Relativistic Quantum Mechanics
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Lecture - 39
Unpolarised and Polarised Cross-Sections

In the previous lecture, I set up the formulism for calculating the cross-section for the process e plus e minus going mu plus mu minus. And I did the kinematics and phase space integrals to reduce the expression to the stage, where the only thing which remains to be calculated is the square of the matrix element. I am going to work it out now first in the unpolarised case and then in the polarised case. And we will see the various features related to many properties of the Dirac spinors emerging in this calculation.

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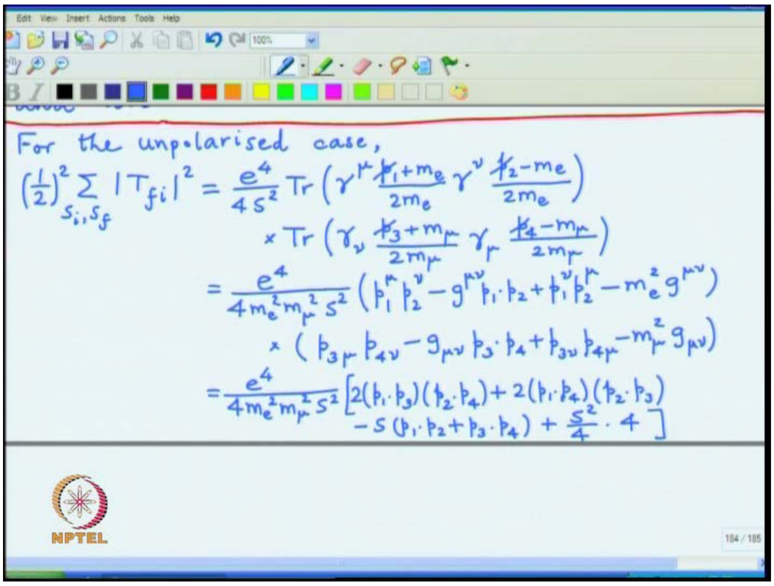

So, first we just have to follow the usual procedure to write down T f i square. And, unpolarised case amounts to averaging over the initial state and summing over the final state. Averaging over each of the e plus and e minus producing a factors of half, and then there is a summation over all the spins degrees of freedom. And this is the object to be calculated, we easily convert it to the standard form from the expression for T f i. And that gives e raised 4 by 4 S square; these are all the factors corresponding to the charge. And S square comes from the photon propagator. And then the fermion propagators produce two trace factors: one for the electron line, and another for the muon line. So, the electron line, for example, I can insert this in x mu for the photon vertex. And then it is p 1 splash plus m e divided by 2 m e for the projection operator of the initial state electron. And then it got contracted with the second factor for the vertex and the projection operator for the initial state positron, which has a p slash minus m structure. So, this is what becomes of the electron line.

And then this object has to be multiplied by the corresponding factor coming from the muon line. These are contracted appropriately with the photon indices and we have to keep track of which index goes to which place. Here the muon is counted as a variable function for V and mu in the final state. But, the muon has a wave function u bar, while the electron had a wave function u. And, for that reason, the projection operators are in different locations. Hence, the corresponding factors look almost the same, except that the index mu and mu are interchanged. And, this is the complete expression for the unpolarised case. We just have to evaluate these traces, which is not at all difficult.

By now, the procedure has become a standard. So, take out various common factors. And, the trace produces an overall factor of 4. And then the first trace gives all the dot product in alternating sequence. At first, the mu gets contracted with T, which gives p 1 mu p 2 nu. Then mu gets contracted with nu; the chain flips – g mu nu p 1 dot p 2. The third one is p 1 mu gets contracted with p 2. So... And then there is a contribution corresponding to the mass term, which is minus m e square times g mu nu. So, this is the first factor. And, the analogous second factor is easily seemed to be of the structure, which is essentially the same, except for which change of labels between electron and muon. And so that is the result.

