Control System Design
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Lecture – 14
Bode plots

Hello, in the previous clip we took a look at Nyquist's stability theory, the mathematical proof for Nyquist stability theory and subsequently, how Nyquist stability theory helps us to determine the stability of a closed loop system given its open loop transfer function. We subsequently took a look at a few numerical examples of Nyquist plots for a few chosen transfer functions.

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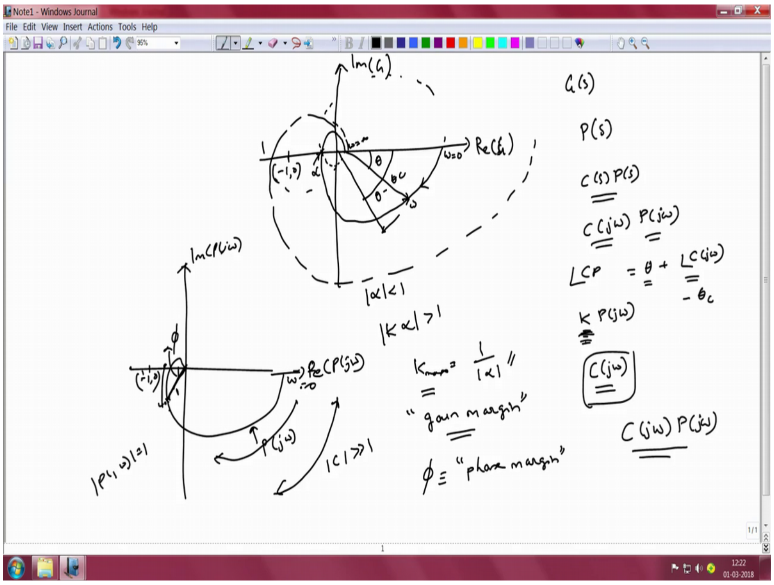

Now, in general let us assume that the Nyquist plot for our plant looks something like this; this is real part of G, this is the imaginary part of G. Let us assume that it looks something like this. The exact shape is depended on the exact structure, but for the moment I am assuming that this is the kind of shape that it might have. And it so, happens that in this case it avoids the critical point minus 1 comma 0. And let us also assume that this particular open loop transfer function G of s has no poles on the right half of the complex plane, which means that it is a stable open loop system.

In which case the fact that this is now not encircling the point minus 1 indicates that our closed loop system would be stable. I want to remind you that generally we draw the

Nyquist plot only from omega equal to 0 to omega equal to infinity, so omega increases along this curve. We do not draw it for negative omega, because in that case the curve would simply be a reflection of this curve about the real axis. So, there is no new information to be gleaned by drawing that curve.

Now what are you interested in as control engineers? I highlighted that you are interested in two things: the primary thing you are interested in is in achieving certain specific performance requirements, we want to be able to reject disturbances by a specified amount or to or be able to achieve a certain amount of robustness to variation in the parameters of a plant or perhaps track references with a desired accuracy. These are the kind of things that we are really interested in.

But, as we discussed in our interest to design controllers with high gain; so there is a possibility of us to destabilize the closed loop system and this is where the Nyquist plot would help us to determine, whether in our attempt to meet our performance requirements our closed loop system is still stable or not. So, one can therefore ask what would happened to the Nyquist plot of the plant. Let us say this is the Nyquist plot of the plant and the plant transfer function let us say is P of s what would happen to the Nyquist plot of the plant when we bring in a controller. In which case what we would have to do is, plot the Nyquist plot of C of s times P of s and then look at whether C of s times P of s encircles the point minus 1 or not and hopefully if our design is done carefully enough it would not encircle the point minus 1.

In general what would C of s do? So, I want to remind you that when you are done Nyquist plot of C of s times P of s, we are substituting s is equal to j omega and we are drawing the Nyquist plot for C of j omega times P of j omega. P of j omega is a complex number. So, we can indicate that as this complex number at some particular frequency omega. So, C of j omega another complex number which multiplies this complex number.

