Relativistic Quantum Mechanics
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Lecture - 41
Bound state decay, Non-relativistic potentials

In the previous lecture, I described the calculation for production of a bound state by combining relativistic perturbation theory with non-relativistic bounds state wave function, and the total transition matrix element in that case just turned out to be the transition matrix element for production of f f bar pair multiplied by the wave function at the origin of the bound state. And now to complete these cross sections calculations we have to perform the usual tricks and evaluate this T f i instead of just the relativistic production cross section, but that is just a trivial multiplication factor by mod of psi at r equal to 0. But the remaining part is the same as what we had encountered before. So, now, we calculate it explicitly.

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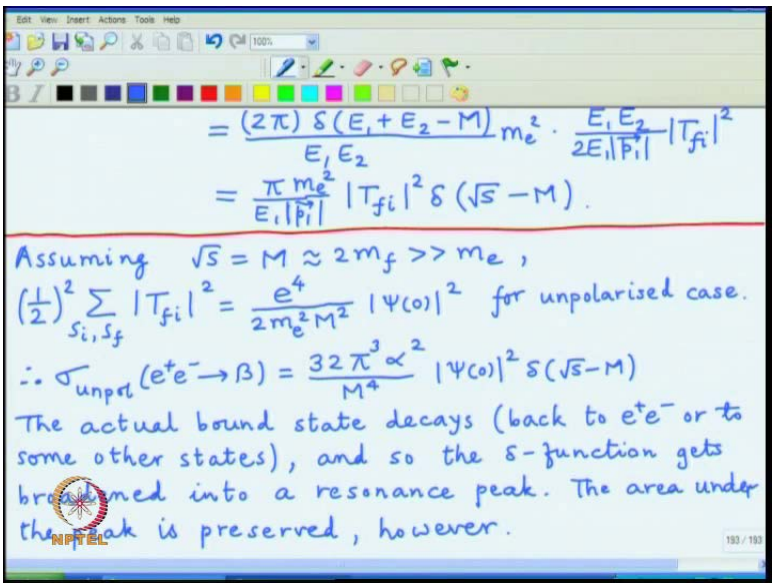

So, assuming square root of s which is also equal to the mass of the bound state which is roughly equal to two term mass of the fermion, and we will take this to be much heavier then the electron, and so we can drop the factors of electron mass square compared to the electron momentum square in particular E 1 and magnitude of p 1 both of them are equal to half of m.

So, then the usual unpolarized cross section requires average over the initial spins and sum over the final spin. We have done this calculation before; I will just write down the answer and which is E raise to 4 divided by 2 times m e square times M square and then the whole thing. This is the matrix element for the f f bar production multiplied by psi 0 and its absolute value square which appears, and this is a result for unpolarized case. And then we can plug it back inside to get sigma unpolarized for e plus e minus going to B; various factors cancels out, m E square cancels E 1 and p 1 are both equal to m by 2, and the factor of E raise to 4 can be converted to alpha square by multiplying by 4 pi square.

So, then the result is counting all the factors of two's, it is 32 pi cube alpha square divided by m raise to 4 mod psi 0 square remains, and we have the delta function also which remains. So, this is explicit result for e plus e minus going to f f bar bound states. In practice this bound state is not going to be stable; it can decay back into the e plus e minus by reversing the time evolution, or it can decay into some other particles if that is possible as well. So, the actual bound state decays, and so the delta function actually becomes smeared or broadened into resonance peak.

And to estimate how much is the broadening we have to actually calculate the decay probability, but even when the peak broadens the area under the peak is persevered, however. And so they calculate that quantity which we calculated represents the area under this particular resonance peak; we just have to find out the shape of the curve explicitly if you want to compare these numbers to experiments. So, now let us go on to the calculation of the width of this peak are equivalently the decay right of this bound state.

