**Relativistic Quantum Mechanics**
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**Lecture - 42**
**Lagrangian formulation of QED, Divergences in Green's functions, superficially divergent 1-loop diagrams and regularization**

In the lectures of this course, I have developed the subject of quantum electrodynamics roughly in the order in which it was historically formulated. This is instructive, because it underlines the various problems that occur in development of a theory either because of new experimental discoveries or because of theoretical inconsistencies and then what kinds of modifications are needed to overcome the hurdles. But nowadays the theory of quantum electro dynamics is described more as a finished product. So, we do not start at the historical level first which say the Maxwell's equations, then the Dirac equation, then efforts to combine the two and calculate various processes, and then come to a comprehensive picture. But rather than that we write down the full theory in one shot and then derive various consequences depending on the application.

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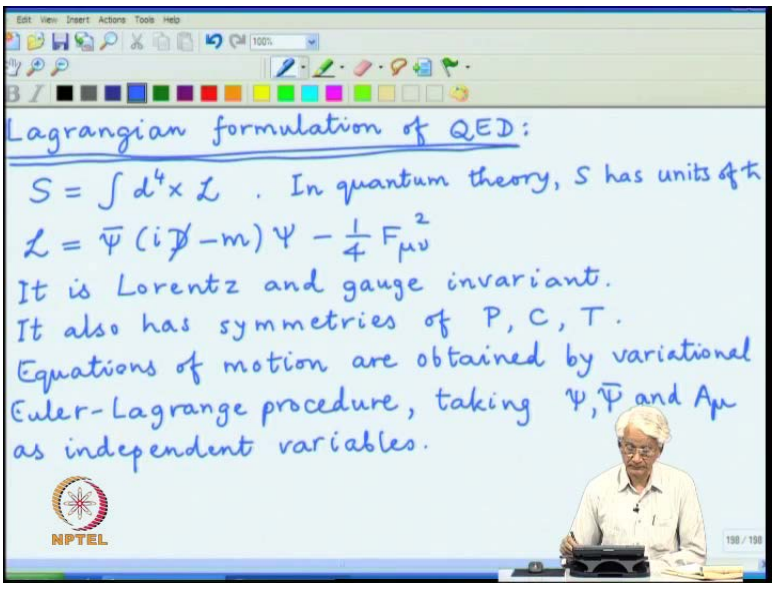


So, this formulation is often described in the Lagrangian framework, and here we start down with writing down a Lagrangian, derive the equations of motion from that Lagrangian and then investigate the consequences. This is the way it is now given in

most text books, but as I said historically this was not the sequence of development of QED. In fact, many of the processes were calculated and matched with experiment even before the full formulation was completed. So, in this more conventional framework the theory now starts by defining the action which is integral of a Lagrangian density, and in quantum theory it is useful to know that S has units of angular momentum which is the same as units of the planks constant.

And that plays an important role in trying to do calculations with S because we are trying to investigate the quantum effects which are characterized by appearance of planks constant in all the observed phenomena. So, the Lagrangian density for QED is psi bar i D slash minus m psi minus one-fourth F mu square, and this has all the symmetries which we have already discussed. So, it is Lorentz and gauge invariant which is seen in the fact that all the indices are contracted; f mu square is just shorthand for F mu nu contracted with F mu nu. And gauge invariants automatically is taken care of by appearance of covariant derivatives; on top of that it also has the discrete symmetries which we discussed parity charge conjugation and time reversal.

So, everything which follows from this action will follow the rules of these symmetries, and in particular we can construct all the equations which we have used by taking this Lagrangian and following the Euler Lagrange variational procedure to derive the equations of motions. So, the equations of motions are obtained by variational Euler Lagrange procedure, and to apply that procedure we have to define which variables are the independent variables and which are the dependant variables.

