Control System Design
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Lecture – 13
Nyquist stability theory (Part 3/3)

Hello, in the previous clip we took a look at Nyquist Stability Theory and how it could be employed to determine the stability of a close loop system. Or more generally how can it be employed to determine whether the 0's and poles of a given complex function f of s is within a certain bounded region, a region bounded by curve c in the complex plane.

Now, how do we mathematically establish like Nyquist stability theory to mathematically establish the theory we need two concepts, one is a notion of analytic functions and the second is what is known as the Cauchy integral formula. So, let us briefly define what analytic functions are and then subsequently also state Cauchy's integral formula and then use both to establish Nyquist stability theory.

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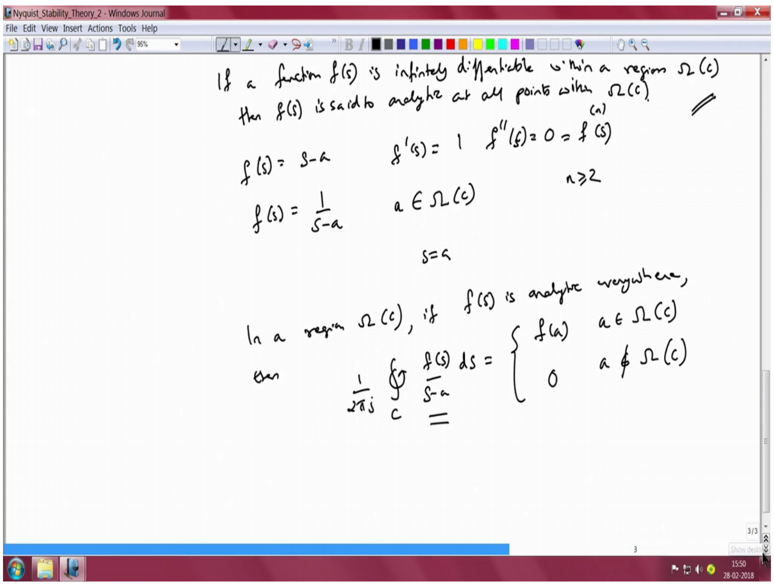

If a function f of s is infinitely differentiable within a region omega bounded by the curve c, then f of s is said to be analytic at all points within omega c. So, if the function f of s is complex differentiable at all points and we should be able to differentiate it as many times as you wish, then we call such functions as analytic functions within this region c.

So, what are some examples for analytic functions if you take f of s to be equal to S minus a. Then we see that f prime of s is equal to 1 f double prime of s equal to 0 and at equal to all other derivates of s for n greater than or equal to 2. So, we see that this function is infinitely differentiable and this is infinitely differentiable regardless of which specific point we are evaluating it at.

On the other hand if you were to look at a function of the chi f of s is equal to 1 by S minus a. And if a happens to be a point that belongs to the region omega which is bounded by the curve c. Then we see that it is not differentiable at the point S equal to a, because this function blows up at a particular point, but it is differentiable at every other location. So, what we can say therefore, in this case is that f of s is differentiable is analytic everywhere except at S is equal to a. So, it is a fairly straight forward concept if a function is differentiable then it is an analytic function if it is not differentiable at certain points then it is not an analytic function. We shall employee this motion of analytic functions to now state Cauchy's integral theorem.

The theorem states that in a region omega bounded by the curve c, if f of s is analytic everywhere meaning that it is differentiable everywhere. Then 1 by 2 pi j times the contour integral over the curve c of f of s by S minus a d s is equal to f of a if the point a belongs to the region bounded by the curve c and is equal to 0 if a does not belong to the region bounded by the curve c. So, this is Cauchy's integral theorem and we shall employee this theorem along with the notion of analyticity as we have defined above to now, prove Nyquist stability theory.