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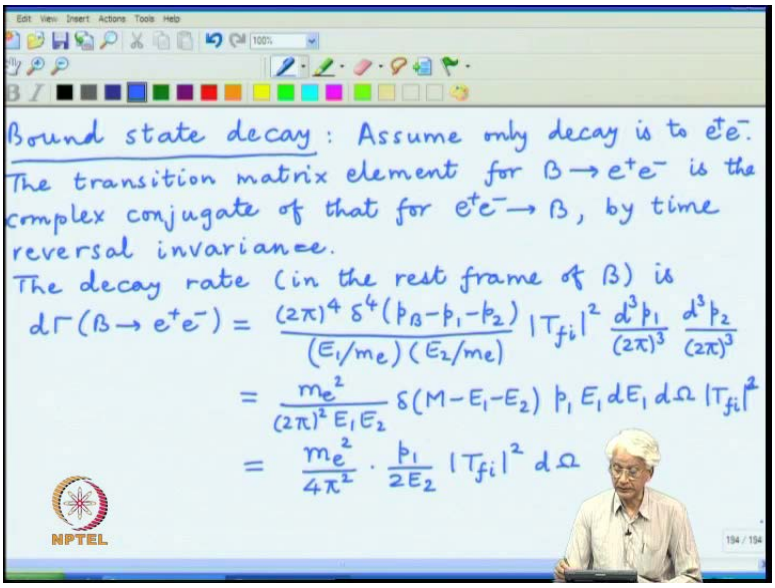

We can easily calculate the decay process which is thus the inverse of the production process, because for that we have the complete machinery at hand. If there are other roots of decay we will not be able to calculate with formalism that has been set up already, and in this simple problem I will ignore that root or equivalently the bound state decay's only into e plus e minus and not into anything else; that is the assumption. So, assuming only decay is to e plus e minus; in that case the transition matrix element where B going to e plus e minus is the complex conjugate of that where e plus e minus going to B. And this is nothing but the property of time reversal invariance which our theory obeys, and since we are interested in only mod t square the complex conjugate does not really matter.

So, we will just use mod t square calculated earlier, and now do the calculation for B going to e plus e minus by the same token; first calculate mod t square, then plug it back into the formula for the decay rates by putting in all the delta function. And the phase space area and integrating over all the things which are not observed, and that gives the formula for the decay rate which we will calculate in the rest frame of B. So, the initial flux we do not have to worry about the functions; wave functions are non-relativistic, everything is normalized to one. And we have the differential result in terms of all the machinery there is overall energy momentum conservation. Then there is the normalization for the electron and positron spinors.

Then there is this matrix element square, and then the integrals over the particles in the final state. Again the box normalization factors of v cancel out if you put them in explicitly, and we have done most of this stuff before. We will just repeat the same thing. There is a m e square by 2 pi square E 1 E 2 in terms of all the constant factors; one of the delta function say d cube p 2 can be just integrated out. So, there is an energy delta function which remains that gives delta of M minus E 1 minus E 2. The other d q p can be written in terms of differential angles as well as p square d p, but p square d p we can rewrite in terms of, say, p 1 E 1 d E 1, because E 1 d E 1 is the same as p 1 d p 1 by the dispersion relation, and then there is the angular d omega, and that takes care of all the phase space factors, and finally we have the usual T f i square from the matrix element.

And now noting that E 1 and E 2 are equal; they are just each of them m by 2, we can do the delta function integral rather trivially. So, also there is some cancelation of other parts of e. So, there is only m e square by 4 pi square which remains; then there is a factor of p 1 divided by 2 times E 2 that comes because E 1 and E 2 are equal. So, the delta function has actually m minus 2 E, and integrating it produces extra one half, and then there is T f i square and d omega. So, this is the simplification after doing all the phase space integration, and now to calculate the total rate; well, we will again do the unpolarized case where we will sum over all the possibilities of the e plus e minus directions of spins.