And in this particular language taking the variables psi psi bar and A mu as independent, it is important to note that psi and psi bar are treated as independent variables in this lagrangian framework which is different than the Hamiltonian procedure which we used earlier where psi bar was treated as psi dagger multiplied by gamma 0. Here they are independent variable which is indeed necessary to obtain the correct equations, because the term in the Lagrangian has only first derivative when fermion fields are involved. For the photon it is a secondary term each of mu nu has one derivative, and so there is a more standard prescription to get the equations for that.

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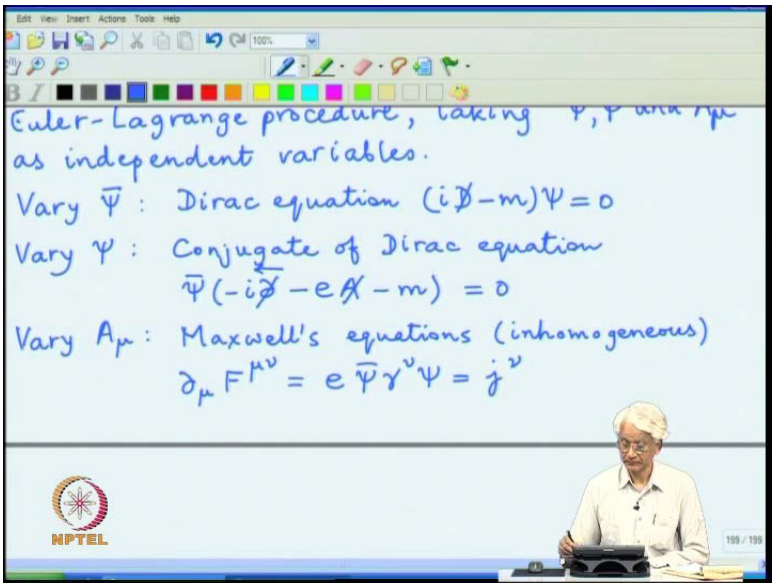


So, let me just write down the equations which are easily seen. So, the Dirac equation is easily obtained by just varying psi bar; psi bar does not have any derivatives acting on it, and so the Dirac equation appears rather immediately which is i D slash minus m, psi equal to zero. So, this is very simple. I should mention that the covariant derivatives have a very specific structure in the notations; I have used where D mu is ordinary derivative plus i e A mu in the units in which h cross n c r set equal to one. So, this is the Dirac equation. Now, one can vary psi; it is more convenient in this particular case to first do integration by parts. So, this derivative in D slash now acts on the left on psi bar with a change of sign, and then the equation does not have any derivatives of psi remaining in it.

So, that gives this conjugate of Dirac equation which can be written as psi bar minus i del slash with the derivative acting in the opposite side minus e A slash minus m equal to 0. And so this is again the standard form one can one obtains this equation from the Dirac equation by the usual procedure of taking Hermitian conjugates and changing the orders of derivatives from one side to another depending on where psi and psi bar appear. So, this is also standard result, and the third equation appears where I vary A mu. Now A mu appears in two parts; inside of mu nu it is acted on by derivatives and inside psi bar d slash psi it does not have any derivative, but it just couples to psi bar and psi.

And now by differentiating by D mu A nu over f mu nu you get the result that this gives the Maxwell's equations or rather only the inhomogeneous parts of Maxwell's equation. The homogenous part is already solved exactly by the definition of f mu nu, and this equation is then D mu F mu nu is equal to e times psi bar m nu psi, and this is the quantity which we had called the current. So, all this equations are implicit in this total Lagrangian, and the normalizations have been chosen so that we get the same normalization which we had used earlier in the Hamiltonian description where various things were put together. And so this is a rather complete description; you can get all the ingradients from it. The classical equations or this Euler Lagrange equations if you solve they give the classical behavior, and then the fluctuations around this classical solutions give the quantum effects.

