Control System Design
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Lecture - 11
Nyquist stability theory (Part 1/3)

Hello. In the previous clip we looked that feedback control problem as one which allows us to get the output of plant x of t to track a reference r of t in the presence of uncertainties either in the plant model or in its environment in the form of disturbances and I have sketched here the basic block diagram for the feedback problem. We have a sensor that measures the output of the plant that you want to control which I have called P here and then, compares it with the reference in the differences, manipulated by a high gain function h and the output of the high gain function is fed back to the plant.

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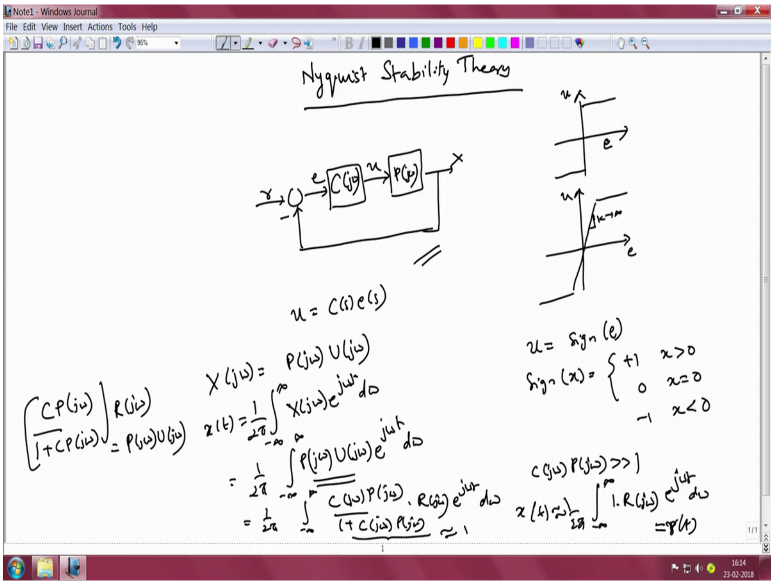

Now, what is quite interesting is to note that practically every single control engineer is ultimately implementing this block diagram in one form or another. So, it is the high gain of the controller that ensures that x of t would be equal to r of t. So, what are the typical controllers? So, we might have come across the most common controller is the on off control which we see in our refrigerators, in our air conditioner and many other such thermal systems. So, what do we have there? We have the output u of the controller to be related to the input e in terms of a step change.

So, if e is greater than 0, u is plus 1 if e is less than 0, u is minus 1 one can view this kind of a control action as the limiting case of a linear controller with saturation. This is the error and this is the output u and in the limit that the slope here k tends to infinity. We would have on off control. So, one can therefore view on off control as another version for the high gain controller, where the gain k is tended to infinity and there is saturation that limits the output of the controller.

So, this is therefore a simple example which one might not have realized has a version of high gain control, but is actually the case. So, in the case of on off control, you would have the control lot to be off the phone, u is equal to sign of e where sign is a sign function where given by sign of x is equal to plus 1. When x is greater than 0, when x is equal to 0 and minus 1 that x is less than 0, but this is not necessary the only way in which we can implement high gain control. There are several other control loss that have been developed over the last half a century. Also one other simple extension to what we just discussed now namely on off control is what is known as sliding mode control.

In sliding mode control, we would have you to be the sign of not the error e, but rather some dynamic function of error. For example, e dot plus e. Now, what happens in this case is that when this kind of a control law is applied, then the system moves to the surface e dot plus e equal to 0. So, this is a surface in the face space of error and its derivative.

Now, this particular term that we have inside the sign function is a choice of the control designer and we choose such a dynamic function of the error that it is stable and the roots of this characteristic polynomial are sufficiently fast for the error to decay down quickly. So, this kind of control strategy which is a general version of on off control is called sliding mode control. So, we see that even in case of sliding mode control, we have this high gain relating the input to the output let us responsible for getting the reference to be track by the close loop system.

Now, all these, the two techniques that I talked about here are non-linear control techniques are several others. Some of them are linear, some of them non-linear, but it is satisfying to underscore the fact that as far as getting x of t equal to r of t is concerned, all of them attempt to achieve high gain for the controller. That is also true in case of linear control with an exception that the input output relationship for a linear control. Linear control system would be a linear system itself.