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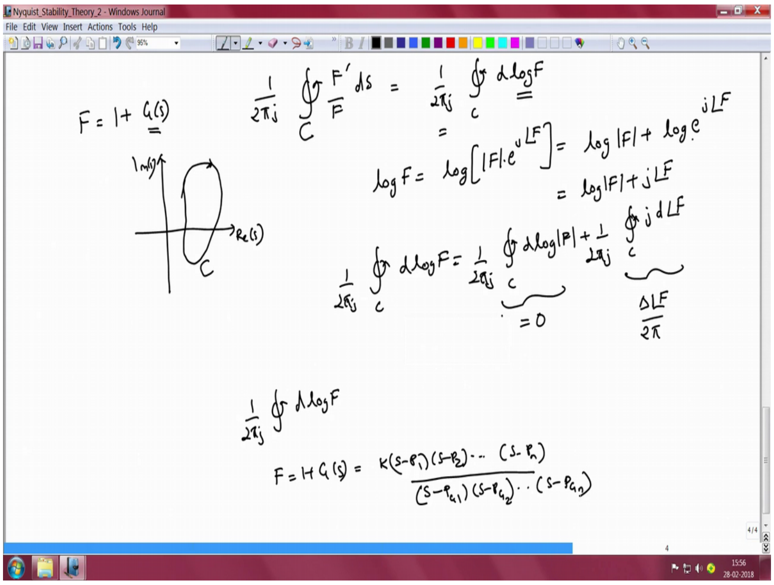

To prove Nyquist stability theory let us integrate, let us evaluate the integral 1 by 2 pi j times the contour integral of F prime by F d s where F is a function the that we are interested in namely 1 plus G of s, where G of s is our open loop transfer function. We want to determine how many of the 0's of F which are essentially the close loop poles are on the right half of the complex plane. So, we undertake the contour integral of F prime by F and since Nyquist stability theory is a general theory that is not applicable only to determine stability, but more general you determine if the 0's of F are inside this close contour C or not C can be any contour in the complex plane.

Now, we see that we can write this integral as 1 by 2 pi j times contour integral over c d of log of capital F. Because by definition d of log of capital F is F prime by F times d F c and we can subsequently further simplify this expression by noting that log of capital F is equal to log of magnitude of F times e to the power j angle of F. And this in turn is therefore, equal to log of magnitude of F plus log of e to the power j angle of F, this in turn is equal to log of magnitude of F plus log of e to the power x is essentially x.

So, therefore, this is going to be equal to j times angle of F therefore, we can rewrite this integral as 1 by 2 pi j a contour integral over c d log of F as equal to 1 by 2 pi j contour integral over c d log of magnitude of F plus 1 by 2 pi j contour integral over c j times d of angle of F. Now, if you notice this integral since our contour integral essentially looks at d of log of magnitude of F and starting from some particular location yes on this contour and then goes round that contour once, we see that the initial and the final magnitudes of F are identical. Therefore, d log of magnitude of F integrated over the contour will be equal to 0, this integral will not be 0 d of angle of F will be the net change in angle of F.

And I shall call that delta of angle of F, and I have the two terms j: one in the numerator other one in the denominator. So, the cancelling though I would get this to be equal to delta of angle of F divided by 2 pi. Now, we know that 1 by 2 pi j times the contour to integral of d of log of F can also be written in a slightly different manner by noting that our F is equal to 1 plus G of s. And by factorizing this we can write this as some constant times S minus P 1 times S minus P 2 and so on and so forth up to S minus P n, where P 1 to P 1 are the n 0's of 1 plus G s which are also the close loop poles of our system divided by S minus P G 1, because these are the poles of G of s times S minus P G 2 and so on and so forth up to S minus P G n.

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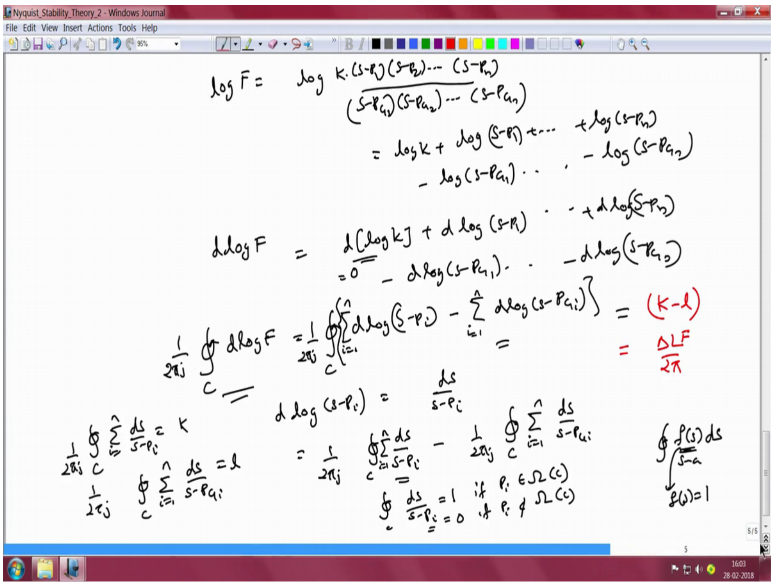

