Relativistic Quantum Mechanics
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Lecture - 38
Photon Polarisation Sums, Pair Production through Annihilation

In the last lecture, I derived the Klein-Nishina result for scattering of photons from electrons essentially at rest, which is the so-called Compton effect. In that calculation, I already assumed that the electrons were unpolarised or rather whatever polarization they may have it was not observed. So, we had to sum over the various spin states of the electron by introducing corresponding projection operators. Now, this result can be simplified further by assuming the same criterion for the photon; that means the photon is also unpolarised. And then we have to average over this epsilon polarization vectors by the same kind of technique; means projection operators and sum over the allowed states. So, let me illustrate that.

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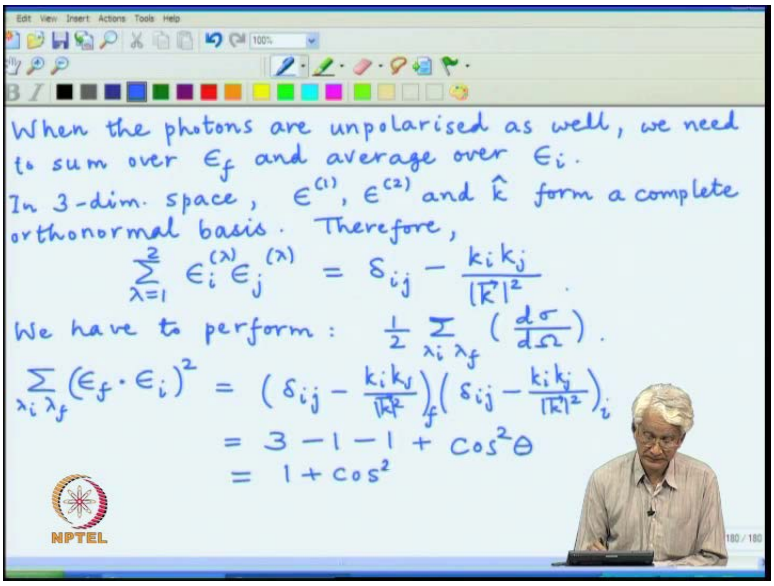

So, when the photons are unpolarised as well, we need to sum over the final state polarization and average over the initial state polarization. And we have been using the physical basis in which there are only two transverse polarization vectors allowed for the photons. So, this gives the same kind of a completeness and orthogonality relations, but now, in the three-dimensional space… So, in three-dimensional space, the two polarization directions, which I can just call epsilon 1, epsilon 2, and the direction of propagation of the photon k – they together form a complete orthonormal basis. And just as in case of any such complete orthonormal basis in a linear vector space, we can therefore, write down the completeness relation, which says that, the sum over all the three directions with components unspecified, you will get the identity operator. And I am writing the same relation; but the 3 components instead of writing on the same side, the two polarisation I am writing on one side. And the contribution corresponding to the direction of propagation I have shifted on to the right-hand side, which is the same thing. And to get the unit vector, I have divided all by the normalization. So, this is the description of the physical basis and it helps us to sum over now this quantity lambda.

And, we have to basically perform the calculation, which is a half from the averaging of the initial part; and sum over both the polarisation calculation and then whatever quantity, which we calculated, which was d sigma by d omega. And now, there are various terms in this d sigma by d omega. One part is just a constant; and in that particular case, this both these sums lambda i and lambda f will give a factor of 2; one of them will cancel out, but the other factor survives. And that remains overall factor of 2 in the calculation of the unpolarised cross-section. But, the nontrivial term is the one which depends on the epsilon symbols themselves. And there we have to put in the various factors explicitly; and the object, which is of interest, is the average of this dot product between epsilon final and epsilon initial whole thing squared.

So, by explicitly writing these quantities in terms of lambda i and lambda f, everything occurs twice. We sum over the quantities. And then we therefore, get one factor of this delta i j minus k i k j by k square for the final part; the other one for the initial part. And the two parts are contracted with each other, because of the dot product. And the dot product importantly is now in 3-dimensional space. So, we can now write this object as this delta function minus k i k j by k square, which can be called a transverse delta function contracted with itself. And so writing this thing explicitly, the product of ((Refer Time: 07:28)) delta i j's contracted together basically gives 3; delta contracted with k i k j gives k square, which cancels with the denominator. So, there are two contributions of the cross-terms, which produce minus 1. And then the final term k i square and k j square together – they again cancel the denominators. And therefore, these objects will give…

The final term coming from this k i k j on one factor contracted with other has to be calculated by taking into account that, the one of the factor belongs to the final photon and another belongs to the initial photon. So, the 2 k i's though I wrote by the same symbol, are actually not the same. And so when I take k i final dotted with k i initial, it actually produces the cosine theta of the angle between the two. The normalization does cancel out, but the result of that calculation is that, this object is now cos square theta; and which can be written as 1 plus cosine square theta. So, now, we can go back and substitute this quantity in the equations derived in the previous lecture.

