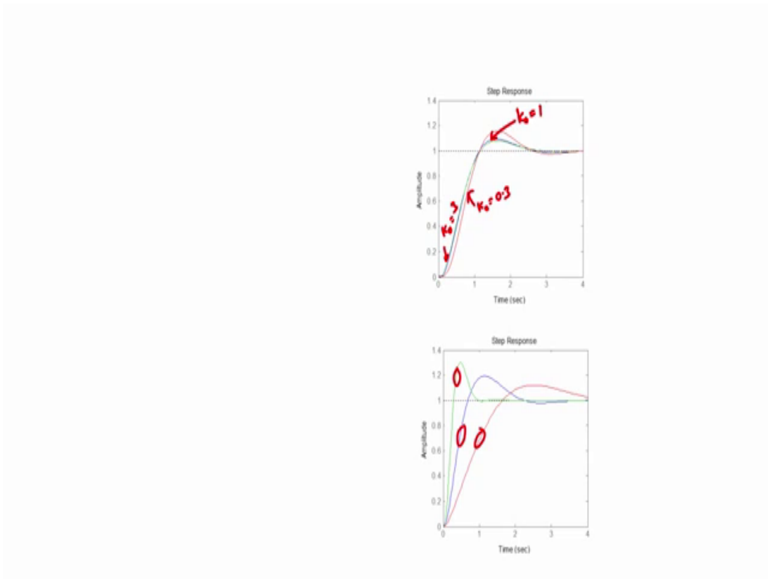

And, that is revealed by the closed loop response of our system here, where we see that the response of our system when the gain is at its nominal value and that is indicated by the blue trace here. So, this is when K naught is equal to 1 and when the gain is at it is maximum value and that is indicated by the green trace here and that is for the case when K naught is equal to 3 and the red curve here is for the case when K naught is equal to 0.3 and all of all of these three sit almost exactly one on top of another. In contrast the one degree of freedom control design resulted in a response where the transient response was widely different for the nominal case which is indicated by the blue curve here and the case when K naught was 0.3 and the case when K naught was 3.

So, this robustness that we have accomplished in the response of our closed loop system to variation in plant parameters is entirely due to our particular design of the feedback controller in combination with the pre filter and this was facilitated in a very transparent manner by using root locus as a tool for our design.

Thank you.