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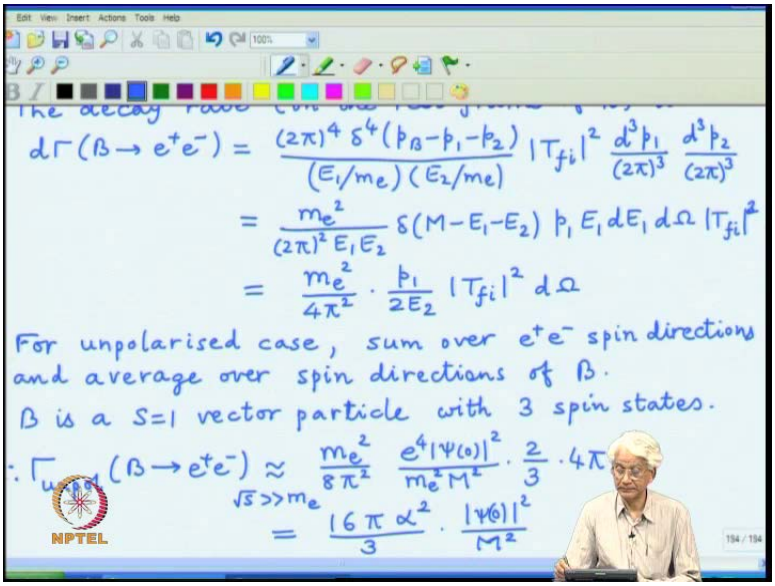

So, for the unpolarized case now we have to sum over e plus e minus spin directions and average over spin directions of B. And here there is a little difference compared to earlier cases, because B is s equal to 1 vector particle with three spin states corresponding to helicity values of plus 1, 0 and minus 1. So, the result then is for the total decay width. Well, we again calculated in the limit of square root of s much larger than m e, and in this particular case for example, p 1 is the same as E 1 which is same as E 2. So, these extra factors cancel out, and we just have to put in the remaining terms.

So, what survives here is first m e square divided by 8 pi square. Then there is a T f i square which we calculated earlier. It is e raise to 4 psi 0 whole square divided by m e square m square, and there was a half, but that half corresponding to an overall averaging factor of one-quarter. So, you have to remove that factor, because that is not present here. So, removing that one quarter and the half factor produces two in the numerator, but now we have to divide by this three spin states of the b. So, this is the contribution of the taking care of all the spin indices, and then there is 4 pi from the angular integral, because nothing in this expression depends on the angle.

It is isotropic distribution, and now everything can be plugged in to write it in terms of the fine structure constant 16 pi alpha square by 3 and then mod psi 0 square divided by m square. So, this is the result for the total decay width. It is explicitly calculable again provided we know that the wave function at the origin. And this wave function at the origin generically is not known, but even if we do not know that we can still relate the production cross section as well as the decay width because both of them have this mod 0 square, and the other one part will be cancel out.

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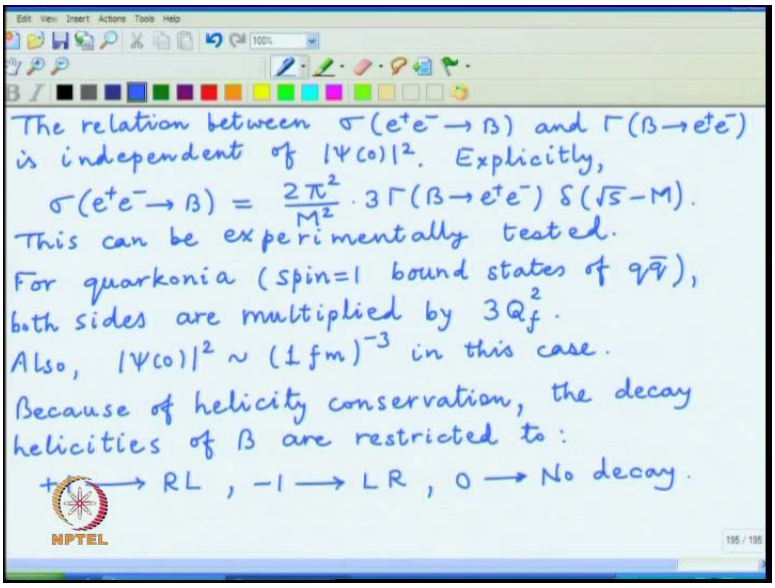

So, if we do that then the relation between sigma is independent of mod psi 0 square. Explicitly we have 2 pi square by M square three times the decay width and the delta function. So, this is a relation which can be actually tested without worrying about mod psi zero square, because sigma will give the area under the peak, and this gamma will tell you how long the particle will leave and both of them are experimentally measureable quantities. So, in case for states which are called quarkonia which means these are the spin equal to one bounds state of a quark and an antiquark; we have to include the extra factors which we have not counted here. And those factors are factor of three for the number of colors of the quark and factor of q square for the electrical charge of the quark which does not have to be the same as the charge of the electron.