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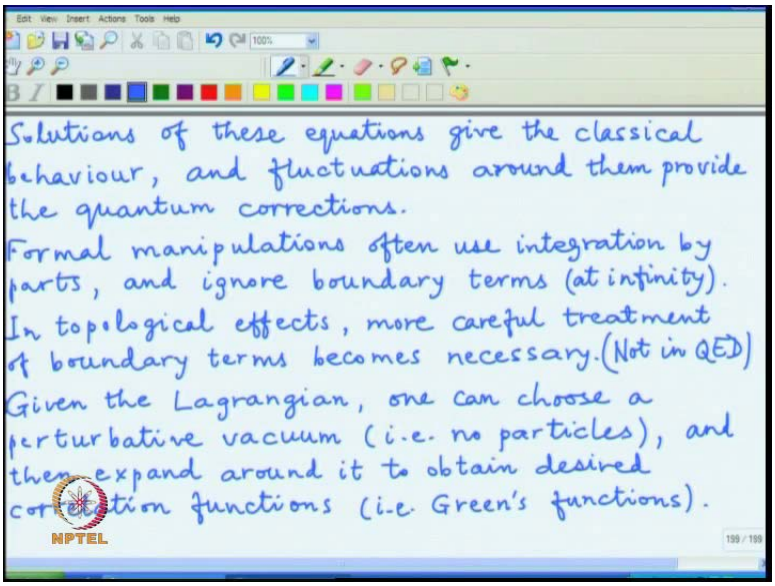


So, solutions of these equations give the classical behavior and fluctuations around them provide the quantum corrections. I also want to state that in doing these formal manipulations with action or the Lagrangian density, so they often use integration by parts in dealing with the theory. And then they ignore the boundary terms arising from these integration by parts; typically they are sitting at infinity and you assume that nothing peculiar is going on at infinity. But there are certain theories where there are so called topological effects more careful treatment of the boundary terms becomes necessary. Fortunately in case of QED this trouble is not present, and we can just ignore the boundary terms in dealing with this Lagrangian framework.

Furthermore given the Lagrangian one can choose a perturbative vacuum which means no particles, and then expand around it to obtain desired correlation functions what we also called the Green's. And to do that all one has to do is a simple Taylor series expansion of various terms which appear as interactions in the Lagrangian and then build on it given a certain number of external states which define the Green's function, and order by order one can proceed in this particular calculation.

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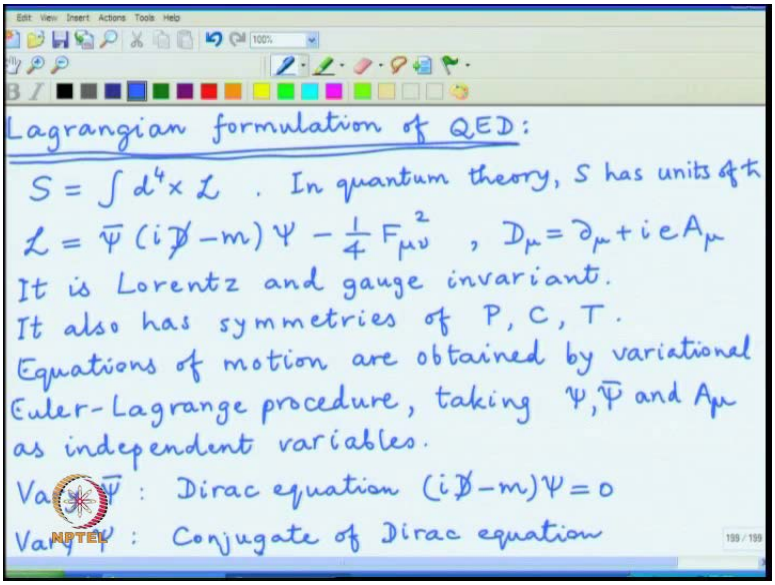


The Feynman rules which we used for this procedure they follow from the Lagrangian in a straightforward manner; that is because we constructed the Feynman rules by looking at the equations of various interactions and the equations are already present in the Lagrangian. So, it is no surprise that the Feynman rules will automatically come from the Lagrangian. So, all this machinery now can be quickly summarized given the Lagrangian we have the Feynman rules and then we can calculate any particular correlation function in perturbation theory which we have illustrated by working out several specific examples.