Since, you can write F in this particular manner we know that log of F will be equal to log of constant times S minus P 1 times S minus P 2 and so on and so forth up to S minus P n divided by S minus P G 1 times S minus P G 2 and so on and so forth up to S minus P G n. Which intern can be expanded out as log of k plus log of S minus P 1 etcetera etcetera up to plus log of S minus P n minus log of S minus P G 1 and so on and so forth up to minus log of S minus P G n.

Therefore you would have d of log of F to be equal to d of log k plus d of log S minus P 1 and so on and so forth. Plus d of log S minus P n minus d log S minus P G 1 and so on and so forth minus d log S minus P G n. Now, we note that d log k is going to be equal to 0 because k is a constant. So, if I were to compute the contour integral 1 by 2 pi j over c d log F, I would get this to be equal to a contour integral over c times 1 by 2 pi j times d log S minus P i summation i going from 1 to n minus summation i going from 1 to n d log S minus P G i.

So, both these summations are within the contour integral, now we note that d log S minus P i for any index y is essentially going to be equal to d S by S minus P i. Therefore, we can simplify this contour integral us being equal to 1 by 2 pi j times contour integral d S by S minus P i sigma i going from 1 to n minus 1 by 2 pi j times contour integral over c sigma i going from 1 to n d S by S minus P j i.

Now, if you look at any one of these terms contour integral over c d S by S minus P i we note that it is of the form contour integral f of s by S minus a d S with the particular function f of s being chosen equal to 1. Therefore, if the point P i is inside the contour c you would have this term d S by S minus P i to be equal to 1. If P i belongs to omega of c is on the other hand is P i is not part of the region bounded by the curve c this is going to be equal to 0. Therefore, I would have 1 by 2 pi j it times contour integral over c sigma d S by S minus P i to be equal to like 1 from 1 to n to be equal to k.

Where k of the close loop poles which are the 0's of f of s or inside the contour c likewise 1 by 2 pi j contour integral over c sigma i going from 1 to n d S by S minus P G i will be equal to l, where l is a number of poles of the open loop system which are also the poles of f of s that are inside the contour c. Therefore, you would have that this expression 1 by 2 pi j contour integral d of log of F to essentially be equal to k minus l. So, if you remember from the previous slide we found that this expression 1 by 2 pi j contour integral d of log of F is also equal to delta of angle of F divided by 2 pi. And this putting this thing together P are able to establish Nyquist stability theory which states that.

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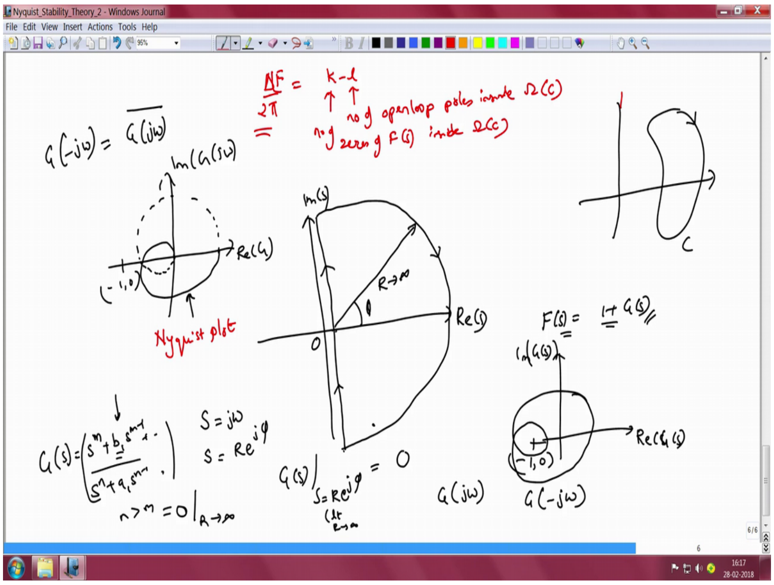

Delta of angle of F divided by 2 pi is e equal to k minus l, where l is the number of open loop poles inside omega c and k is a number of zeros of F of s inside omega of c. So, delta F which is a net delta of angle of F which is a net changing the angle of F divided by 2 pi is essentially the number of encirclements of the complex number F of the origin in the F of s plane.