And also roughly we know the magnitude of mod psi 0 square for this particular states, because the bounds states are typically of the size of Fermi, and if you consider just like a box normalization then mod size 0 square is about inverse of a fermi cube in this case. So, one has reasonable idea about what to except, and that is how many of this process is for production of new quarks where historically searched for. I can also mention one caveat for these decays which is a consequence of helicity conservation equivalently the fact that the mass of electron is totally negligible at the energy scales which we are looking at the decay helicities of B are restricted to the choices that helicity plus 1 will always decay into R L for E plus L E minus helicities. And that is just conservation rule

for the component of angular momentum along the direction along which e plus and e minus travel.

They travel in opposite direction, but we are in the center of mass frame. So, it just defines one possible axis. Similarly if it is minus 1 then the decay will be of the type L R, and the unusual part is that if the helicity is the third possibilities 0 that states does not decay. Where in other words the decay rate if you calculate in full detail will be suppressed by powers of m E m divided by where m where is the mass of the electron, m E is the energy of the electron; currently it will be the ratio of the electron mass to the bound states mass, and that is generally small. So, this is a conservation law which plays its role that out of the three helicities only two of them really contribute to the decay. And finally, we can put together various factors to see the experimental signature in terms of a resonance peak.

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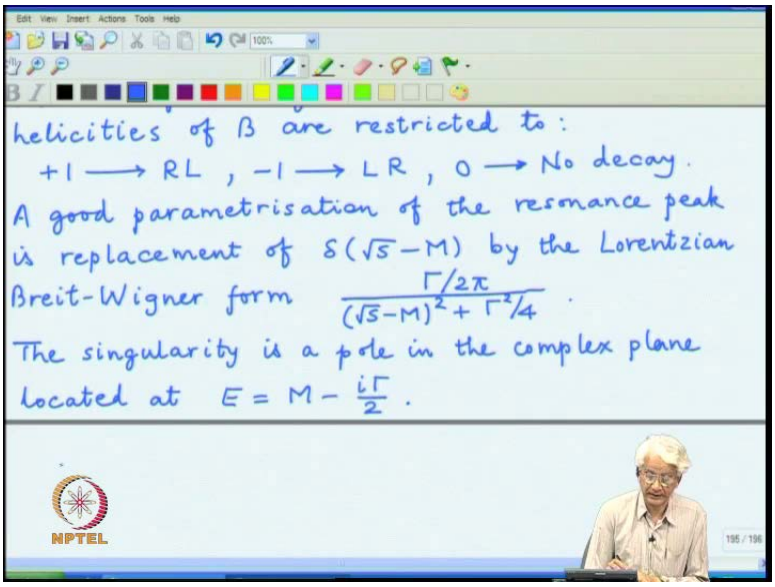

So, a good parameterization of the resonance peak is replacement of the delta function of square root of S minus M by the so called Lorentzian Breit-Wigner form, whose algebraic expression is gamma divided by 2 pi square root of S minus M thing whole square plus gamma square by 4. And this particular form is chosen so that the areas under this broaden form is the same as the area under the delta function it normalizes to one. The center of the peak is at the location of the delta function, but the nonzero value of gamma gives it a width, and the algebraic expression is that of Lorentzian.

And with this structure one can interpret that the singularity which was at square root of S equal to M producing a delta function, now it can be labeled as the singularity of this new form is a pole in the complex plane located at root s is characterization of the energy. So, we can write it E is equal to M minus i gamma by 2, a particular sign is picked for the imaginary part so that the time evolution given by E raise to minus I E t will get a exponentially decaying factor coming from gamma which is what the decay process should give. So, this is a replacement, and if we now do that one can see the change in the shape of the peak rather easily, and one can only describe it in terms of two parameters; one of them is the height and another is the width, width is dictated by gamma already which we have calculated.