Now, to calculate it, we have to do the various dot products. And, that can be done in a straightforward manner depending on what gets contracted with what. First, we will contract p 1 mu p nu with the corresponding p 3 p 4 terms on the other side. So, it gives p 1 dot p 3 and p 2 dot p 4. It occurs twice in the whole cross product. First, p 1 p 2 with one mu nu giving the one combination; and when the indices are nu mu, they give the second combination. So, this term gets an overall factor of 2. This is a similar term, which comes from the other cross part, which gives 2 times p 1 dot p 4 and p 2 dot p 3. So, these are all the terms with mu and nu indices sitting on p 1 and p 2. And then there are contributions from terms, which involve g mu nu, which we have to explicitly evaluate as well. So, those terms involve factors of energy and momenta explicitly.

Again, various dot products have to be calculated. And, g mu nu contracted with any one of this p 1 p nu just produces dot products of those objects. And, we just have to work out the overall coefficients of the total terms. So, there is the terms, which behave like p 1 dot p 2 multiplied by p 3 dot p 4; no, it is the plus relative sign and not a product. So, for example, one get p 1 dot p 2 and the other one will produce p 3 dot p 4 when the various things are contracted. And, the overall combination has to be calculated. I will just write down the result. And, this term has a coefficient minus S and plus. There is an overall squared term, which gives S square by 4 times 4. Here I have explicitly used the structure that, the m square can be converted to S by using the dispersion relation between energy and momenta. And, m can be completely eliminated in favour of this overall variable S. So, that is the result.

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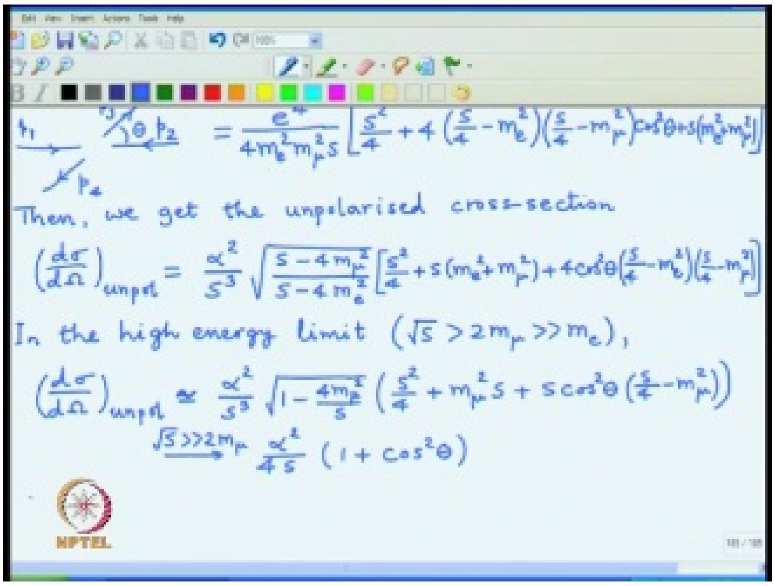

And, it can be simplified further to the form, where that various dot products can be explicitly evaluated in terms of the angles. And, to do that, we have to look at the geometry in the centre of mass frame, where one can take p 1 and p 2 colliding along one axis; and after that, p 3 and p 4 emerging along another axis. And, the two sets of directions make an angle theta. So, the dot product between p 1 and p 3 is essentially proportional to cosine theta. And, including that factor, now, explicitly, in evaluating the various dot products, the result reduces to the structure S by 4 minus m e square is nothing but the 3-momentum square. And, similar for the factor for mu, which accompanies an overall angular dependence cos square theta; and then there is little term left over for the masses. And so this is the completely simplified form in terms of the energy and the scattering angle and the masses. Everything else has been eliminated from the calculation by choosing the specific centre of mass frame as well as the dispersion relations for both electrons and muons.

So, then we have the complete result for the unpolarised cross-section, which can be now written in a simple in a form. There is E to the 4 and there are factors of 16 pi square in the denominator, which can be just written as alpha square; which alpha is the fine structure constant. The masses in the overall proportionality actually completely cancel out. And, one is left with this factor of alpha square by S cube. Then there is a proportionality constant involving the momenta of the muon and the electron. And, that can be written in a form, which involves the energy and the masses.