Therefore, the number of encirclements of the origin in the F of s plane when the complex number S is made to go round the contour c. Once is equal to the number of close loop poles which are essentially the number of zeros of capital F of s minus the number of open loop poles which are the poles of the open loop system. This in effect establishes Nyquist stability criterion, although in this particular form it is applicable for any other contour c as well.

So, this contour need not necessarily be the right half of the complex plane what could be any contour in the close contour in the complex plane. And, this statement can be employed to determine how many of the close loop poles are inside that contour and how many or outside the contour. What is worth noting once again is that this number of encirclements does not directly give us the number of close loop poles which are the zeros F of s that are inside omega c. It only gives the difference between the number of close loop poles and the number of open loop poles.

Now, how do we then get the number of close loop poles separately? To do that it is assumed that we know the number of open loop poles that are inside that contour c and that is assumed to be known because the open loop system is something that we have design. So, either from first principles or from certain strategically conducted experiments it is assumed that we can determine how many of it is poles are inside the contour c.

With that knowledge we can then look at the number of encirclements and then determine how many of the close loop poles are inside the region omega of c. Just one more point that I want to make regarding the number of encirclements, the encirclements could either be clockwise or anticlockwise. So, which one do we take to answer this question one needs to simply note that if we take the variable S in the complex plane. In the contour clockwise direction then the statement for Nyquist stability theory would be that the number of contour clockwise encirclements of the origin in the F of s plane would be equal to the number of close loop poles minus the number of open loop poles.

On the other hand if we choose to take the variable S in the clockwise sense, then the statements on Nyquist stability theory would be that the number of clockwise encirclements of the origin in the F of s plane for points S on the contour c would be equal to the number of close loop poles minus the number of open loop poles. So, depending on the direction along which we take the complex number S around contour we need to look at the direction in which we are encircling the origin in the F of s plane. So, as we discussed the real utility of this theory is not from it is able to determine the location of close loop poles in some specific closed region c omega of c, but rather to determine the stability of the close loop system.

To determine the stability all that we need to do is to pick the c to encompass the entire of the right half of the complex plane. So, if I were to write draw the complex plane here then we distort this curve c to be a very large curve whose one edge is very close to the imaginary axis, but does not exactly coincide with the imaginary axis. And the other site is a large d shaped curve and the radius of the semicircular d shaped curve or is tended to infinity. So, that we are able to cover the entire of the right half of the complex plane and check whether any of the poles of the surface would be in this plane in this side of the complex plane or not. Now in this case we choose to take the variable S in the clockwise sense although in some of my previous demonstrations to took it in the clock anti-clock wise sense in this case we chose to take it in the clock wise sense. And have been taken S how long this contour in the clock wise sense we now look at the number of clockwise encirclements of the origin in the f of s plane.

So, what is F of s? F of s is essentially 1 plus G of s. So, if I were to draw f of s plane then I would have the real part f of s the imaginary part of f of s here. I know look at the number of encirclements of the origin in the f of s plane what you available to us from experiments is G of s. And we have to undertake this extra calculation to compute the complex number 1 plus G of s. In order to avoid this extra computation that we would have to undertake what is commonly done is to look at the encirclement of not the origin, but rather the point minus 1 comma 0 not in the f of s plane, but rather in the G of s plane.

So, if I were to plot the real part of G of s versus the imaginary part of G of s for values S on this contour c, I would get some curve. And I look at the encirclement of the point minus 1 comma 0 of this curve, which is exactly identical to looking at the encirclement of the origin in the F of s plane. So, to minimize this computation of 1 plus G of s given G of s I simply look at the encirclement of the point minus 1 comma 0 in the G of s plane rather than looking at the encirclement of the origin in the f of s plane.