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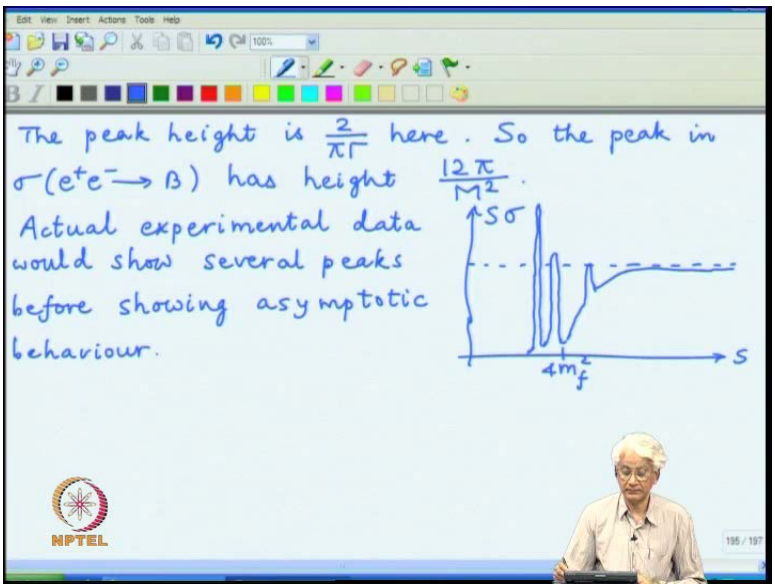

And the peak height is the value at square root of S is equal to M which is nothing but 2 divided by pi gamma in this particular formula. And if you now go back and plug in the relative constants which where there in the relation between sigma and gamma basically this formula is smeared the delta function by this Lorentzian it becomes 2 by pi gamma the product of that with; this remaining part just cancels out the value of gamma completely, and we just have a peak value of sigma given in terms of pure constants and no dependency on the various transitions matrix elements.

So, peak in sigma e plus e minus going to B has height 12 pi divided by m square, and this is very easily identifiable. It just depends on the mass of the bound state, and

experimentally it is also very easy to check. What one sees in actual experiments is the behavior of the cross section as a function of energy. Actual experimental data would show several peaks corresponding to many different bound states being produced one after the other. They are all radial excitations of the lowest state, and ultimately the cross section would show the asymptotic behavior which in this particular case is described as one over s as we have calculated above.

So, one has a schematic structure if one plots the cross section or rather s multiplied by the cross section versus s; in that case the asymptotic behavior is a constant. So, some value which dependence on alpha the fine structure constant, and the charge is of the particles, and the numbers of colors and all those kind of degrees of freedom. And there is a threshold which is 4 times m f square; there might be some sharp delta function before reaching the threshold which can show up in the data. They might be more than one, but the higher states will have a lower value of the wave function at the origins of the peak will be reduced. In height it will become border as well, and then one has a gradual square root of raise to the final cross section, but it can happen that there might be another peak of some bound state which is above the threshold.

It will again decay, but it can exist for a short time and show up as a peak, and then asymptotically it goes to some constant value as we see. So, this is a kind of generic feature which if you open a description of e plus e minus cross section data you will find pictures of this type; often well known cases are the production of the charmonium states and also the bottomonium or upsilon states. So, that is as much as I want say about the e plus e minus going to fermion, antifermion pair and vice versa. There are many things which can be deduced and verified easily in this particular situation in real experiments.

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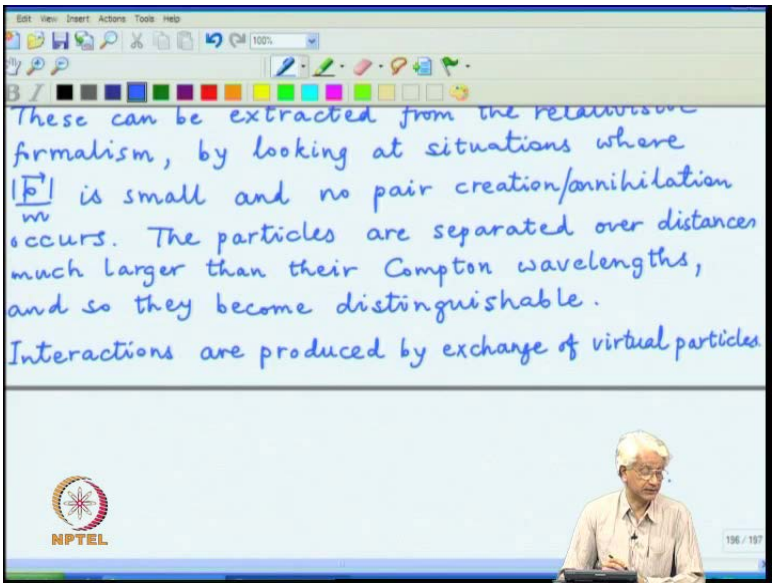