And, this is the ratio of the muon momentum to the electron momentum. And then there is this overall factor, which we calculated above. It can be written in a little different form, but it is a same thing. So, it is S square by 4 plus S times m e square plus m mu square plus 4 cos square theta. And, these are the factors of the momenta of electron and muon. So, this is the expression in full detail. It is convenient to look at various limits of this expression, where the result looks simpler. And, I can do those things one by one.

So, in the high energy limit, which specifically means that, the total energy is much larger than the masses; and actually, muon mass is much larger than the electron mass anyway. So, it is good to keep the muon in the calculation, but drop the electron mass altogether to a very good approximation whenever the energy is larger than 2 m mu to be able to produce muon pair in the annihilation process. So, in that particular case, the cross-section is approximately alpha square by S cube. We will drop the electron mass altogether, but keep the muon mass around. So... And, this can be further simplified if one decides to neglect the muon mass as well.

So, then it just becomes a simple number. And essentially, the cosine square theta factors are the only thing, which comes out. So, it will become alpha square divided by 2 s. And then the square by 2, which came out and cancelled these overall factors. And, the answer is this 1 plus cos square theta; the factor is 4. So, this is a simplified form. And, it shows a very simple angular dependence 1 plus cosine square theta. And, the overall magnitude is just 1 over the total centre of mass energy multiplied by a fine structure constant square. So, this is the result.

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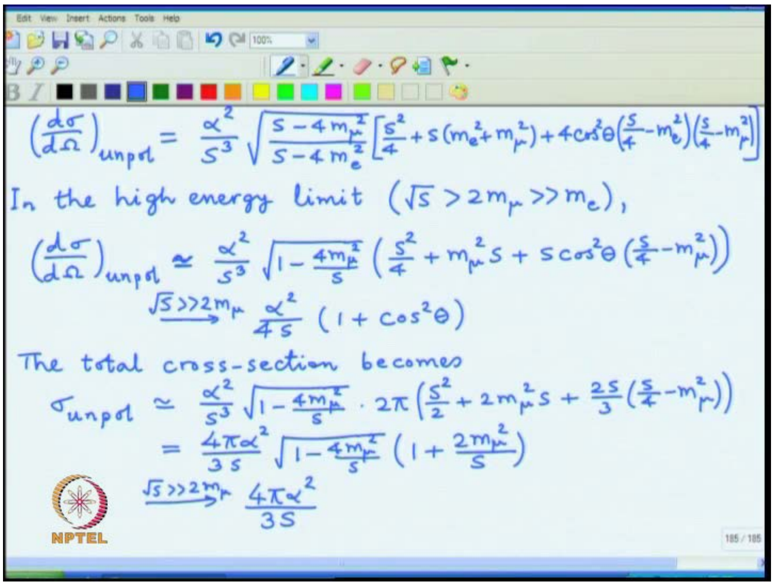

One can go further and calculate what is the total cross-section by integrating over the angle. Again, first, just neglecting the electron mass and then we will remove the muon mass as well. So, the first step is just alpha square by S cube square root of 1 minus 4 m mu square by S. Then the angular integrals or the angle phi just produces 2pi. And, the angle over cosine theta will change the various normalization to... 2 in case here. And, here this number will be two-thirds; even which can be written as 4 pi alpha square by 3 s cube square root of 1 minus 4 m mu square by S; this factor and then inside there is 1 plus 2 m mu square by s. And, the result for root S becoming much larger than 2m mu is rather trivial. It is 4 pi alpha square by 3 S. Again I cancelled some factors. It is now cube here. Everything else drops out. So, this is the contribution from these various integrals. And, one can see the various factors emerging.