So, what we shall have do in order to determine the stability of the close loop system is to plot G of s for values of S along this d shaped contour. And look at the number of times it encircles the point minus 1 comma 0 in the G of s plane. Now, what are the kind of values that S would assume on this contour since this straight edge of the contour is very close to the imaginary axis along this contour you would have S to the equal to j omega. Now, along the d shaped contour since it has a radius R in the centered at the origin of the complex plane you would have S to be a complex number of the form R e power j phi where phi is this particular angle.

Now, since we are dealing with transfer functions which are generally strictly proper. In other words the degree of the numerator polynomial is always less than the degree of the denominator polynomial, we see that G of s for S of the form R e power j phi. Where the radius R is tended to infinity in the limit R tends to infinity will be essentially equal to 0.

So, therefore, all the points on this d shaped contour will collapse to the origin in the G of s plane. Simply because my G of s which would be on the form S power m plus b 1 S power m minus 1 and so on by S power n plus a 1 S power n minus 1 and so on where n is greater than m will assume a value 0 in the limit R tends to infinity. So, what we are left with therefore, is to simply compute G of s for values of S on the straight line. Now for the part of the straight line which starts from the origin and goes all the way to j infinity our G would be of the form G of j omega. Since, S would be equal to plus j omega on that straight line on this straight line however, which starts from minus infinity and comes to the origin we would have S to be equal to minus j omega.

So, you would be evaluating G of minus j omega however, there is a further simplification that is possible by noting that since G represents the transfer function of a real system. When we write it down as a ratio of two polynomials as we have done here, we note that all the coefficients a 1 a 2 a 3 b 1 b 2 b 3 and so on are all real numbers and what that indicates to us is that G of minus j omega would be equal to G of j omega bar.

Where G of j omega bar represents the complex conjugate of G of j omega, in other words if I were to graph G of j omega for omega going from 0 to infinity then G of minus j omega we look very similar to this curve with the exception of this curve will be reflected about the real axis. So, therefore, we do not even need to obtain G of j omega along the negative imaginary axis, because that curve will be simply a mirror image of the curve that would obtained for omega going from 0 to infinity.

So, what is therefore, generally done is you would simply compute real part of G versus the imaginary part of the G for omega going from 0 to infinity a typical curve might look something like this. It might or might not encircle the point minus 1 comma 0 and it will collapse to the origin because all the points on this d shaped curve would get mapped to the origin in the G of s plane. And for points that are on the negative imaginary axis the curve would simply be a reflection of this curve about the real axis.

So, we can easily complete the entire Nyquist plot if we simply evaluate it for omega going from 0 to infinity. And this curve G of j omega for omega going from 0 to infinity is called the Nyquist plot and by looking at the encirclement of the point minus 1 comma 0, we can tell the difference between the number of close loop poles and the number of open loop poles that are on the right half of the complex plane. So, to look at the practical utility of Nyquist stability theory let us take a few numerical examples.

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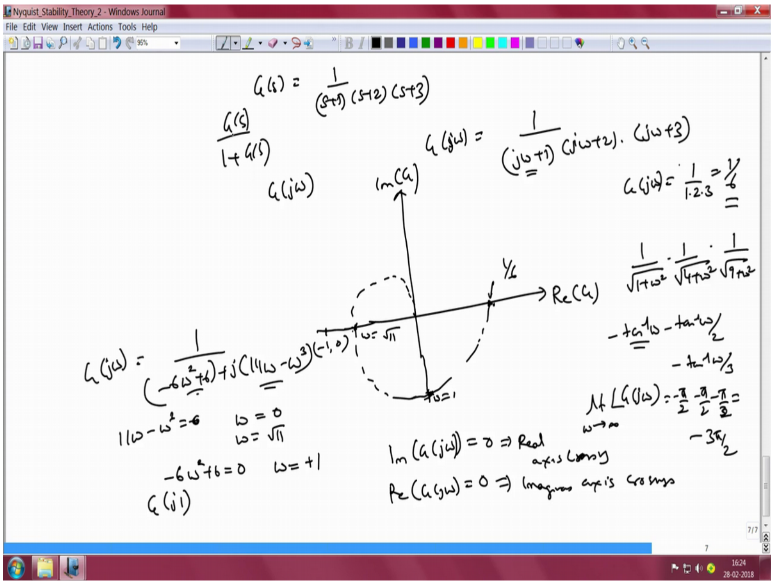