Now, when two complex numbers are multiplied two things can happen. One is the magnitude of the product would be increased or decreased depending on the magnitude of C of j omega, and the second the net angle of the product. So, the angle of P of j omega if you want to call it theta, the net angle will be equal to theta plus the angle of C of j omega. So, both the angle will change as well as the magnitude will change.

Now suppose we want to look at the special case, where our controller C of j omega only modifies the magnitude, but not the phase in which case our open loop system would be of the kind k times P of j omega where k is some real constant; how would the Nyquist plot for this kind of an open loop system look like. We note that if P of j omega has this particular characteristic, at each point when we have a controller of gain k, it will increase the length of that complex number by a factor k.

Therefore our new Nyquist plot will be a simply a scaled version of the original Nyquist plot. The question now is how much of scaling is permissible? I want to remind you that as control engineers who are interested in achieving good performance in terms of disturbance rejection or robustness to plant parameter variations. We want our controller gain to be very high, but then there is this concern that if the controller gain is very high, our closed loop system might become unstable. Therefore, the a legitimate question to ask is if we were to simply go with a proportional controller, what is the best we can do in terms of increasing the gain of the controller.

To answer this question, let us assume that the point at which the Nyquist plot of the plant crosses the real axis has a gain given by alpha. Now clearly alpha the magnitude of alpha is less than 1 and that is why our closed loop system in this particular case is stable. Now if our controller want to have a gain k, then the new location at which the Nyquist plot of k times P would cross the real axis would be at k times alpha.

Now the moment k times alpha becomes greater than 1 in magnitude, you notice that the Nyquist plot of k times P will start to encircle the point minus 1. So, let me sketch that out here. Then our k times alpha is greater than 1 our Nyquist plot will look something like this. And if it encircles the point minus 1 you would notice that its mirror image would also encircle. So, you would have essentially two encirclements of the critical point, which would make this closed loop system unstable because you would have two unstable poles for the closed loop system.

Therefore, the best the highest value of gain k that I can have, and still ensure that my closed loop system is stable is given by k max is equal to 1 by magnitude of alpha. This value of gain k, which is the maximum that is permissible before our closed loop system becomes unstable has a special name. So, this is called the gain margin and this term is sort of self-explanatory, because it tells us that in our attempt to increase the gain. We should be sure that in the vicinity of the point where the Nyquist plot crosses the real axis, the gain should not be increased by an amount greater than 1 by alpha. If that is ensured then we will still be able to make sure that our closed loop system is stable.

The second effect that our controller can have as I said, could be that it would add phase it would essentially only contribute to change in the phase of the complex number P of j omega. So, the angle of C times P would be equal to the angle of P time plus the angle of C. Now generally, since we deal with strictly proper controllers most of the controllers add phase lag although there can be frequency ranges that they would where they can add phase lead, but assuming the phase lag what would happen to this complex number.

Let us say at some particular frequency omega, if the controller adds a phase lag of minus theta c, what would happen to this complex number? What would happen is that this complex number would get rotated in the clockwise direction so, that the net angle that it makes no is no longer theta, it is going to be equal to theta minus theta c. So, you see that the effect of a controller that adds only phase lag to the plant transfer function, is to rotate the Nyquist plot at each point by a certain angle that is dependent on the phase lag added by the controller.

Now, let us ask a question complementary to the one that we asked just a few minutes back. If we had a purely proportional controller the a controller that does not add any phase, but only amplifies the gain the open loop gain. Then we found that there was an upper limit to the extent by which that gain can amplification is possible and we called that upper limit to be the gain margin.

Likewise, if our controller were to simply contribute to phase lag and not modify the gain characteristic of the plant, which I shall call which is essentially P of j omega. So, the x axis will be real part of P of j omega, the y axis will be imaginary part of P of j omega. A complementary question that we can ask is in the vicinity of the frequency at which the Nyquist plot crosses the negative real axis, what is the maximum phase lag that is permissible by the controller.

To answer this question we note that if we pick a point in other words a particular frequency at which the magnitude of P of j omega is equal to 1. So, this radius is therefore, equal to one unit, then the maximum phase lag that the controller can provide, and still ensure that our closed loop system is stable is given by the angle that this complex number makes with the real axis, I shall call that angle as phi. Now if the angle by which the phase or the location where the magnitude is 1 is rotated is more than phi, then we would have points of magnitude greater than 1 getting rotated over to the real negative real axis. And therefore, the new Nyquist plot will end up encircling the critical point.