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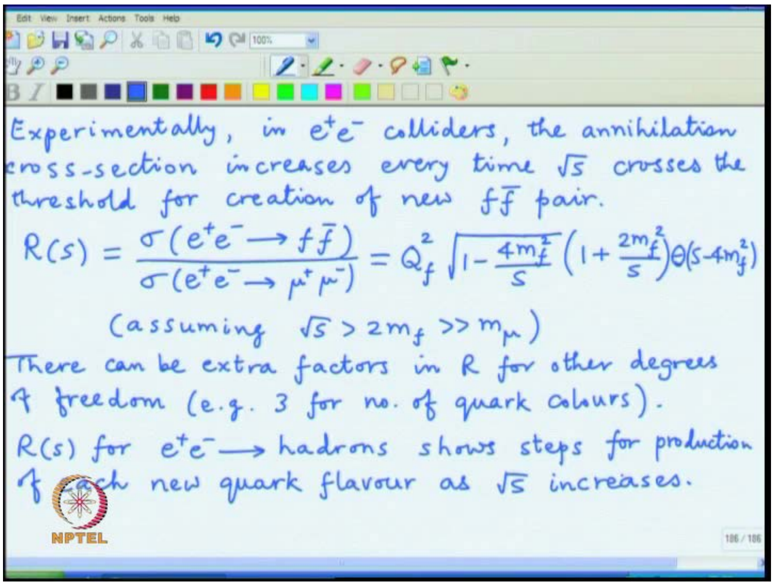

What experimentally is observed is not this muon thing, but the same process. Every time, the energy of the e plus e minus collider becomes large enough to produce a new pair of particles. So, every time, the threshold for a production cross, you will get a corresponding addition to that cross-section. And so the cross-section just keeps on increasing with steps occurring whenever S or rather root S crosses 2 times the mass of a new particle. And, that result is often expressed in terms of a ratio, which is denoted as R. So... And, generically, this is just labelled as f f bar pair. And, experimentally, it is much easier to compare the contribution of the new pair to the standardized calculation of mu plus mu minus production, which we calculated.

And, this is the object, where all the overall factors of various numbers and energy can sell out. And, the result looks quite simple. When there is a factor of the corresponding charge of the nu for muon, it does not have to be the same as the muon charge; muon charge happens to be 1. We will ignore the muon mass in this particular case; and because f is going to be still heavier than muon. And so the overall factor just becomes this numerical constant. There is a slow correction of this m square by s to the constant one. But, much more important is this – a square root arise, which starts at 0 when S is equal to 4 m f square. Below that energy, the particle cannot be produced in terms of this pair creation process. And, that can be explicitly included by putting a step function. And, this square root rise is a kind of characteristic signature, which shows, that kind of increase in the cross-section is a signature, where the new particle is going to be created.

And then one can go and look for the specific properties of that particular particle. But, it is the first signal, which can be easily seen in experimental data.

And, there can be extra factors in R for other degrees of freedom, which we have not included in the calculation. But, maybe, there in real process. For example, there is a factor of 3 for number of quark colours. So, when a quark gets produced, it can be produced in any one of the three colour and we have to sum over the possibilities. So, R is actually larger by a factor of 3 compared to this formula in case of quarks. And, that can be easily included. And, what can be then seen is the successive steps in case of hadron. So, for example, R s for e plus e minus going to hadrons shows steps for production of each new quark flavour as square root of S increases. And, the lowest hadrons are all very light and there you cannot separate them easily between the various processes.

And, most of the time, the up-down and strange quarks get produced altogether. But, then one sees for heavier quarks; there is a threshold for production of charm hadrons. And, after that, there is another threshold for production of bottom hadrons. And, the easiest way to see the various signatures including the value for the degrees of freedom parameterized by number of colours as well as the value of the electric charge; one can easily measure the height of this step and see what the process contributes. So, this is as much as one can say about this behaviour of this high energy cross-section. One can analyze the behaviour near the threshold itself.

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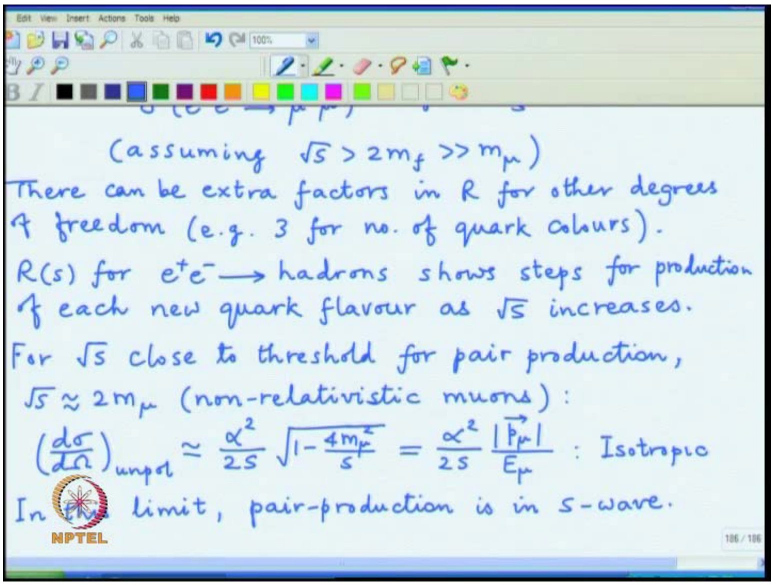