Now I want to describe somewhat unrelated topic but which can be easily extracted from the calculation machinery which you have setup, and that is how to obtain non-relativistic potentials for a given pair of particles. And that is something which can be extracted from the relativistic formalism which we have setup by looking at situations where all the velocities are small and also no pair creation or annihilation occurs. So, you have to take a particular limit of our calculation to get such results. There is one more thing which is necessarily included in this non-relativistic framework that the particles are separated over distances much larger than their Compton wavelengths, and because of that reason their wave functions do not have much overlap with each other.

And so they become distinguishable by their position coordinates, and we really do not have to worry about exchange interactions; for example, we can talk about the potential between two electrons. One at position x 1, and the other at position x 2, and we do not have to worry about what happens when these two electrons interchange positions, because that never occurs in a non-relativistic situation. So, we will have to correspondingly drop all the contributions which are related by interchange and only keep the ones which keep the locations of the particles fixed.

So, the interactions are the potentials or consequences not about these particles which are remaining stationery; they are moving at a very low speed. But they will be produced by exchange of virtual particles which get emitted and absorbed and not included in either

the initial states or the final state. So, let us just work out this elementary stuff for the case of interaction between two electrons.

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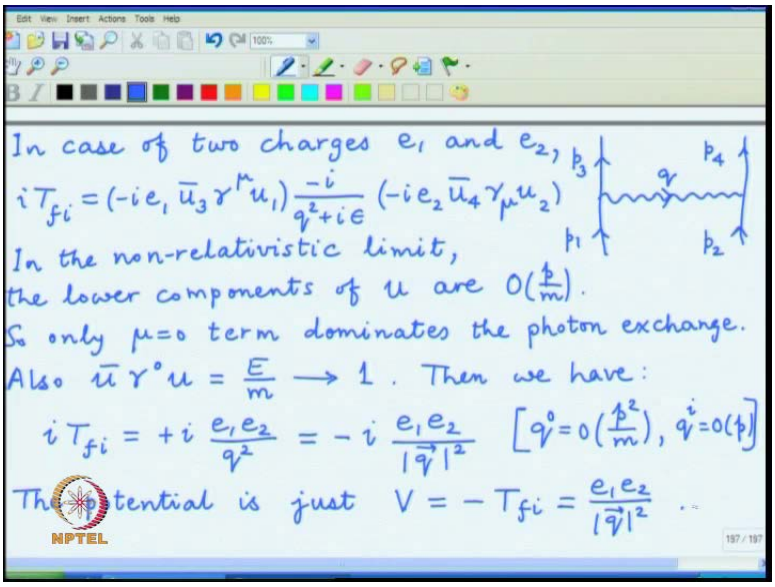

And let us have two charges E 1 and E 2; what this result gives us are rather simple Feynman diagram. There is p 1 going to p 3 and p 2 going to p 4 and in between there is an exchange of a photon which will carry some momentum q which is fixed by the momentum conservation at the vertices. And we will not include the exchange diagram where p 3 and p 4 are interchanged, because the locations of the two charged particles are fixed, and they do not allow the exchange process where this two legs p 3 and p 4 are crossed, okay. So, then we have the well known expression which I have been writing many times is minus I e for the first particle, then the wave functions u bar 3 gamma mu u 1.

Then there is a photon propagator minus i q square plus i epsilon and contracted in the Lorentz index with the others vertex are the other side which is minus i e 2 u 4 bar gamma nu u 2, and this is the full relativistic result. Now we have to simplify this by taking the non-relativistic limit; in the non-relativistic limit the lower components of u are order p over m and so negligible compared to the upper components which are order one. And now the gamma matrices have the structure that gamma 0 is diagonal and gamma i are off diagonal.