Therefore, the maximum phase lag that can be tolerated on part of the controller, in the vicinity of the critical point is equal to phi and this phi has a special name it is called as the phase margin. So, in our interest to design controllers with adequately high gain and possibly also design dynamic controllers; we should make sure that in the vicinity of the frequencies where the magnitude is close to 1 and the phase is close to minus 180 degrees, our controller gain should not exceed 1 by alpha and the controller phase should not phase lag should not exceed phi. And this is required in order to ensure a stable closed loop system.

What this Nyquist plot also reveals to us is that, it is not a wise decision to go with an overly simplistic controller such as a simple proportional controller. The reason for this is because a simple proportional controller is limited in the magnitude of gain that we can achieve to 1 by alpha, which means that even at frequencies where stability is not a concern namely in the frequency range around omega equal to 0. And a little bit beyond that, the stability is not a concern we can only increase the gain by a factor k equal to 1 by alpha before our closed loop system becomes unstable.

But such a modest increase in gain in that frequency range may or may not allow us to meet our performance requirements; namely of rejecting disturbances achieving robustness to plant parameter variations and so on and so forth. Therefore, it is wise not to go with a simple proportional controller as a choice for our control design, though it looks extremely simple and as engineers we value simplicity. But yet, but instead we would it would be wise, but instead it would be wise for us to go with controllers C of j omega, which have a very high gain in the frequency range where stability is not a concern.

But then as we approach frequencies where the phase lag of the plant is close to minus pi and its gain has come close to 1 it is good to tune down the gain of c. So, in this frequency range the magnitude of c can be much greater than 1, and the phase of c can also be relatively large phase lag of c can also be relatively large, but in this frequency range where we are near the critical point its good for the magnitude of c to be small as well as the phase of c to be also small.

So, what the Nyquist plot clearly reveals to us is that, it is not wise in general for us to have a static gain as our controller though it is preferred in terms of simplicity, but rather to have a dynamic gain. In our words a controller that exhibits dynamics and we choose this controller to have adequately gain high gain at frequencies, where our performance targets need to be satisfied, but at frequencies where stability alone is a concern we make sure that the gain of C of j omega is not very high.

. So, having talked about Nyquist stability theory and the use of Nyquist plots in determining the stability of a closed loop system, I want to emphasize the fact that the Nyquist plots are the final courts of appeal as far as stability of the closed loop system in frequency domain is concerned. So, there can be occasions where by looking at the open loop transfer function; one is not very sure whether the closed loop system would be stable or not. In order to resolve this question one only needs to plot the Nyquist plot for that open loop transfer function and then check the number of encirclements of the critical point, and the number of encirclements would be equal to the number of closed loop poles minus the number of open loop poles.

So, if our closed loop system is stable, then the number of encirclements of the critical point should be the negative of the number of open loop unstable poles that we have on the system. Despite the Nyquist plot being the final court of appeal as far as the stability of the closed loop system is concerned when we are doing design in the frequency domain. And hence being of central importance to us as a tool for design it has an important disadvantage.

So, when we are trying to do design, we are essentially trying to design obtain the structure for our controller C of j omega. So, each time we come up with a certain controller structure based on our particular performance requirement, we need to plot the Nyquist plot of C of j omega times P of j omega and confirm whether this Nyquist plot is encircling the minus 1 point or not. And if it is encircling then we have to go back and change the controller structure so, that this encirclement is avoided.

The problem though with the Nyquist plot of C of j omega times P of j omega is that, it looks nothing like the Nyquist plot of P of j omega alone. Therefore, as engineers it is not intuitive for us to anticipate how C times P the Nyquist plot of C times P would look if you are given the Nyquist plot of P alone indeed. Even the Nyquist plot of P is not any simple intuitively obvious curve except for the simplest of cases for example, a first order plant or such examples, the Nyquist plots are not familiar curves such as circles or straight lines. And hence they are not easy for us to visualize and draw in a good manner.