Now I would like to mention a new feature which is much more clearly apparent in this framework, and it is also dealt with much more easily in this particular framework. And that is this in calculation of this Feynman diagrams divergences may appear in the language which we have used. They appear inside momentum integrals, and we need to

develop a procedure to deal with that. So, now let me disgrace onto a discussion of how this divergences appear in calculations of Feynman diagrams.

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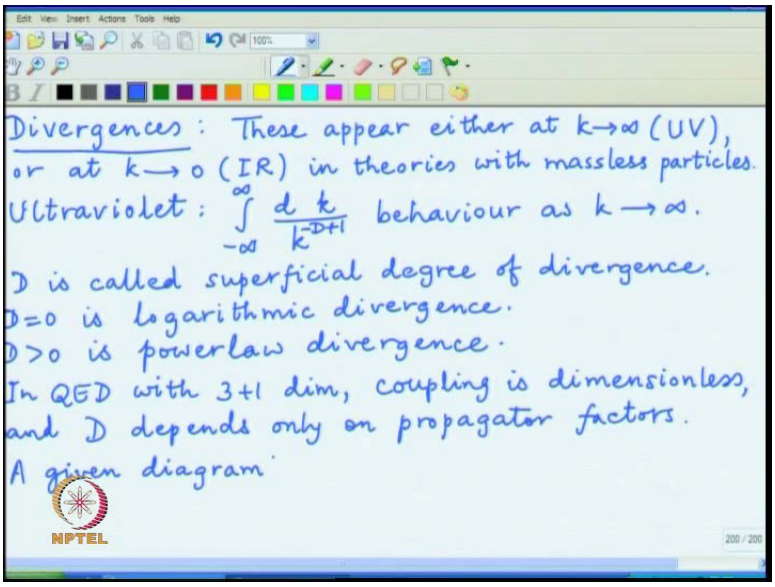


These divergences appear in two particular limits; once when the momenta go to infinity which is often denoted as ultraviolet divergences or at momentum going to 0 which are referred to as infrared divergences. And in particular these infrared divergences appear only in theories with massless particles, because we will soon see that the origin of the divergences is the various factors of propagators. And only when mass is 0 the k square plus m square which appearing in the denominator in all these propagator factors can become singular QED; we do have to worry about this infrared divergences, because the photon is a massless particle. But let me first deal with this ultra violet part which is a consequence of having momentum integrals going from minus infinity to infinity of some dimensionality divided by some power of dimensions.

And typically the angular integrals are all thrown out, or rather they are all finite, and we already have to worry about the radial integral, and in that particular case I can drop this D from the exponent and leave only this behavior as k going to infinity. So, whenever the value of D; so D is called superficial degree of divergence. And to be dimensionally correct I need to change these factors to k minus D plus 1, and in particular D is equal to 0 is logarithmic divergence, and D greater than 0 is power law divergence. So, to understand this divergence we need to estimate what is the value of D for a given

Feynman diagram, and that follows from using the various rules of counting powers of momenta in each of the factors that appear inside Feynman diagrams.

In case of QED in QED with 3 plus 1 dimensions the coupling is dimensionless and D depends only on the propagator factors. So, we just have to now work out what those various propagator factors are in a particular diagram. So, in a given diagram there are vertices which I will denote as v, then there are internal propagator lines I, external state lines E and closed loops. And we will only talk about irreducible ones and also independent of each other and denote them by L. So, now there are various relations between D and this numbers of V, I, E and L.