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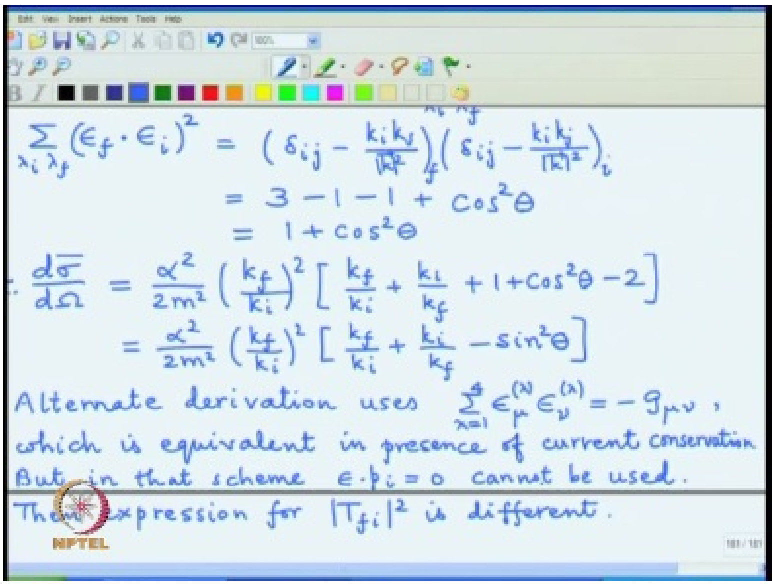

And so we have the result for unpolarised cross-section for the photons as well. There is an overall factor of 2, which I am explicitly taking into account in changing the normalization. So, it is instead of alpha square by 4 m square, it is alpha square by 2 m square, which came in. And then there is k f by k i square the whole thing unchanged. Then there is k f by k i plus k i by k f, which is also unchanged. Then there was a factor of 4 times this object multiplied by half coming from the averaging.

And so this 4 and half takes into account the total overall normalization outside. And this 1 plus cosine square theta remains. And that quantity is 1 plus cosine square theta. And then there is minus 2, which was the last term and which can be now written as k f by k i plus k i by k f minus sine square theta. So, this is the result for the unpolarised cross-section. And that is the one, which is kind of easy to measure in experiments without bothering to detect many of the final state or initial state variables.

I should make one comment that, this calculation in many text books is also done in a different manner. So, that calculations uses the property that, this summation over lambda. But, this lambda is no longer restricted to the physical degrees of freedom. They go over all the four components and produce a contribution, which is the metric instead of the identity operator. And that is equivalent in presence of current conservation. Or, in other words, it is the word identity, which will ensure that, the result of using this Lorentz covariant description of the polarisation tensors is the same as just counting the physical transverse degrees of freedom.

The only thing one has to watch out for is… When one does this Lorentz covariance scheme, one has to keep all the 4 degrees of freedom. And the criterion, which we used quite often in simplifying the algebra of this matrix element T f i that, the polarisation vector dotted with p i equal to 0, which is true for the physical basis, but not true for this 4-Lorentz component basis . So, one has to do the same calculation without explicitly using this condition epsilon dot p i equal to 0. So, then the expression for T f i square is different. And one has to then take this different expression and then average over the polarisation tensor using this metric as a completeness relation. And one then gets the same result for d sigma bar by d omega. But, these are equivalent procedures; one has to just keep in track what are the explicit symbols, which are being used and what do they correspond to in terms of physical variables.

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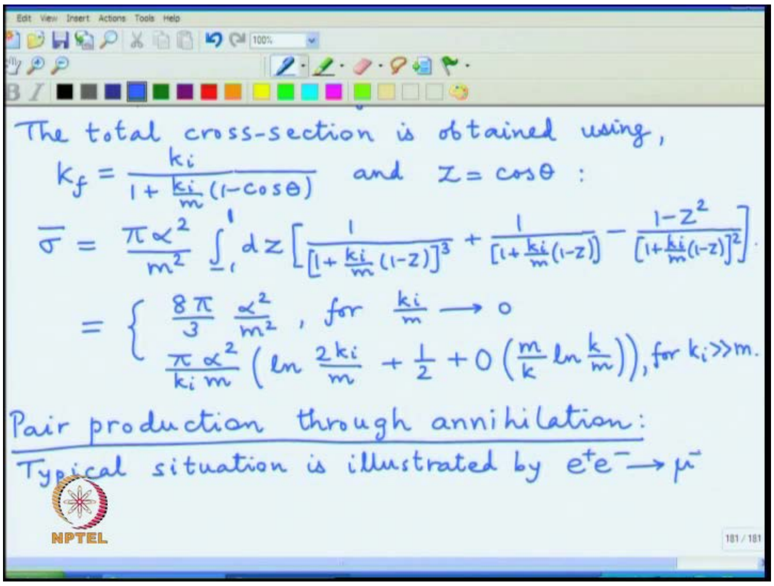