So, if I have a controller see here. So, if I have a controller c, then if c represents the linear system, I can obtain its transfer function C of s and the relationship between e and u is therefore code would be given by u is equal to C of s times e of s and if my plant is also linear, then we would be dealing with linear control systems. Now, one would naturally want to wonder if this motion of high gain is also valid in case of linear control systems. It is so, but it is not easy to visualize it if one were to stick to the Laplace domain over, it is obvious when one goes to the Fourier domain. So, if one were to look at C of j omega, so one were to obtain the transfer function and represent in the Fourier domain and represent the plant also in the Fourier domain as P of j omega, then we would have X of j omega to be equal to P of j omega times U of j omega and X of t to be equal to integral minus infinity to infinity 1 by 2 pi X of j omega e to the power j omega t d omega and if I were represent it in terms of u, I would have this to be 1 by 2 pi integral minus infinity to infinity P of j omega u of j omega e to the power j omega t d omega.

Now, I can represent this term P of j omega times U of j omega in terms of the reference r and the plant transfer function P as C times P by 1 plus C times P of j omega times r of j omega. This would be equal to P of j omega times U of j omega. If I were to replace that in the equation, I would get X of t to be equal to 1 by 2 pi integral minus infinity to infinity C of j omega P of j omega by 1 plus C of j omega times P of j omega times r of j omega e to the power j omega t d omega.

So, what we see therefore is that in the frequency range, where C of j omega times P of j omega is much greater than 1, then the term here would be very nearly equal to 1 because C of j omega C times p divided by 1 plus C times P will be approximately equal to 1. So, if the frequency content of the reference is such that it is matching with the range within which C times p is much greater than 1. We would have x of t to be approximately equal to 1 by 2 pi integral minus infinity to infinity 1 times r of j omega e to the power j omega t d omega and that is equal to r of t. So, indeed therefore that a high gain for the controller is necessary even in case of linear system analysis to get x of t to be equal to r of t, however there is one central concern for all the different control strategies that have been developing the past.

So, while the trick for getting x of t to be equal to r of t is very simple and is common across all control techniques and for all plants that one might is to consider the really important bottleneck that every one of these controllers have to address is the issue of the stability of the close loop system. It can be easily shown, but when your controller gain is very high, your close loop system can potentially become unstable which means that some of the poles of the close loop transfer function would go on to the right half of the complex plane. It is therefore imperative for us as control engineers to simultaneously where 2 hertz. One is the hertz that you would were when you are trying to do performance engineering and another words getting the output of our plan x of t to track the reference r of t with the desired accuracy.

The second had that you would where is one of stability engineering where in we try to make sure that in our attempt to get x of t to be equal to r of t with the desired accuracy, our close loop system has not become unstable in which case our entire effort are getting the controller to have a high gain would be few tile. It is therefore very important for us to have tools at our disposal which will tell us if our close loop systems are stable or unstable. So, let us remind ourselves what transfer function we are talking of here.

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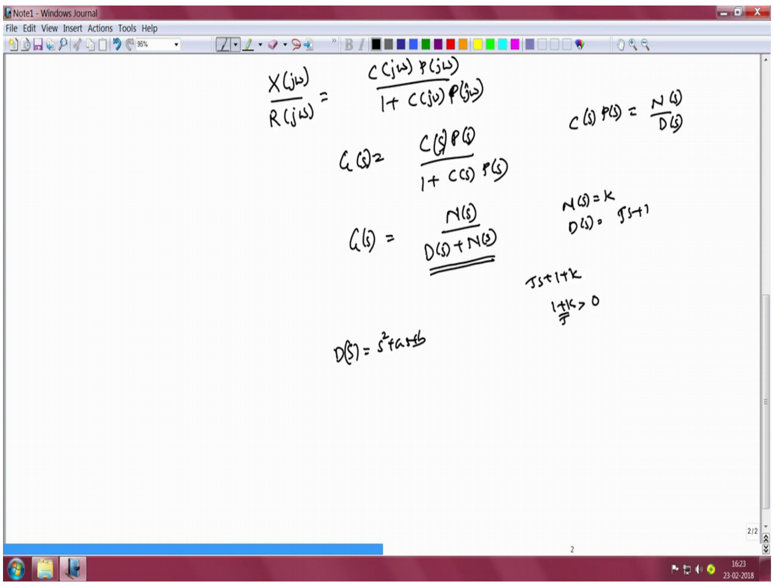

When we are interested in the stability, the transfer function that we are looking at is the one that relates the output X to the reference R and that is going to be given by C times c of j omega times P of j omega divided by 1 plus C of j omega times P of j omega. It is stability of this system that we are interested in and to pass the stability of the system, it is useful to represent the transfer function in the Laplace domain rather than in the frequency domain.

So, the frequency domain representation that we look that in the previous slide was very useful to convince ourselves, but X of t would be equal to R of t when C times P is very large. So, C times P therefore had a very nice physical meaning of a gain which needed to be large in order for our control performance objectives to be met, but as far as stability objectives are concerned, it is useful to come back to the Laplace domain in which case you would have the transfer function G of s to be equal to C of s times P of s divided by 1 plus C of s times P of s. Now, one has to determine a stability of this transfer function.