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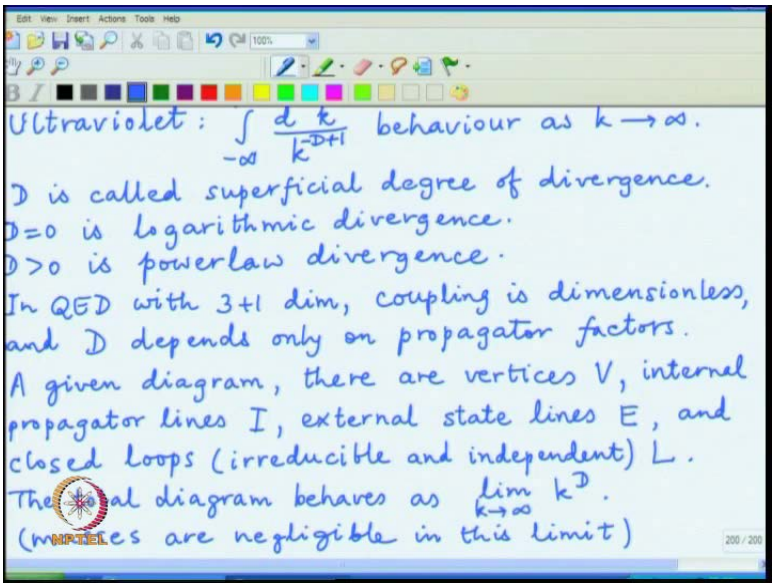


So, the diagrams; so total diagram behaves as limit k going to infinity k raise to D from the structure which we have assumed, and we want to avoid the positive or 0 powers of D. So, we just have to count the overall powers; the masses are negligible in this particular limit, because the momentum is going to infinity, and we just have to do a simple rule of putting the various factors together. So, now we can do this counting by looking at all the factors of propagators which involve momenta, and we will do it a step by step.

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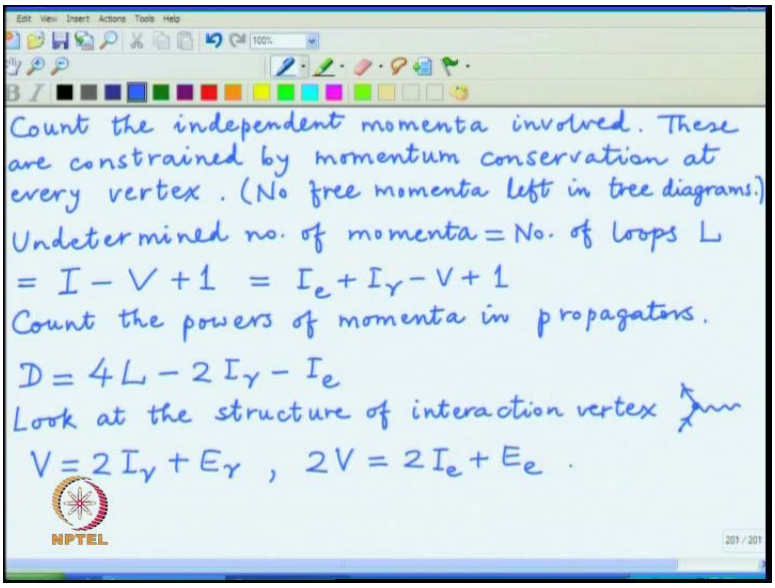


So, the first step is to count the independent involved in the particular process, and these are constrained by momentum conservations at every vertex, and we know in particular that in case of tree level diagrams there are no free momenta left in tree diagrams once all these conservation rules are applied. So, counting is very simple, every internal line brings in one. So, independent momentum show undetermined number of momenta; it also happens to be the same as number of loops, and these are the momenta which by the rules of quantum theory they are not fixed, and they are not observed. So, we must sum over all the possibilities, and by summing over all the possibilities are where these integrals over momenta will arise, and they can leave to divergences according to the value of D. And we just look at the momenta which are introduced by drawing the general diagram.

