Control System Design
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Lecture - 12
Nyquist Stability Theory (Part 2/3)

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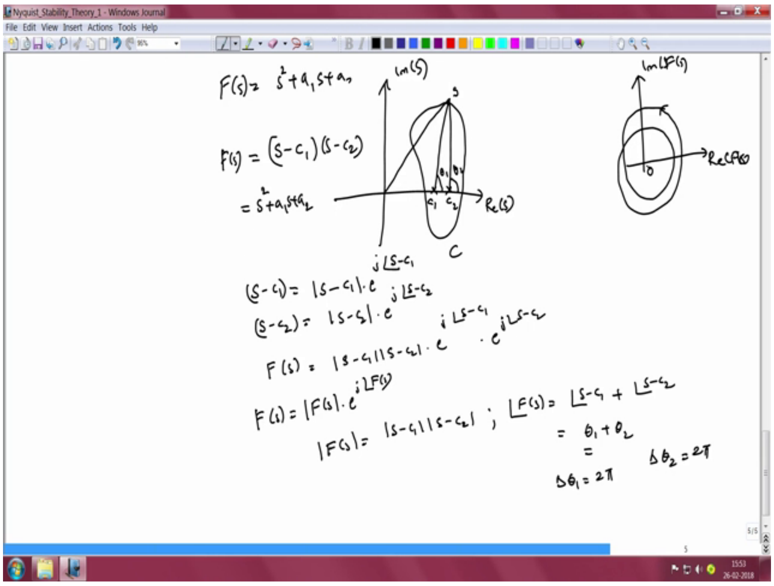

So, let us now take another example namely of a quadratic polynomial; F of s equal to s square plus a 1 s plus a 2. Now we want to tell whether the zeros of this quadratic equation or inside or outside a certain region in the complex plane. Once again that region is c bounded by the curve c is a real part of s is the imaginary part of s. Now unlike in the previous example we cannot buy inspection locate the 0 of this polynomial, we have to compute we want to solve this quadratic equation and the roots of a quadratic equation of course, can be obtained with by solving it. Once we have it we can potentially locate them in this complex plane and tell whether there inside or outside, the curve c. But once again we impose this constraint that we are not allowed to locate them in the complex plane.

So, we are not allowed to see where exactly the r fall in relation to the curve c, but yet we want to tell whether there inside or outside the region bounded by the curve c how do we do it? So, this is once again a significance because in practice we are really not given the roots of the polynomial, but yet we want to tell whether these roots are within a certain region or not. So, let us assume that we can factorize the polynomial into 2 roots, and let these roots be s minus c 1 and s minus c 2.

So, let F of s be equal to s minus c 1 times s minus c 2, which 1 expand it looks the way it is originally given namely s square plus a 1 s plus a 2. Now there are three possible cases 1 is when c 1 and c 2 are both inside the curve c, the second case is when 1 of them is inside the other is outside, and the third case is when both of them are outside the curve c. So, let us consider all these three cases and see what happens.

So, let us take the first case when both c 1 and c 2 are inside the curve, now we note that if we pick a complex number s on the curve c, this is the complex number s. Therefore, s minus c 1 would be this complex number, s minus c 2 would be that complex number therefore, we would have F of s to be the product of these 2 complex numbers.

Now, when we have a product of 2 complex numbers, I can write s minus c 1 the complex number s minus c 1 as the magnitude of the complex number which is namely magnitude of s minus c 1, which represents the length of this particular complex number times e to the power j times the angle of this complex number angle of s minus c 1, which is the angle theta 1 in this figure. Likewise, I can write s minus c 2 as magnitude of s minus c 2, which represents the length of this complex number starting from c 2 ending at this point s, times e to the power j angle of s minus c 2 which is represented by the angle theta 2 here.

Therefore the product is given by F of s equal to magnitude of s minus c 1 times magnitude of s minus c 2, times e power angle of s minus c 1 times e power j times angle of s minus c 2. Now if I want to represent the left hand side also in terms of magnitude and phase, I would have F of s to be equal to magnitude of F of s times e to the power j angle of F of s. By comparing the left hand side and the right hand side I would get that the magnitude of F of s, would be equal to magnitude of s minus c 1 times magnitude of s minus c 2. And the angle of F of s to be equal to angle of s minus c 1 plus the angle of s minus c 2, which according to our notation is theta 1 plus theta 2.