So, in order to do that we know that for the systems that would be working with, you would have our controller and plant to both be represented by linear time invariant ordinary differential equations. Therefore, both C of s and p of s would be the ratio of two polynomials. So, when I take their product, they two would be the ratio of two polynomials. Let us say N of s and D of s in which case G of s would be equal to N of s by T of s plus N of s.

Now, we look at the roots of the denominator polynomial and we need to make sure that the roots of the denominator polynomial which are the poles of G of s are stable or in other words, they are on the left half of the complex plane. If they are on the left half of the complex plane, the impulse response associated with this particular transfer function would, how would all comprise decaying exponentials and therefore, our close loop system would be stable.

Now, the question is how do we go about determining whether the roots of D of s plus N of s are stable or unstable. Well if the degree of the denominator polynomial is 1, it is obvious. So, if let us say N of s for some constant k and D of s for some term of the kind tau s plus 1, then we can by inspection tell that D of s plus N of s would be equal to tau s plus 1 plus k and we can clearly tell the conditions under which this system would be stable, namely that 1 plus k by tau should be greater than 0.

So, it is possible even by just visual inspection to determine whether a first order close loop system would be stable or unstable. For a second order, close loop system it is not that straight forward. One has to solve a quadratic equation because in that case D of s would be of the form s square plus s plus b and N of s could be either a constant or could be some linear function of s or at most a quadratic function of s. And therefore, 1 D of s plus N of s would be a quadratic polynomial and one can still determine its roots by analytically solving this quadratic equation.

If our D of s is 3rd degree polynomial, then it is still possible to obtain the roots of 3rd degree polynomial. There are several techniques available to factorize 3rd degree polynomials and therefore, be able to tell where the roots are and whether those roots are on the left half of the complex plane or the right half of the complex plane. If you come to the next level of complexity namely 4th degree polynomials, then it is possible to factorize 4rth degree polynomial into two quadratic polynomials in another word, 2nd degree polynomials and then, subsequently obtain the roots of these two 2nd degree polynomials. We are not sure whether our close loop system would be either 4th degree or 3rd degree polynomial. Depending on the complexity of the system, it could be a fairly high order system in which case let us stay let us say we have 5th degree polynomial for D of s.

Now, there are theorems in mathematics that say that it is not possible to factorize a general 5th degree polynomial and the same is applicable for higher degree polynomials as well. Therefore, our initial attempt to determine the stability of the close loop system by identifying the locations of the close loop poles directly by factorizing the denominator polynomial is not going to take it very far. It can take us at most 2 addressing the stability concerns up to a quad by quadratic or quadratic polynomials, 4th degree polynomials, but not beyond that and there is no guarantee that our plant times.

The controller would be would have a denominator polynomial that would be of 4th degree and not more than that. Also, we note that as engineers we are not just interested in determining the stability or answering this. Yes or no? Question is whether it is stable or unstable, but we want to be able to improve its stability, we want to be able to engineer the performance such that it has a desired amount of stability and the desired speed of response. This approach of determining the 0s of the denominator polynomial does not lend it so very well to our engineering intention of being able to tune the parameters of the close loop system to achieve a desired performance. So, therefore what other approach to be have at our disposal, one other approach is what is known as a routh array which you must have come across in your undergraduate education. The routh array looks at the denominator polynomial and uses a Routh table in order to determine whether that polynomial is stable or has stable roots or not. As powerful as it is, Routh array has this issue that it assumes that the coefficients of this denominator polynomial are constants; however as I discussed control system designers often have to deal with plants whose parameters might be uncertain and which might change with time and so on.

Therefore, while Routh array is definitely useful in the contacts when we have fairly good knowledge of the plants parameters, it does not work as well when we have uncertainties associated with the plants parameters. Hence, what we would look at now is an alternate technique of determining the stability of the close loop system and that is based on the Nyquist stability theory. So, the subject of today's clip would be nyquist stability theory. Now, if we come back to this problem of determining the stability of the close loop system, we note that it is not really necessary for us to tell exactly where the close loop poles are. In another word, it is not really necessary for us to tell the exact roots of the denominator polynomial of the close loop system. All that is necessary or expected of us is to check whether any of these poles are on the right half of the complex plane or not. If this one decision can be made without even knowing whether, without even knowing exactly where these poles are, then we are good. We would know whether our close loop system would be stable or not.

So, Nyquist stability theory answers precisely this question, but it does more than just that. It allows you to tell whether you have close loop poles in any part of the complex plane or not.