So, for root S close to threshold for pair production, we can again take the complete expression. The electron mass is still negligible. But, we will keep the muon mass finite and see what happens. And, in this particular case S, root S is roughly equal to 2 m mu. We can choose. And, this corresponds to non-relativistic muons. And then the cross-section can be simplified to this differential cross-section, which is also alpha square by 2S. The 2S comes from combining the overall factor with the remaining dependencies of the matrix element. The dominant factor is this square root, which rises from 0 at S equal to 4 m mu square to 1 when S becomes very large. And, there is other factor of S square by 4, which I have already cancelled with the overall factor outside to produce this result.

And, the angular dependence – explicitly, the cosine square theta term had a factor of S by 4 m minus m mu square. And, that factor basically becomes 0 in this particular case. So, this is actually all that remains of the cross-section; and which can be rewritten as alpha square by 2S. What is there is nothing but the ratio of the momentum of the muon to the energy of the muon. And, this is a rather special result. In particular, one can easily see that, because the angle dependence has cancelled out. This is an isotropic cross-section. And so in this limit, pair-production is in S wave. And, generically, if you remember the non-relativistic scattering theory in quantum mechanics, the cross-section behaviour goes like momentum raised to power 2l plus 1. In this case l is equal to 0 and this is just proportional to the magnitude of the momentum. And, that is all which appears.

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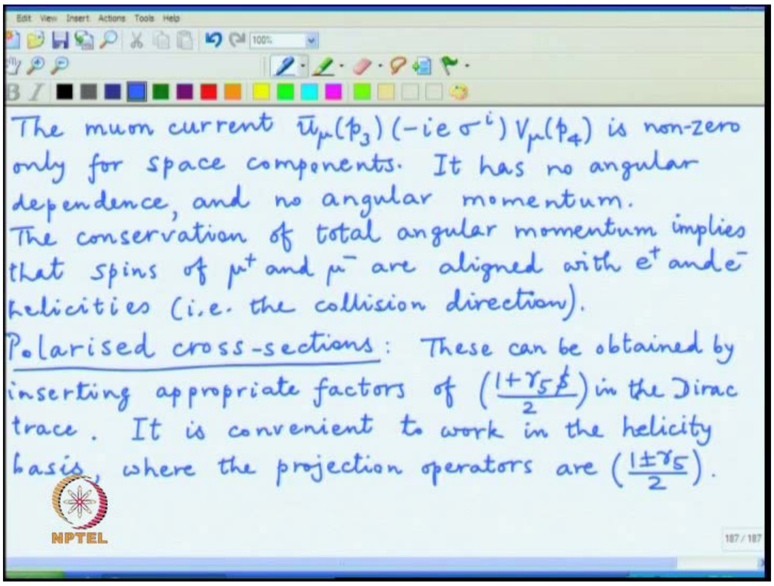

One can observe some special features, which lead to this isotropic behaviour if one can just look at the actual matrix elements involved. And, in this case, the current has a particular structure, which becomes very obvious. This has only nonzero contribution from the space components, which involve the spin. The result being that, u and v are the upper and the lower components of the spinors. And, the matrix gamma 0 couples only the diagonal parts. So, it cannot couple this off-diagonal u to v connection. We have to include this vector matrices alpha or gamma. And, they will only give the nonzero contribution. And, the other part is 0. And, in particular, it has... If you now look at this particular structure; it has no angular dependence; and therefore, no angular momentum. So, all which is involved is a spin degree of freedom. The space part has no appearance in this structure at all. And so the cross-section is completely isotropic.