Now, what happens if we take the complex number s on the counter c in the counter clockwise sense? What happen to the complex number s minus c 1. You can think of this complex number as a kind of a dial, in a distorted clock. So, when we take this point s around the counter c, this dial will go round once by and change its angle by an amount 2 pi. So, when we take the complex number once around the change in theta 1, which I call delta theta 1 will be equal to 2 pi 2 pi radians. Likewise, when I take this point s around this counter c, you can once again imagine the complex number s minus c 2 to be another dial of this distorted clock. Then when this end point goes round this close counter, we can imagine that this dial will also undergo 1 rotation or change its angle by an amount 2 pi.

Therefore, if both the zeros namely c 1 and c 2 are inside the contour, and I want to compute F of s; let us say I want to graph real part of F of s verses the imaginary part of F of s. If I want to plot it for every point s on the counter c, what I would see is that the new curve would encircle the origin twice. Therefore, if both are inside you would have 2 encirclements of the origin in the F of s plane.

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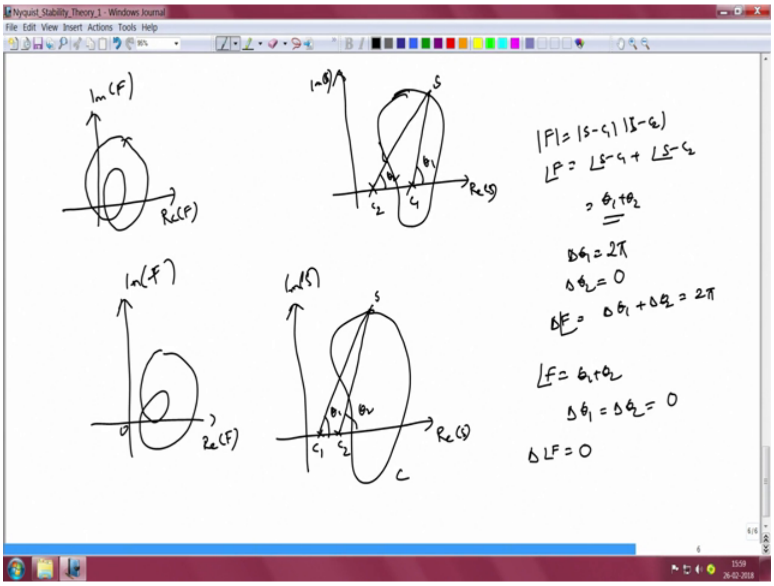

Now, let us take the second case. The second case is 1 when 1 of them is inside the other is outside and without lose of generality I have assumed that the 0 c 1 is inside this counter and the 0 c 2 is outside the counter. So, s minus c 2 would be this complex number s minus c 1 would be that complex number, we know that magnitude of F will be equal to magnitude of s minus c 1 times magnitude of s minus c 2. Angle of F would be equal to angle of s minus c 1 plus the angle of s minus c 2. If we call the angle of s minus c 1 as theta 1 and angle of s minus c 2, as theta 2 then this would be equal to theta 1 plus theta 2 and this is an important relationship that we are exploiting.

Now, suppose we want to take the point s around this counter c, we see that 1 of the dials namely s minus c 1 is inside the counter and therefore, the net changed angle delta theta 1, when we go round the counter once will be equal to 2 pi radians. On the other hand if you look at the change in angle of the complex number s minus c 2. Since c 2 is outside the counter we can easily see that as we move the complex number s, the angle will probably initially increased, then start to reduce become negative, and finally come back to its original value.

In other words the net change in angle will be equal to 0; what this therefore, reveals is that the net change in angle of F which is the net change in angle of delta theta 1 plus delta theta 2 will be only equal to 2 pi. If one of the zeros alone is inside the counter and the other is outside the counter net change in angle of F the third case is when both the roots c 1 and c 2 are outside the counter c.