This problem is further complicated because the incorporation of a controller makes even less intuitive for us to draw the Nyquist plot of C times P and therefore, anticipate how this Nyquist plot might differ from Nyquist plot of p alone. Therefore, as design engineers we desire tools that are a little bit more transparent and more intuitive compared to the Nyquist plot, although we still acknowledge the centrality of Nyquist plots in our ability to decide whether the closed loop system is stable or not.

It is in this context that we stumble upon Bode plots. Bode plots are essentially design tools which simplify the Nyquist plots which contain exactly the same information as what is in a Nyquist plot, but allows the control designer to visualize the effect of a certain controller structure on the overall plot much more easily. So, what is a Bode plot?

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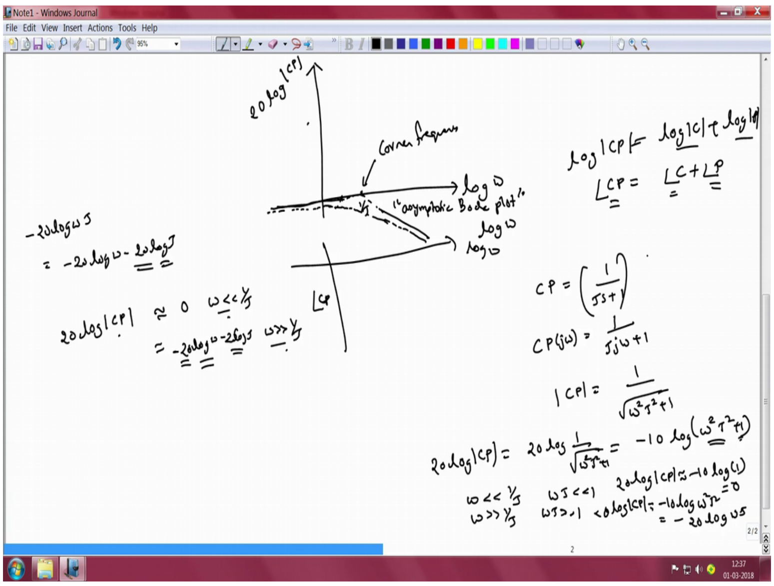

So, Bode plots essentially refer to a pair of plots. In one plot we plot the log of the magnitude of the open loop transfer function namely C times P and historically we multiplied with a factor 20. In order to represent the y axis in terms of decibels and the x axis would be log of the frequency.

Likewise, there is a second plot also which is part of the pair that constitute the Bode plot, and in this plot we plot log of frequency versus the phase of the open loop transfer function C times P. It is worthwhile to spend a few minutes to justify why the y axis in one plot is log of C times P and the x axis is log of omega and why in the other plot the; we are just plotting the angle of C times P and not the log of the angle for instance.

. So, as I said in the previous slide the Nyquist plot is all that we need for us to complete our design, but the problem with the Nyquist plot is the issue of lack of intuition as far as how the controller might modify the Nyquist plot of the plant. Therefore, and that is because we are firstly drawing a polar plot, and secondly we are having a multiplication of two transfer functions, and the polar plot of a product is not intuitively easy to imagine if you are given the polar plots of the individual transfer functions, namely that of the controller and the plant.

Therefore, instead of multiplication which is not as intuitive an operation, if we could convert it to addition then an addition of two complex numbers is far more easy to visualize than multiplication of two complex numbers and that is the reason why, the y axis in the first Bode plot is the log of magnitude of C times P. Because log of magnitude of C times P is equal to log of magnitude of C plus log of magnitude of P. So therefore, the transformation of the coordinates so that the y axis of the Bode plot now plots the log of magnitude of the product of C and P, allows us greater intuition in anticipating what would happen to the Bode plot of C times P when we are given the Bode plots of P and the Bode plot of C.

This is also the reason why the y axis or the second plot is just the angle of C times P, because the angle of C times P is essentially equal to the angle of C plus the angle of P. Therefore, if I am given the angle of P and the angle of c, I can easily anticipate what the angle of the C times P would look like. Before the y axis or the magnitude plot is a logarithmic plot because logarithmic the logarithmic function transforms a product into a sum. And the sum is a more intuitive operation for us to in terms of visualizing how the sum would be related to the individual parts.