So, every internal line gets a new momentum label. Then every vertex introduces one delta function which eliminates one degree of freedom for a momentum. So, we have I minus V, but all the vertices cannot be counted as constraints, because one of the constraints refers to the overall total momentum conservation which depends only on the external links and not on the internal momenta. So, actually only V minus 1 delta functions can be eliminated in a diagram, and so the net number of undetermined momenta is I minus V plus 1. And sometimes it is useful to separate the contributions of electrons and photons from this diagram, and this is a number of loops equal to L. So, this is one relation between the number of loops and the number of lines and vertices.

Now we can also count the powers of momenta in propagators, and that gives the degree of divergence D. Now it has various sources. One of the sources is this momentum integration which is left over and which must be summed ups, and so the first term is just 4 times the number of loops. Every loop is in 4 dimensional momentums integral and one has to take care of the numbers, but then there are negative powers in the denominator coming from the propagator. And those are two powers for every photon the propagator is 1 over k square, and one power for the electron, because the propagator is one over k slash neglecting the mass. So, this is the expression for the degree of divergence, but clearly there are too many unknowns here which do not have enough information to fix the value of D from these two equations alone.

So, we need more equations, and to do that we look at the structure of interaction vertex, and in this particular theory it has a very specific form. It involves two fermion lines and a photon line, and so there is a relation following from this between the number of vertices and the number of propagator lines. So, first let us look at the number of photon lines. So, each vertex will have one photon line, but this photon line could come either from an internal line or from the external line. Every internal line gets counted twice because a photon goes from one vertex to another. So, we have V is equal to 2 I gamma, but if it is an external line it gets counted only once, and so there is a relation between the number of vertices and the number of internal photon lines and external photon lines.

So, that is for the photons; we can do the same job for the electrons, but now at every vertex there are two electron lines. So, if you count the number of electron lines there will be 2 V from counting the vertices, but on the other hand they can be counted also in terms of internal propagators. The internal line will be counted twice and the external one will again get counted once. So, we have these two relations, and I want to stress that many of these relations are just topological, and it is only in the last two equations we are using the property of the particular interaction vertex; the other ones can be used in a more general theory also. So, now we have enough unknowns and enough equations so that we can eliminate whatever we want.

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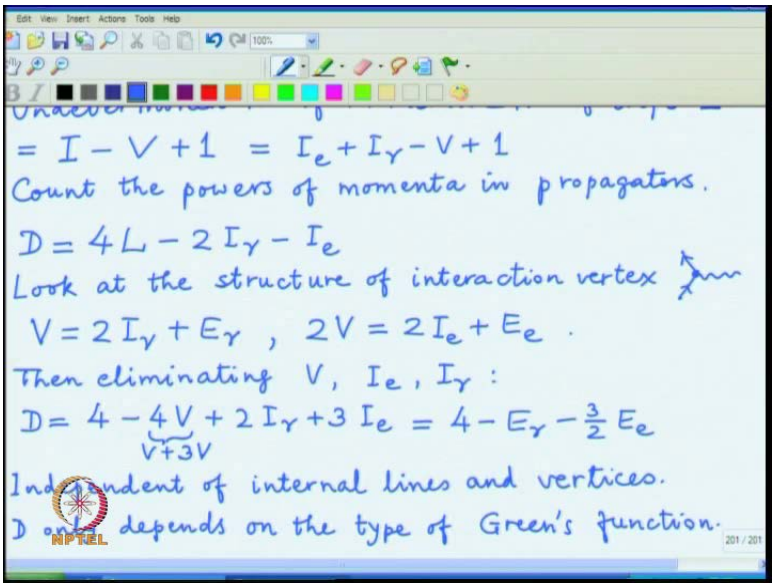


And so we will now look at what is the value of D, then eliminating V, I e and I gamma want to get rid of all these objects. So, then we get D is equal to 4 times the number of loops, and well, we just have to do a little bit of algebra. So, it is for which I will kill explicitly outside and from this one which is there in the diagrams, and then there is minus 4v, and that has to be rewritten in terms of internal and external lines. So, let me just 4 minus 4 v, and then there is a little bit of cancellations between the values of this photon and an electron internal lines.