The spin part is completely dictated by then the conservation of total angular momentum. And so that implies that, the spins carried by these muons – they have to be aligned with the angular momentum of the initial electrons and positrons. The electrons and positrons are highly relativistic. All they contribute are actually helicities and not the total angular momentum. We have been dropping the expression for mass of the electron everywhere or rather treating them as massless. So, the spins of mu plus mu minus add up to an angular momentum, which has to be the same as the angular momentum provided by the electron. But, the electrons have only helicities, which is directed along their initial direction of collision. So, the mu plus and mu minus – even though they are emitted isotropically, their spins are along the electron e plus e minus collision directions. So, aligned with e plus e helicities, which is the collision direction. So, this is the result in the non-relativistic case; one can work out the details in a simple manner.

We can work out some more details in the non-relativistic case. In particular, bound states can appear, because the particle and antiparticle actually can produce bound states analogous to positronium in case of e plus and e minus. But, that is little more technical; and I will discuss the bound state problem in the next lecture. Right now, that may go to another possibility, which is allowed in this particular e plus e minus collisions. And, that is having the beams e plus and e minus polarised in specific directions. And then we can worry about what will happen to the different possibilities, where the e plus and e minus have the same polarization, opposite polarization. One can be called left and right and so on and so forth. And of course, if one adds up all the different pieces, one must get back the total unpolarised cross-section anyway. So, we have a total check. But, in this particular case, I can illustrate how to include the polarization factors as well. And, we have seen the machinery to do that.

So, the factors needed to project the polarization directions are 1 plus gamma 5 S slash by 2 in the Dirac trace for the polarization direction S mu. This projection operator we worked out in the Dirac basis. It is little more convenient to work not in the Dirac basis, but what is the helicity basis or the Weyl basis; in which case, the protection operators for helicity are 1 plus or minus gamma 5 by 2. This simplifies the calculation a little bit, because the S is cleverly taken along the direction of motion and one does not have to do too complicated calculation.

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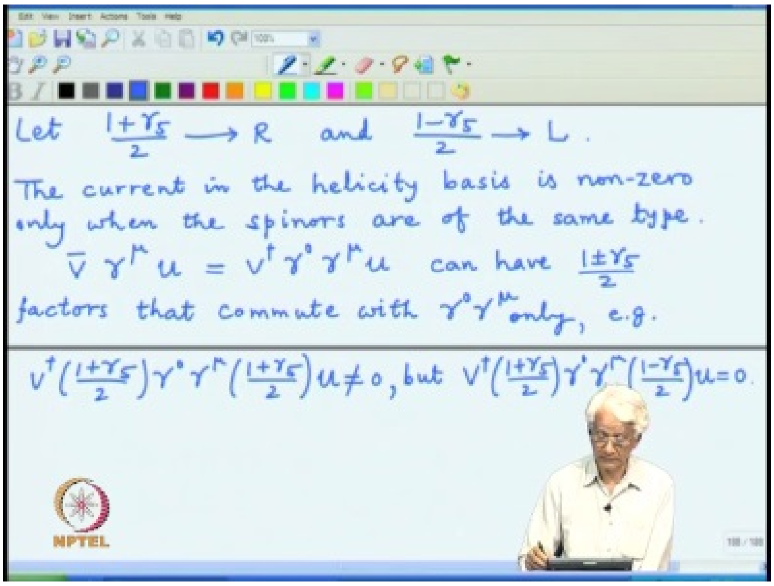

And, I am going to denote these factors 1 plus gamma 5 by 2 to something, which is a right-handed helicity and 1 minus gamma 5 by 2 be denoted by the left-handed helicity. So, we have the two helicities; we again have the particle and antiparticle degrees of freedom on top of this. And, those will make up the 4 complete components. But, now, in doing the calculation, we will project both by this energy projection operators, which give p slash plus or minus m divided by 2 m as well as the helicity projection operators, which will now be 1 plus or minus gamma 5 by 2. Now, the interesting point is to look at this structure of the interaction, which we have in the helicity basis. And, the typical objects are the currents. So, the current in the helicity basis is... And, the nonzero contribution comes only when the spinors in the current are of the same type. The reason being that, there are the structures.