And for the same reason we would have the y axis of the plot to be simply the angle of the product itself, because the angle of the product is essentially the sum of the angles of the two controllers namely that of C and of P. Second the question is why is the x axis log of omega why is it not simply linear omega? To answer this question let us take a simple example of a open loop transfer function C times P, that is of first order namely let it be of the kind 1 by tau s plus 1. So, C times P of j omega would be equal to 1 by tau times j omega plus 1.

So, the magnitude of C times P would be equal to 1 by square root of omega square tau square plus 1. Now if we want to compute 20 log magnitude of C times P, we would get it to be equal to 20 times log of 1 by square root of omega square tau square plus 1, which is essentially equal to minus 10 times log of omega square tau square plus 1.

Now, we note that in this expression, when omega is much less than 1 by tau we would have omega times tau to be much less than 1, in which case 20 log magnitude of C P would be approximately equal to minus 10 times log of 1 and that would be equal to 0. On the other hand that omega is much greater than 1 by tau or in other words omega tau is much greater than 1, then we can ignore the number 1 in comparison with the term omega square tau square. And we would have 20 log magnitude of C P to be equal to minus 10 log omega square tau square, it is equal to minus 20 log omega tau.

This can be simplified further. So, minus 20 log omega tau would be equal to minus 20 log omega minus 20 log tau now tau is a constant. So, this term here is therefore, going to be a constant, what we notice is that 20 log magnitude of C P is approximately 0 in the limit omega much less than 1 by tau, and is approximately equal to minus 20 log omega, minus 20 log tau in the limit omega greater than greater than 1 by tau.

So, we see that if we choose the x axis of the Bode plot to be log omega, then we would have the Bode plot of the transfer function C times P to be in these particular limits, my omega is much less than 1 by tau it is a straight line of slope 0 and magnitude also 0, and my omega is let us say the point 1 by tau is somewhere here. Omega is much greater than 1 by tau we would have this to be another straight line of slope minus 20 dB per decade or of slope minus 20, mx intercept of minus 20 log tau. So, that would be another straight line.

So, we see that by having the x axis to be log of omega. We are at least in some limits able to represent the transfer function C P of j omega as two straight lines and what is often done is to extend these straight lines with dotted lines if you wish, which meet together at the point 1 by tau. The point 1 by tau is called the corner frequency and these two this combination of the two straight lines is called the asymptotic Bode plot of the transfer function C times P.

How will the actual Bode plot look? Tactual Bode plot would look very close to the asymptotic version for frequencies that satisfy these two limits. So, in these limits they will be very close to the asymptotic versions its only in the vicinity of 1 by tau namely when for frequency that are two times to five times 1 by tau or half or one fifth of 1 by tau, that there would be an appreciable difference between the asymptotic case and the actual Bode plot.

So, if and it would the actual Bode plot would look something like this if for a moment we were to ignore as control designers, the small difference that exist on either side of the corner frequency, for all practical purposes we would have transformed what was earlier a fairly complicated non intuitive curve in the Nyquist plot in to a pair of broken straight lines. And since, our physical system would be a cascade of terms of this kind we would have terms of this kind, either in the numerator or in the denominator essentially our Bode plot would get transformed to a set of broken straight lines.

And likewise the Bode plot of the controller would also be a set of broken straight lines, this makes it all the more easy for us to anticipate how the Bode plot of the final open loop system C times P would look like. So, the y axis of the magnitude plot is the logarithmic plot because it transforms the product to a sum, the x axis is a logarithmic plot, because if we choose it to be a logarithmic plot then the Bode plot of the magnitude characteristic would be a set of would be a set of straight lines. And the y axis of the angle plot is chosen to be the angle of just C times P, because the angle of a product of C and P is equal to sum of the angles of C and P.