So, if I want to draw the complex plane once again, and want to sketch the region bounded by this counter c, I shall mark of the roots of F of s as c 1 and c 2 let us assume that both of them are outside the counter. Now the complex number s minus c 1 is this complex number the complex number s minus c 2 is that complex number for any point s on this counter c. Now the angle of F will be equal to the theta 1 plus theta 2 where theta 1 is the angle made by complex number s minus c 1 with respect to the real axis and theta 2 is angle made by s minus c 2 with respect to the real axis.

Now, if we take this complex number in the counterclockwise sense once, we see that both these dials are outside this distorted clock. So, the net change in angle delta theta 1 and the net change in angle delta theta 2 when we go round this counter when the complex number s is taken round this counter once will both be equal to 0, which means that the net change in the angle of F will be equal to 0. Therefore, what we see from all these arguments is this very nice indicator that tells us whether the zeros of this polynomial F of s are either inside or outside of this counter c.

So, what all that I need to do is once I am given this counter c, I compute F of s on the counter. So, I compute real part of F and imaginary part of F and I plot it for all the different values of s that I can assume on the counter c. Now if the encirclement of this curve is once. So, it encircles only once something like this then we have only one of these zeros inside the counter.

On the other hand if in the F of s plane there is no encirclement at all then we can be sure that neither of the two zeros are inside this counter c. Therefore, by looking at the number of encirclements of the origin in the F of s plane it is possible for values of s that are on this counter c. It is possible to tell how many of the 0 or F of s are inside the counter and how many are outside the counter, can be generalized this to other higher degree polynomials of course, yes.

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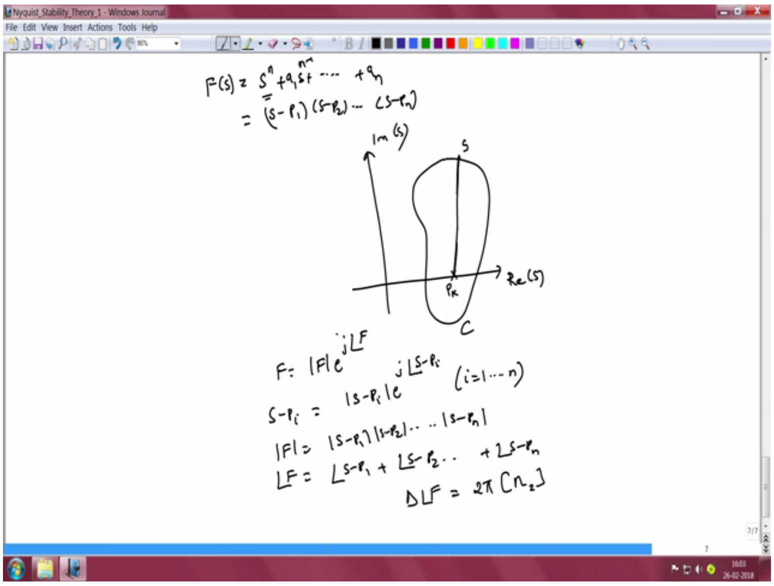

So, let us take the next case, where we have a polynomial F of s to be a general nth degree polynomial namely s power n plus a 1 s power n minus 1 plus so on and so, forth plus a n. Now we know that this polynomial has n roots, I shall call them s minus p 1 s minus p 2 up to s minus p n, but the real challenge here is that we cannot explicitly compute the locations of these roots for n greater than or equal to 5 analytically this is not possible. But yet what we want to tell is whether the roots of whether any of the roots of this polynomial are inside the counter c.

So, once again I shall mark out the counter c in the complex plane; suppose I have this counter c close counter we want to be able to tell how many of the roots of this polynomial are inside c and how many are outside c. We note that F is equal to magnitude of F e power j angle of F and s minus P i is equal to magnitude of s minus P i right e to the power j angle of s minus P i where i goes from 1 to n. So, what this indicates to me therefore, is that the magnitude of F is equal to the product of the magnitudes of each of these factors. So, s minus p 1 times s minus p 2 so on and so, forth times s minus P n. The angle of F is the sum of the angles angle of s minus p 1 plus angle of s minus p 2 and so, on and so, forth plus angle of s minus p n.