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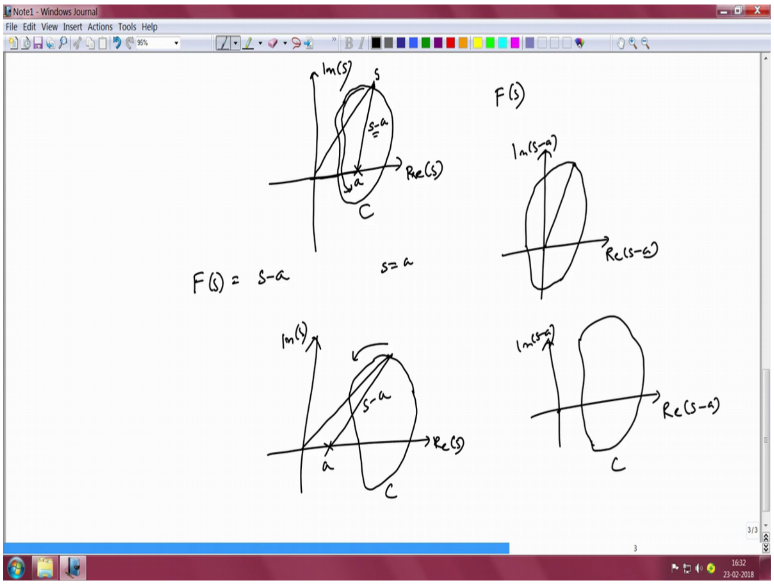

So, if I have a general complex plane, what Nyquist stability theory allows me to determine is whether you have any poles of a particular close loop transfer function in any given region within the complex plane. So, if I were to demark at a close region within the complex plane by its boundary C, then I can apply Nyquist stability theory to tell me whether a function F of s has poles or 0s within that region and the power of this technique is that I can do it without actually having to factorize F of s and determining exactly the locations of its poles and 0s. So, in order to illustrate how Nyquist stability theory works, let us take a numerical example. Let us start with a very simple function F of s is equal to s minus a. Now, what is expected to start with is to verify whether 0 of this turns of 0 of this polynomial F of s equal to s minus a is within this close contour c or not. While you might say this is a very simple trivial problem because I know that 0 of this polynomial is s is equal to a. I just locate this point s is equal to a in the complex plane. Wherever it could be, could either be here or it could be there it might depend on the exact numerical value of a and I just see whether this point is inside this contour C or outside that contour. So, visually I can tell.

Well that is so, but suppose we want to impose further restriction that you are not allowed to locate within the complex plane, but yet you are asked to tell whether a is within c or not, how would be do it ? So, this is where we would have to imply Nyquist. The concepts of Nyquist stability theory, the reason that we want to do this is because in real life our F of s which would be the denominator polynomial of our close loop system cannot be factorized. So, we cannot tell exactly where its roots are, but yet by doing a few simple tests based on Nyquist stability theory, we want to be able to tell how many of the 0s of F of s are within the contour C and in the case when this contour C encompasses the entire of the right half of the complex plane, then we would have, we would be in a position to tell how many of the 0s of F of s or on the right half of the complex plane.

So, let us start off with this function F of s. For the moment let us assume, but the point a is inside the contour. Now, let me consider a general complex number s on the contour. So, the complex number s is represented by this particular phase or this particular phase or here is the complex number a. Therefore, this complex number is essentially s minus a. Now, if I want to take this complex number which is on the contour C and take it round the contour in the counter clockwise direction once, what will happen to this complex number s minus a if I were to graph this complex number s minus a. So, in other words the real part of s minus a versus the imaginary part of s minus a in the present configuration, the complex number looks something like this.

Now, when I take the variable s around the contour C, what would happen is that this, the end of this complex number would trace a curve that is similar in shape for this particular case to the curve C and what is more significant is that it goes around once and therefore, l circle the origin. Once the angle of this complex number s minus a changes by 2 pi if the point a is inside that contour. On the other hand let us take the other scenario where the point a is outside this contour in which case once again if we were to consider a point s on the contour, you would have s minus a to be this complex number and when we take the point s in the counter clockwise sense around the contour, once there what would happened to this complex number s minus a. If I want to keep track of the change in angle of this complex number, what I would notice is that the net change in angle of this complex number s minus a would be 0.

So, in another words, it would execute a motion that would trace again a curve similar to see in the plane that plots real part of s minus a versus the imaginary part of s minus a, but in this case it will not encircle the origin. In another words, the net change in face of the complex number s minus a will be equal to 0. In this case, therefore I have this one nice test where in even if I cannot locate the point a in the graph paper where I have drawn my curve c on the complex plane, I can still determine whether a is inside that curve or outside that curve by looking at whether that curve encircles the origin in F of s plane or does not encircle the origin in F of s plane.