There is one final reason why we choose the axis to be log of magnitude of C P versus log of omega and angle of C P versus log of omega. This is not evident at the moment from the numerical examples that we could consider, but we will revisit this point at some later instant in this course towards the end of this course, where we will show that the slope of the magnitude characteristic in a Bode plot is proportional approximately to the phase lag of the same characteristic. And what this reveals. Therefore, is that if I able to simple draw the magnitude characteristic, I do not really need to take extra care to also draw the phase characteristic. I can derive the phase characteristic directly from the magnitude characteristic by looking at the slope of the magnitude characteristic at each of the frequencies.

Of course, this relationship is only approximate, but the degree approximation is generally quite good in most cases. So, these are the therefore, the three reasons why Bode plots are preferred over Nyquist plots. The first is that it converts a product to a sum. And therefore, makes it more intuitive to anticipate the effect of having a controller transfer function on top of the plant transfer function, on the overall stability of the closed loop system.

The second is that it allows the curves the Nyquist curves are not intuitive, the Nyquist curve was C times P is not intuitive whereas, the curves are now transformed to a set of broken straight lines, where the slope of the straight lines change at the corner frequencies of the plant or the controller. And thirdly the magnitude the slope of the magnitude characteristic of in a Bode plot is proportional to the phase characteristic. We will establish this rigorously mathematically towards the end of this course. So, theses reasons are motivations for us to employee Bode plots as the canvases for doing our control design.

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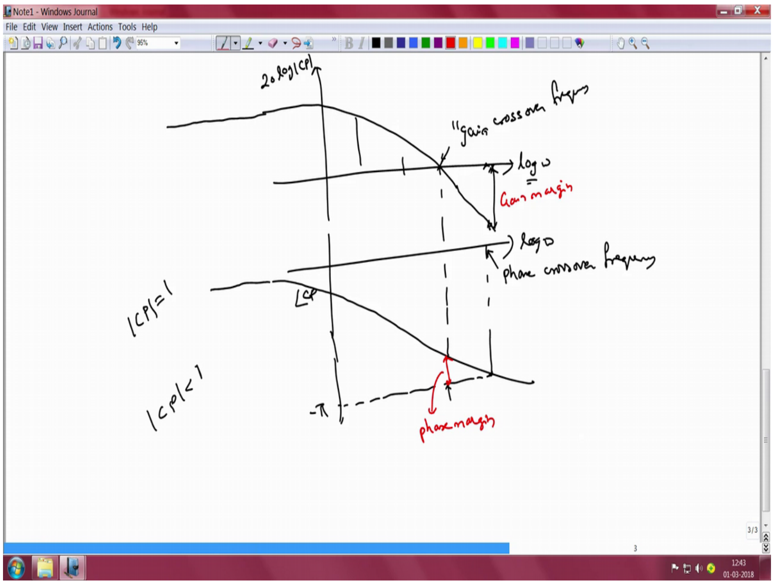

So, if you able to draw the Bode plot of the typical plant that we considered a few slides back, for which case we drew the Nyquist plot, then the Bode plot might look something like this. At low frequencies this is log of omega, the y axis is 20 log magnitude of C times P and this is again log of omega and this is angle of C times P. The magnitude characteristic looks something like this. Note that in the Bode plot the frequency omega equal to 0 cannot be located, because omega equal to 0 is located at minus infinity since the x axis is a logarithmic plot.

. So, the omega equal to 0 cannot be located, but at low frequencies assuming that the plant has a constant gain, then the Bode plot of the plant look something like this where the slope of the Bode plot changes near the corner frequencies of the plant. So, if one were to the draw phase plot it would start at a phase close to 0 degrees, and then decrease and then continue decreasing further. Now there would be a frequency at which the phase crosses minus 180 degrees you see this is minus pi radian, and that frequency the frequency at which it causes minus 180 degrees is called the phase crossover frequency.

Likewise, there is a frequency at which the magnitude curve crosses 0 dB or in other words magnitude curve has a magnitude of C times P to be equal to 1 and that frequency is called the gain crossover frequency. Now the difference in phase between the actual Bode plot of C times P and the angle minus pi. So, this difference is essentially the phase margin, this is a definition that we have simply borrowed from the Nyquist plot.