Now, if we want to consider a general complex number s on the counter c, then if let us say the k-th 0 p k is inside this counter then the complex number s minus p k is given by this particular phrasal here, and when s goes round the counter once we see that s minus p k the clocks dial will change its angle by 2 pi. On the other hand if the same complex number p k were outside the counter, then the complex number s minus p k will change the angle by 0. Therefore, the net change in angle of F would be equal to 2 pi times the number of zeros that are inside the counter.

Therefore, this general analysis tells us by looking at the change in angle of the complex number F, when we take the variable s around the counter c. This can also be reduced by looking at the number of encirclements of the origin. The change in angle of F when we go round the counter c, once can also be reduced by looking at the number of encirclements of the origin in the F of s plane. They are both equivalent and that will tell us the number of zeros of this polynomial F of s that are inside that counter c.

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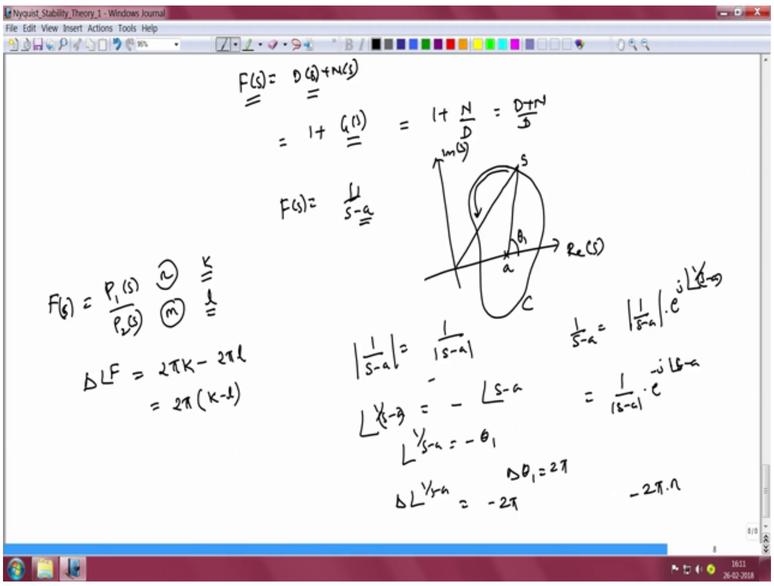

Now, in our particular application we are dealing with the function F of s is equal to D of s plus N of s, and we can repeat the steps that we just undertook for this particular polynomial and tell how many of its zeros are inside the counter c and how many are outside the counter c. Of course, from a practical perspective a general counter c is of no use, what we are interested to tell is whether our close loop system would be stable or not. In other words we are interested to tell whether any of the zeros of this polynomial are on the right half of the complex plane or on the left of the complex plane. And to do that we have to distort this counter c to include the entire of the right half of a complex plane, and judge how many of these zeros are on the right half of the complex plane.

So, it is a straight forward extension generalization of whatever we discussed whatever we have been discussing so, far. However, what often happens in practice is that, we are not given the function polynomial F of s is equal to D of s plus N of s, but instead we are given from experiments the open loop transverse function G of s directly. This can be evaluated experimentally or from mathematical modeling. So, it is easy to obtain directly G of s and not D of s plus N of s.

In other words what is given to us is 1 plus N by D, but without factorizing G of s into its numerator polynomial and denominator polynomial. And what we want to know is whether the principles of micro stability theory which we have explored so far which allows us to tell how many of the zeros of F of s are inside a certain counter c can be apply also for this more general case where I have F of s to be the ratio of 2 polynomials. Namely, it will be of the form D plus N by D how do we generalize whatever we have discussed for this particular case?

So, to do this let us first consider again a simple example namely F of s is equal to 1 by s minus a. So, here the numerator polynomial D plus N is just 1 denominator polynomial is s minus a, this is just a simple example that I have considered. To do this let us first draw the counter c I have drawn the counter c here, now the challenge is to determine whether the point a, is inside or outside of this region.

Well 1 straightforward way is to just locate it in this scrap, because we know that the pole of F of s is simply equal to s is equal to a, but assuming that we are not permitted to do that can we indirectly infer whether the point a is inside or outside the counter c. To develop the arguments let us first assume that it is inside the counter c ok.