Let us say that G of s, were equal to 1 by S plus 1 times S plus 2 times S plus 3. Firstly, we note that there are no open loop poles on the right half of the complex plane the open loop poles are at minus 1 minus 2 and minus 3. So, there all on the left half as a complex plane therefore, this represents a stable open loop system to determine whether 1 plus G is stable or not we plot G of j omega for omega going from 0 to infinity and look at the encirclement of the point minus 1 comma 0 in the G of j omega plane.

So, if we have to compute G of j omega we substitute S is equal to j omega and we get the expression to be 1 by j omega plus 1 times 1 by j omega plus 2 times j omega plus 3 we note that where omega is equal to 0. So, if you were to plot real part of G process the imaginary part of G we note that when omega is equal to 0, we would have the G of j omega to be equal to 1 by 1 times 2 times 3 which is 1 by 6. Now, 1 by 6 is a positive real number which indicates that it is a complex number with phase equal to 0 and magnitude equal to 1 by 6.

So, be locate it on the real axis positive real axis of the G of j omega plane and this is the point 1 by 6. Now as omega goes from 0 to infinity we note that the magnitude of each of these terms for instance 1 by j omega plus 1 would be square root of 1 plus omega square. The magnitude of 1 by j omega plus 2 is going to be 1 by square root of 4 plus omega square the magnitude of 1 by j omega plus three would be 1 by square root of 9 plus omega square

So, it is a monotonically decreasing function of omega and in the limit omega goes to infinity this magnitude also tends to 0, what about the phase the phase would be equal to the phase of the first term would be equal to minus tan inverse omega. The phase of second term would be equal to minus tan inverse omega by 2 and the third term would be equal to minus tan inverse omega by 3.

So, when omega is close to 0 all these terms or closed to 0 when omega tends to infinity each of these terms go to pi by 2 which implies that the net phase of G of j omega as omega tends to infinity angle of G of j omega in the limit omega goes to infinity will be equal to minus pi by 2 minus pi by 2 minus pi by 2 which is going to be equal to minus 3 pi by 2. In order to obtain the real axis and imaginary axis crossings we determine the location at which the imaginary part of G of j omega goes to 0. This is for real axis crossings and the real part of G of j omega equal to 0 will give us imaginary axis crossings.

In this particular case, if you were to expand out by multiplying the theta. So, would have G of j omega to be equal to 1 by minus 6 omega square plus 6 plus j times 11 omega minus omega cube. Therefore, the real axis crossing would happen when the imaginary term is 0 which implies that eleven omega minus omega cube should be equal to 0 which has a solution omega equal to 0 has one solution the other being omega equal to square root of 11.

So, if you substitute omega equal to square root of 11 and compute G of j omega you will be able to determine the location at which the Nyquist plot crosses the real axis once more. It cross once at omega equal to 0 which we already saw it is going to cross again when omega is equal to square root of 11. Likewise determine the imaginary axis crossing we set the real part to be equal to 0 which implies that minus 6 omega square plus 6 would be equal 2 0, for another words omega would be equal to plus 1 we ignore the negativity term because we are only looking at omega being positive going from 0 to infinity.

So, we substitute omega equal to plus 1 and compute the G of j times 1 that will allow us to locate the point at which it crosses the imaginary axis if we join all these points by a dotted curve then you would get the final Nyquist plot. The phase goes from 0 to minus 3 pi by 2 and it causes the imaginary axis of omega equal to 1 it causes the real axis at omega equal to square root of 11, and it so happens that this particular case it avoids the point minus 1 comma 0.

So, since we have no open loop poles on the right half of the complex plane and the Nyquist plot is not encircling the point minus 1 comma 0 we conclude that a system 1 plus G of s has all of it is zeroes. So, in other words the close loop poles of the overall system namely G of s by 1 plus G of s are all on the left half of the complex plane.