Likewise if you were to continue the magnitude characteristic up to a frequency at which the phase crosses over and evaluate the gain of the magnitude characteristic at this frequency, it is evident from this Bode plot that the gain here is going to be less than 0 dB. In other words the magnitude of C P is going to be less than 1, which is indicative of the fact that our closed loop system would be stable assuming that our open loop system is stable. And difference between the gain of C times P and 0 dB by definition is going to be the gain margin of our open loop system. So, we want to make sure that C times P always has the specified phase margin and the specified gain margin for the closed loop system to be adequately stable.

With this we conclude taking an overview of the canvas namely the tool that we would use for doing our design namely the Bode plot. And in the next clip we shall start performing control design for specified performance requirements. So, let us now take a look at the Bode plot of some common transfer functions. So, let us start with the transfer function that we considered in the previous slide namely P of s is equal to 1 by tau s plus 1.

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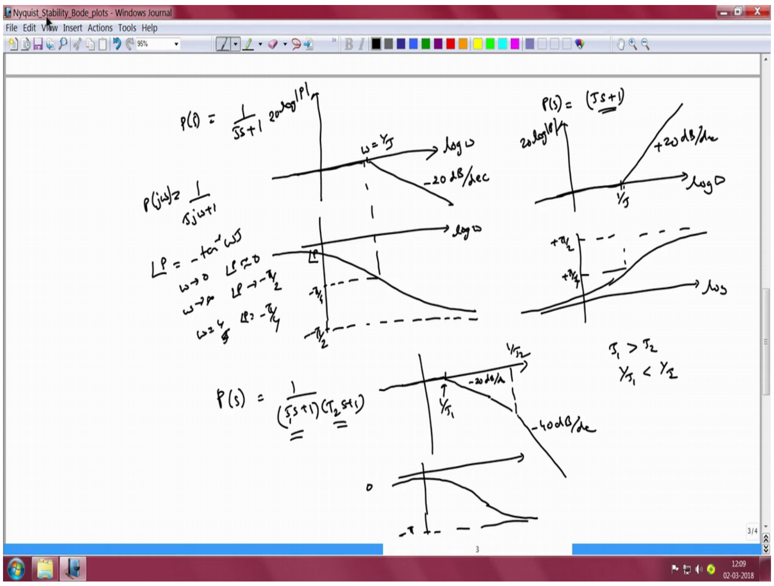

So, if we were to draw the magnitude plot we saw in the previous slide, that its slope is 0 all the way up to omega equal to 1 by tau and its slope reduces linearly with frequency for omega much greater than 1 by tau. So, the x axis is log omega, the y axis is log magnitude of P. And this slope is minus 20 units or equivalently set to be minus 20 decibels in one decade of frequency change. So, one decade represent a factor of 10 change in frequency. So, the slope of this curve is going to be minus 20 decibels per decade.

Likewise, if we were to plot the angle of P as function of log omega, the angle of P of j omega P of j omega is equal to tau times j omega plus 1. So, angle of P is equal to minus tan inverse of omega tau. So, we that when omega is close to 0 angle of P is also close to 0 and as omega tends to infinity, angle of P tends to minus pi by 2 and exactly at omega equal to 1 by tau angle of P is equal to minus pi by 4 therefore. So, at omega equal to 1 by tau the angle is minus pi by 4 at omega equal to as omega tends to infinity the angle tends to minus pi by 2 and it starts at 0. So, the phase response looks something like this, it never crosses minus pi by 2 it only asymptotically approaches the angle minus pi by 2.

Similarly if you consider the same term tau s plus 1, but this time in the numerator of the transfer function. So, if it is a 0, then it is of the form tau s plus 1, then we would have the Bode plot of a term of this kind I want to mention that this is a non causal transfer function. So, we are not really in this case drawing the Bode part of a transfer function of a causal physical system, but simply the Bode plot of this term tau s plus 1; we would have 20 log magnitude of p. So, there should be a 20 here also on this graph, and log of omega. And we see that with very similar arguments we can show that up to omega equal to 1 by tau, the magnitude characteristic of P is a horizontal line coincident with the real axis it is at 0 dB and the asymptotic Bode plot has an upward slope is a straight line with a slope of plus 20 decibels per decade beyond omega equal to 1 by tau.