So, the dominant contribution comes from gamma 0 which connects the two upper components; both of them are order one and the sub dominant contribution couples the upper component with the lower component using gamma i, and we will drop that part. So, in this whole expression only mu equal to 0 term. So, we do not have to sum over these indices, okay. I should correct these indexes, because it is contracted and should not put mu, but it is the same index as on the other side. So, we will need only u bar gamma 0 u, and we already know what that value is. By our normalization convention this object was normalized to E over m, and that becomes one in the non-relativistic limit. It is just that non-relativistic are normalized to unity, and we leaded this Lorentz contraction factor in going to relativistic case which we are just dropping again.

So, that simplifies this whole algebra, and then we get a rather simple expression that i T of f i everything cancels. We have to put some minus i as the overall factor, but all that remains is e 1 e 2 by q square. And now this, sorry this should be plus i here, and we have to now look at the components of the exchange photon momentum. The time component is the energy difference between the electron energy, and the electron energy is the mass plus the next term is p square by 2 m. So, the q the mass gets cancelled by doing a difference between p 3 and p 1. So, q 0 is order p square by m. On the other hand q i is order p, because it is just the difference of the momentum. And so we can simplify this denominator as well it had q 0 square and q i square both of them together, but q 0 square is negligible.

It is only the q i square which survives and with the Minkowski sign this now becomes e 1 e 2 divided by mod q vector whole thing square. So, this is the result for the matrix element, and this matrix element is related to the potential is just V is equal to minus T of f I; the overall sign comes because of the various conventions used. In particular the Lagrangian involves minus V as a particular term while this T f i uses the evolution scheme involving the Hamiltonian, and one just has to check the signs and convince that this is the actual relation. And that then is e 1 e 2 by q square, and this is in terms of the momentum which is exchanged.

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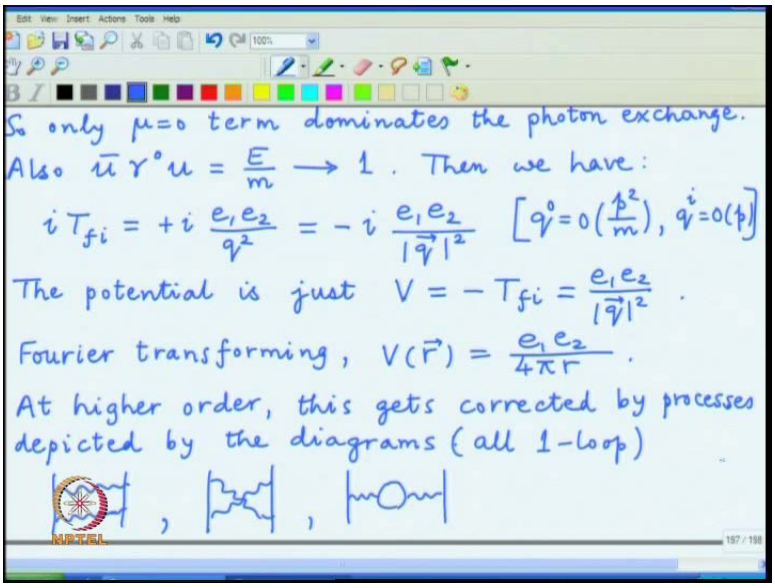

And one can go back to the position space by simple Fourier transform, and one gets the potential in coordinate space as e 1 e 2 divided by 4 pi r which is the usual expression for the coulomb potential. And in more detail one can want to calculate to higher order one can follow the same procedure but involve more complicated diagrams, and I will just illustrate a few of them which are the simplest diagrams which correct this single photon exchange. They are the so called one loop diagrams; one of them is a double photon exchange.

So, all of them are one loop which is like this. Then one can have a cross photon diagram as well, and third possibility is a virtual pair production by the photon and then annihilation, etcetera. And all these diagrams can be evaluated and converted back by the same non-relativistic scheme to construct the correction to this lowest order coulomb potential. And in particular those corrections can be found in standard text books, and in particular these diagrams are one loop, and they give contributions which are order alpha square. The coulomb potential is order alpha. So, this is the next correction to the coulomb potential.