So, the point s could be some point here, I should consider those points s that are on the counter. So, these are the complex numbers that I am interested in s minus a is this complex number. 1 by s minus a is the complex number whose magnitude is equal to 1 by the magnitude of s minus a in other words it is the inverse of the length of this particular complex number. The angle of 1 by s minus a is essentially equal to minus of the angle of s minus a. This is because if I want to write 1 by s minus a as 1 by magnitude of s minus a times e to the power j angle of 1 by s minus a, I can easily show a this equal to 1 by magnitude of s minus a times e to the power j minus j angle of s minus a. Therefore, if I want to call this angle to be theta 1, then the angle of 1 by s minus a is equal to minus theta 1.

Therefore when the complex number s goes round this counter and comes back to its original position, then the net change in angle theta 1 which would happen when this dial want to go round along with the complex number s would be 2 pi therefore, the net change in the angle of 1 by s minus a, will be equal to minus 2 pi. So, what this indicates is that, if any of the poles of F of s are inside this counter then the net change in angle of F of s will be equal to minus 2 pi times the number of poles of F of s inside the counter. So, let us say I have F of s to be the ratio of 2 polynomials, p 1 of s and p 2 of s.

Now, let us say the polynomial p 1 is of degree n, and the polynomial p 2 is of degree n and among this n zeroes of F of s, k of them are inside the counter c and among the m zeroes of p 2 of s l of them are inside the counter c. Then because of these k zeroes that are inside the counter c the change in the angle of F when the complex number s is this taken around the counter once will be equal to 2 pi times k.

And because of the L poles that are inside the counter c the change in angle is going to be equal to minus 2 pi times L. Therefore, by looking at the net change in angle of F, which is equivalent to saying that by looking at the net number of encirclements of the origin in the F of s plane. I will be able to tell the difference namely k minus l between the number of zeros of F of s that are inside the counter c and the number of poles of F of s that are inside the counter c.

Now, this is all that is required for us to extend this technique and apply it for determining the stability of a close loop system.

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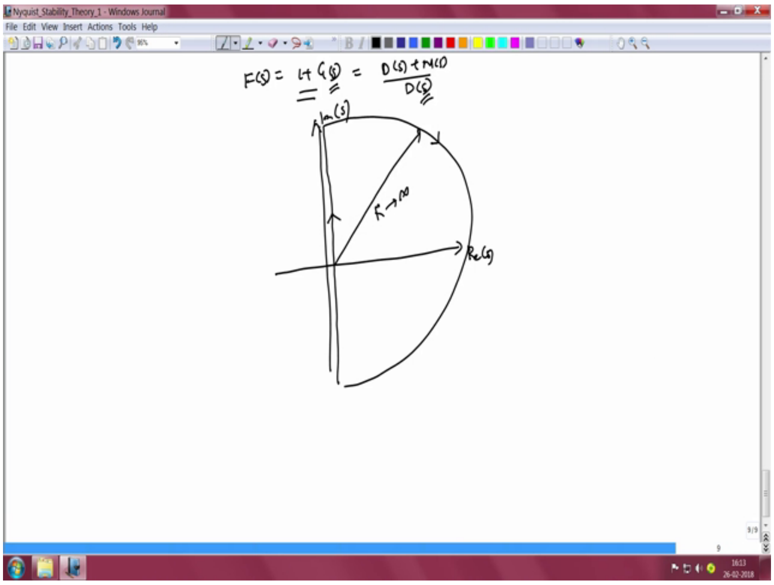

In the practical case we would have F of s to be equal to 1 plus G of s, which we know can be written as D of s plus N of s divided by D of s. Now D of s is the denominator polynomial of the open loop transverse function G of s. And now we have interest to determine if the zeros of 1 plus G of s, which are the poles of the close loop system are on the left half of the complex plane or not, which we can do by once again revisiting the complex plane and stretching our counter c to include the entire of the right half of the complex plane. I do that by shaping this counter to be this D shaped curve where the straight part of the D is very close to the imaginary axis, but just a little bit infinitesimally to the right of the imaginary axis and the semicircular part of this D is at a distance R that is tended to infinity away from the centre of the origin of this complex plane.

In this way I am able to consider all possible positions of the roots of 1 plus G of s on the right half of the complex plane.