As far as the phase response is concerned, since tau s plus 1 is the inverse of 1 by tau s plus 1. The phase of 1 by tau s plus 1 is seem to change from 0 to minus pi by 2. Therefore, the phase of tau s plus 1 will change from 0 to plus pi by 2. So, it will asymptotically approach plus pi by 2, and at 1 by tau the phase will be plus pi by 4 and it will start close to 0. So, the phase response will look something like this.

Now, if I have a second order system which is a cascade of two first order systems. So, for instance if I were to consider P of s to be equal to 1 by tau 1 s plus 1 times 1 by tau 2 s plus 1, then we would have the Bode plot of this to be the addition of the Bode plots of 1 by tau 1 s plus 1 and 1 by tau 2 s plus 1, assuming that tau 1 is greater than tau 2. So, that 1 by tau 1 is less than 1 by tau 2 then we would have 1 by tau 1 s plus 1 times 1 by tau 2 s plus 1 to have a magnitude characteristic whose slope is close to 0 dB all the way up to 1 by tau 1.

And the asymptotic Bode plot will start to reduce in magnitude at a rate given by minus 20 dB per decade between 1 by tau 1 and 1 by tau 2, and at 1 by tau 2 there is a further reduction brought about by the pole at 1 by tau 2. And therefore, the slope will be higher it will now reduce at minus 40 dB per decade beyond 1 by tau 2. So, this is how the magnitude characteristic would look.

And as far as the phase characteristic is concerned, each of these terms 1 by tau 1 s plus 1 and 1 by tau 2 s plus 1 contribute to a phase lag of minus pi by 2 when omega tends to infinity and each of their phases is close to 0 when omega is close to 0. So, the rough phase plot would start at 0 radians and would asymptotically approach minus pi radians. So, unlike the first order system a second order systems phase lag goes from 0 to minus pi instead of 0 to minus pi by 2.

It is worth noting that the phase curve does not have a simple intuitive shape unlike that of the magnitude curve. But as we discussed this is not a major concern, because we would see later that we can derive the phase curve directly approximately from the magnitude curve itself. So, there is one other possibility for the structure of a second order system and that is when its poles are complex.

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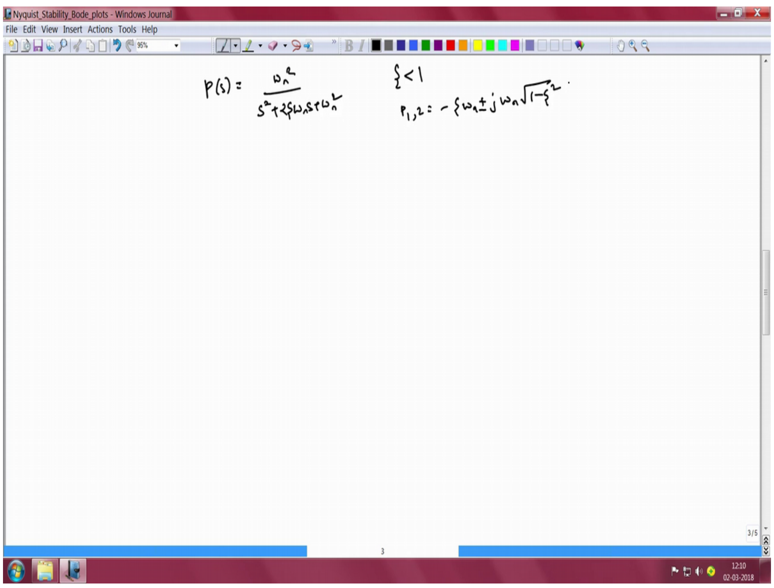

So, a second order system whose poles are complex can be written out in this particular form a square, plus 2 zeta omega n s plus omega n square. And for a structure of this kind where zeta is less than 1, if zeta is greater than 1 we would have two real poles for this plant and that is the case that we discussed in the previous slide. If zeta is less than 1 however, we have a pair of complex conjugate poles which are given by P 1 comma 2 equal to minus zeta omega n plus minus j times omega n square root of 1 minus zeta square.