So, it is two times I gamma plus 3 times I e. So, we got rid of the number of loops. Now we need to simplify this thing further, and we can substitute for v as well, but we will do it in a specific manner which will cancel all this internal line factors. So, this 4 v will be written as v plus 3v. The first we will be substituted from the photon equation and that will cancel this 2 I gamma exactly, and the 3 v will be substituted by three half times the electron equations which cancels this 3 I exactly as well, and the result then looks much simple. So, it is 4 minus E gamma minus 3 half E of the electron.

So, this is now the expression which is independent of internal lines and vertices. It is a peculiar result for this particular theory, but it is extremely useful. So, that the D depends only on the type of Green's functions without worrying about internal details about how many orders you want to calculate the Green's function, and when you are calculating the higher order when new things will go wrong or not, the order of the calculation does

not appear here at all. It depends on the Green's function has how many external line, and that is extremely helpful in doing calculations with this theory and looking at the various kind of divergences that arise and how to find solutions for them.

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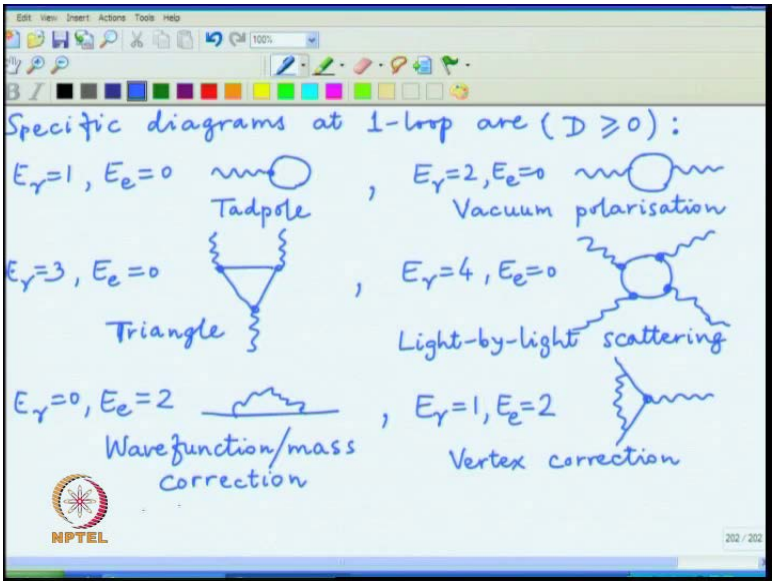


So, before describing the solution let me just describe the specific type of divergences which follow this particular rule, and we will only look at diagrams involving only one loop where these things appear in tree level diagrams. There are no such divergences, and this can be quickly listed; if there is one external photon and no electron there is a diagram which is called the tadpole diagram for the reason of its shape. So, this is tadpole, then one can have E gamma is equal to 2 E e equal to 0; it gives a diagram which looks like a bubble connected by two ends, and this is a correction to the photon propagator. And this is called the vacuum polarization diagram, then there are three photons and no electron labels, and this is a diagram which is drawn in the shape of a triangle with photon attached to each vertex.

Then there is the four photon lines with no electron line, and so that will be a diagram which looks like this, and for obvious reason this corresponds to a process which is called light-by-light scattering, two incoming photons and two outgoing photons. So, these are all the diagrams where there are a number of photon lines but no external electron lines. And then now one can write down the diagrams which have external

electron lines, and the simplest one is two electron lines and one photon loop interacting with it.

And this diagram corrects the fermion propagator and so it corresponds to corrections of either the wave function or the mass which are the two terms involved in the fermion propagator, and the diagram with four fermion external lines has D less than zero. So, we do not have to worry about it. There is just one more possibility left with one photon and two electron lines, and that corresponds to the correction of an electron photon vertex, and this essentially exhausts all the list of diagrams at one loop which have D greater than or equal to 0; everything else will be convergent and it turns out that.