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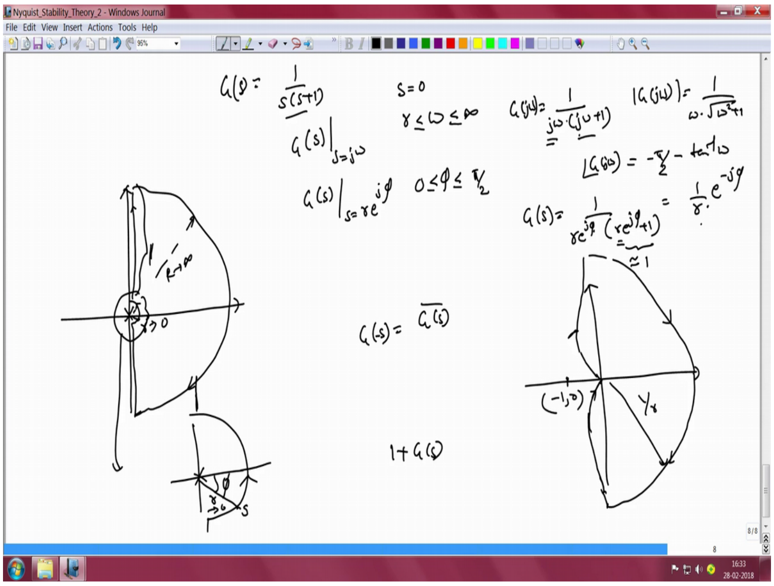

Now let us take one more example G of s is equal to 1 by S times S plus 1. Now this is a small problem with this particular example namely that the point is S equal to 0 is a pole of G of s and it happens to also be a point on the contour c. So, we want to make sure that none of the poles of our open loop system or on the contour c in which case we cannot evaluate the integral contour integral and apply the Cauchy integral formula for such a case. Therefore, what we do since we have an open loop pole that is on the contour c we distort the contour a little bit near the origin by choosing a semicircular path that avoids this point S is equal to 0 and the radius r of that semicircle would be tended to 0.

So, it is very small, but it just avoids the point is S equal to 0 and ensure that this contour does not have any poles on it is on itself subsequently the rest of the contour would be exactly as it was in the previous case the d shaped contour with radius R tended to infinity would be exactly same as before. And the rest of the contour would be very close the imaginary axis exactly as before.

So, what we do in this case is that you would evaluate G of s where S is equal to j omega for this part of the contour. And then for the semicircular part we compute G of s where S would be of the form S is equal to r e power j phi where if you zoom into this region I will draw the semicircle in a slightly magnified form our contour c avoids the origin. And then it goes further up we define this angle phi to be the angle made by some point S on this small d shape contour of radius R with the real axis.

Now, since we would have once again G of s to be equal to G of minus S to be equal to G of s bar for all points on the contour we need to consider phi going from 0 and up to pi by 2. So, we compute G of s for S is equal to R e power j phi for phi going from 0 to pi by 2 and we likewise compute G of s for S is equal to j omega for omega going from 0 to infinity, omega will not be exactly 0 omega will start at value r and go to infinity and the r radius r of course, is tended to 0.

So, if we first compute this term G of s for S is equal to j omega we would have it to be of the form 1 by j omega times j omega plus 1. The magnitude of G of j omega would be equal to 1 by omega times square root of omega square plus 1 the angle of G of j omega would be the angle of 1 by j omega plus the angle of 1 by j omega plus 1. The angle of 1 by j omega is minus pi by 2 it is always a purely imaginary number on the negative imaginary axis therefore, it is always minus pi by 2 and a angle of j omega plus 1 1 by j omega plus 1 will be equal to minus tan inverse omega.

Now, if we substitute S is equal to r e power j phi we would have G of s to be equal to 1 by r e power j phi times r e power j phi plus 1. You note that since r is tended to 0 r e power j phi plus 1 will approximately be equal to 1 therefore, G of s we simply be equal to 1 by r e power minus j phi. So, G of s would therefore, be a circle of radius 1 by r which is a very large numbers since r is tended to small r is tended to 0 and the angle will change from since phi goes from 0 to pi by 2 the angle of G of s will go from 0 to minus pi by 2.