For example, v bar gamma mu u; which can be written as v dagger gamma 0 gamma mu u. And, one wants to introduce these factors of the same type, because the gamma 5 is going to anticommute with both gamma 0 and gamma mu. And so if there is a 1 plus gamma 5 by 2 operating on u, it again has to be 1 plus gamma 5 by 2 operating on v; there is no other choice allowed. If they are of opposite; once you commute through gamma 0 and gamma mu, they will annihilate. So, for example, v dagger 1 plus gamma 5 by 2 gamma 0 gamma mu into 1 plus gamma 5 by 2 into u not equal to 0; but v dagger into 1 plus gamma 5 by 2 into gamma 0 gamma mu into 1 minus gamma 5 by 2 into u is equal to 0. So, this restricts the helicity states, which can contribute to this current. And, the current is the basic part of the matrix element. And so only combination of the helicity can contribute; the other one does not.

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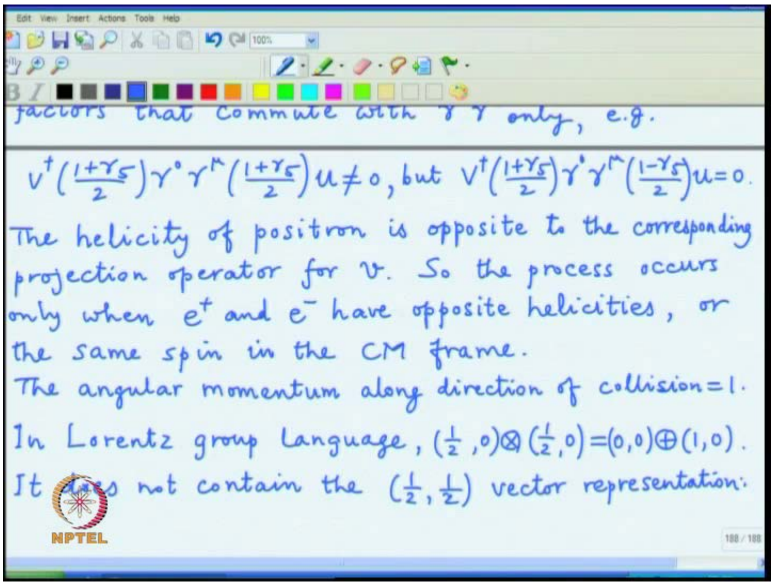

And, what we have chosen as the convention that, the helicity of positron is opposite to the corresponding projection operator for this wave function v. This was our convention between mapping of a particle and antiparticle. The v is a negative energy solution. And, the conversion from the particle to antiparticle language flips everything – momentum, position as well as the helicity. We have seen this and rather built it into a convention. So, the whole reaction... So, the process occurs only when e plus and e minus have opposite helicities; or, in the centre of mass frame, they are moving in opposite directions. So, the spin becomes the same for both these particles. And so the state which contributes actually if you measure the angular momentum along the direction of collisions. So, the angular momentum along direction of collision is 1. Both the contributions of half add up. And, this is a structure directly dictated by the relativistic spinor definitions. One can actually see it in different ways.

And, one way is to just go and look at the representations, which we have in the Lorentz group language. The objects which we have are two Weyl spinors: u and v – both of them have the same projection structures. When you combine these two, you get the states... This is a tensor product. And, the simple angular momentum addition rules says that, this gives these two states: one of them is a helicity one object; and another is a total scalar part. And, what is contributing in this process is the helicity one part. And, the other component, which has the same combinations, but not the same helicity; if you do half, 0 tensor product with 0, half, you will get this contribution half, half, which is the vector representation. And, that is absent in this particular scenario. And so we do not get the combinations, where the spins are pointing in opposite direction for e plus and e minus. Now, the same property is also true for the muons in the final state. It has the same structures and the same transformation properties under Lorentz group. So, they obey the same result, And so now we will look at these various combinations allowed by this helicity selection rule one by one next time.