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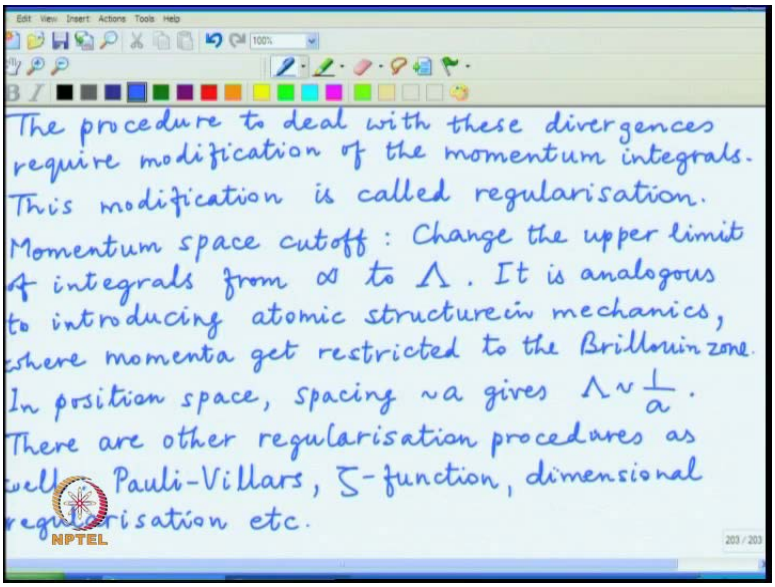


Once we take care of these divergences at one loop no further divergences at higher orders or loops need to be tackled, and this is a particular feature of quantum electrodynamics connected with the fact that the degree of divergence depends only on the number of external lines, and one can give a detailed proof of this result based on extensions of this topological counting rules. I would not do that, but it is a useful thing to keep in mind that one loop we have to take care of a certain number of things and after that there is nothing at all, and of course, at 0 loop which is the tree level there are no divergences by definition.

So, this is a list of all the possibilities of various divergences, and we have to now deal with how to control this particular divergences. And one thing I should add that

symmetries of the theory can reduce the value of D from the superficial value above, and this is a fact because we have not used any details of symmetries in calculating this diagrams, but they are the properties which involve the specific structures of the vertices and propagators. And they can follow the rules so that certain contributions vanish, and then we do not need to calculate those corrections at all.

So, this is as much as I would like to say about the appearance of the short distance divergences. Now come the question of how we are going to deal with them? So, the procedure to deal with this divergences require some or the other modification of the momentum integrals, and this is where various algebraic techniques of how you change this integrals do enter this particular picture, and this modification is called regularization. The easiest among this list to understand is just something which is called momentum space cutoff which means that change the upper limit of integrals from infinity to lambda. Lambda is some number large enough so that it does not affect the calculations which we are doing with a certain value of external momentum on the legs, but otherwise it is not infinity, because if it is really going to infinity things will diverge.

And the common way to understand this thing is an introduction of atomic structure in a mechanics where momenta gets restricted to the brillouin zone and do not go all the way till infinity. So, one can phrase this cutoff in various ways. So, in position space, spacing which is of some size a gives the momentum space lambda which is of the order of 1 over a, and this is a technique which is easy to understand because we have real atomic systems in which such behavior applies when we are trying to do useful mechanics calculations. We have to cut off the momentum integrals, and it works, and we have quantitatively correct descriptions of how the atomic structure plays a role in cutting of these integrals.

And so it is a convenient method to understand the divergences and its regularization, but there are other procedures as well, and they go by various names. One method is called Pauli-Villars, another method is called zeta function regularization, and yet another one is called dimensional regularization, and one can probably think of some other possibilities as well. But the use of this various techniques is limited by ease of algebraic calculations with them and in the present day machinery it is actually the dimensional regularization form which is the most common in controlling the divergences that appear in quantum field theory.