So, you would discuss the Nyquist plot for this case we would have for the first part G of j omega omega going from small r to infinity. It when r is very small G of j omega will of the form 1 by r times square root of r square plus 1 and that is going to be a very large number and the phase is going to be close to minus pi by 2. So, we start here and then the final phase as omega tends to infinity will be equal to minus pi by 2 minus another pi by 2 that is our minus pi. So, the Nyquist plot would look something like this for this part of the curve.

Now, for the other part the phase would start at 0 and go to minus pi by 1. So, that curve would look something like this and the radius of this is going to be equal to 1 by small r. The Nyquist plot is obtained by reflecting this Nyquist plot about the real axis that is going to good look something like this, and we see that this curve always avoids the point minus 1 comma 0. And we have no open loop poles on the right half of the complex plane. So, our close loop system 1 plus G of s the G of s 1 plus G of s will always be on the left half of the complex plane, so our close loop system will always be stable.

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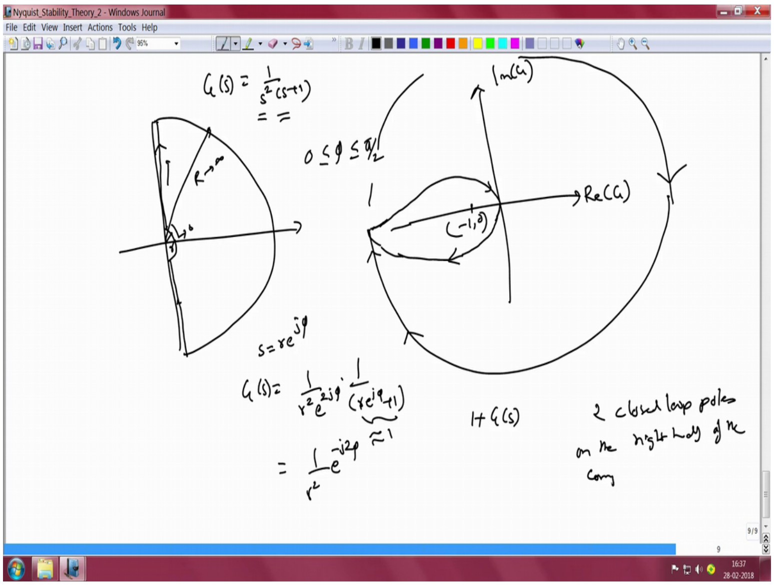

We will look at one final example we shall take G of s to be equal to 1 by S square times S plus 1 in which case once again since we have 2 poles of the open loop system at the origin we have to undertake a similar distortion to the d shaped contour exactly as we had done for the previous example.

So, this is how our d shaped contour is going to look to have a tiny semicircular region of radius r tended to 0 and this big d would have a radius capital or which is tended to infinity and these 2 straight lines are close to the imaginary axis. So, we once again compute G of s for values of S on this straight line and on this semi tiny semicircular part and then look plot it. So, if you were to do that you would see that the phase of this transfer function starts at an angle that is close to that is slightly greater than minus pi.

So, when we are near the origin somewhere here our Nyquist plot and the value would be somewhere here and then as omega tends to infinity G of s tends to 0. So, it would approach the origin in this particular manner and on this tiny semicircular patch you would have S to be equal to r e power j phi. And will be substitute it in this expression you would have G of s to be equal to 1 by r square e to the power 2 times j phi times 1 by r e power j phi plus 1. So, this term 1 by r e power j phi plus 1 can be approximately equal to 1.

So, G of s would be equal to 1 by r square e to the power minus j 2 phi. Since, we would have phi going from 0 to pi by 2 we would have 2 phi going from 0 to pi and minus 2 phi going from 0 to minus pi therefore, we would have this part of the curve being map to this particular semicircular arc. And for the rest of the curve you just have to take the reflection of this part about the real axis and that is going to look something like this. And the point minus 1 comma 0 will always be inside this contour and you see that this contour encircles the point minus 1 comma 0 twice.

So, this part encircles it once that part encircles it once more, so it encircles it twice and we have no open loop poles on the right half of the complex plane. So, what it indicates is that a close loop in the close loop system whose denominator polynomial would be 1 of 1 plus G of s will always have 2 close loop poles. On the right half of the complex plane, on the right half of the complex plane.