There are a couple of things one can work out rather simple mindedly from this differential cross-section regarding the total cross-section. So, that requires just the angular integral. And one has to just express all the various variables in terms of the angles cosine theta. In particular, the final momentum is related to the initial momentum by Compton’s formula, which we have derived earlier just the energy momentum conservation constraint. And a dummy variable of integration, which is convenient is just z is equal to cosine theta. And with that, the total cross-section can be now expressed as integral over the angle phi is trivial; gives the factor of 2 pi. The theta integral is written in terms of this integral over cosine theta. So, it is minus 1 to 1 d cos theta is same as d z.

And then this whole expression in the brackets, which now can be written as… Everytime there is a ratio of k f by k i. There are associated powers of this denominator. So, 1 plus k i by m 1 minus z. In one case, it is the cube, which is appearing; the reciprocal term has just a linear part. And the sine square theta has numerator is 1 minus z square and denominator is square of the same object. So, this is a managable integral. It can be solved in a closed form. There are only simple polynomials at various places. The detail result is a not that important. But, the limiting cases can be written as simple enough formula. And in the first case, this low energy cross-section, which we… or the compton cross-section – it can be just worked out.

In that particular case, this first two terms do not contribute in the low energy limit; it is only the last term. And the result we have already seen; it is just 1 plus cos square theta, which comes from the polarisation tensor integrated over. So, it produces this term 8pi by 3 alpha square by m square for this k i by m going to 0. When it is large; which means the k i is much bigger than m. Then again we can do a similar expansion in terms of all the various terms. The important singularity is the one which is logarithmic, because everything else turns out to be finite quantities. And that result I will just write down; it can be easily verified. And that essentially comes from the middle term producing the log part, the dominant part; and then the subleading terms are the other ones.

So, this gives log of 2 k i by m, which is the difference between z equal to plus 1 and z equal to minus 1 of this log argument; then there is a constant term; and then there is terms, which are supressed by inverse powers of k. So, these are the values for low energy and high energy Compton cross-sections. The high energy because of this inverse factor of k in the denominator, the cross-section actually falls to 0. But, there is a logarithmic correction to that inverse power, which one must take into account in trying to make a detailed analysis or comparison with experiments. So, that is as much as I want to say for this Compton scattering process.

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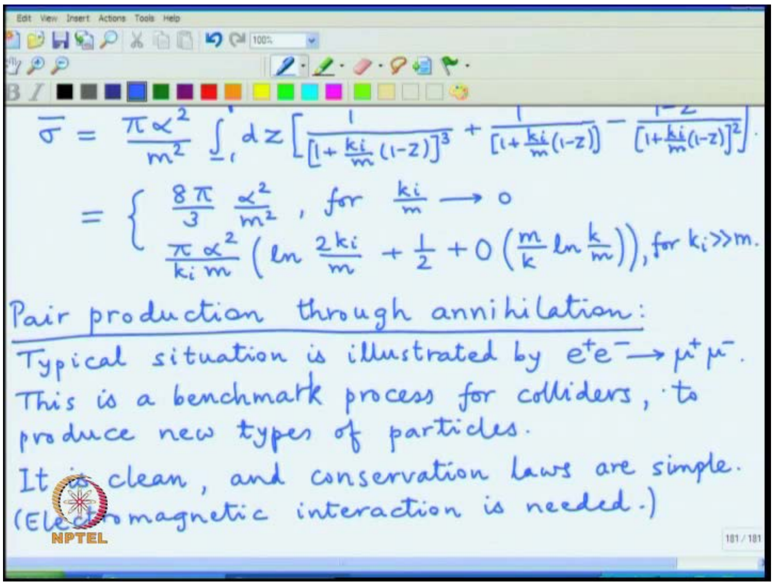

Now, I would like to discuss another process, which is pair production through annihilation. And the typical situation is illustrated by e plus e minus going to mu plus mu minus; where, this muons are essentially another set of fermions, which have the same charge as the electron, but a different mass. And in particular, the calculation, which I am going to describe can be used for any other productions, cross-sections for… Fermions may have different charges; it may have different masses. And the reason this process is important that, it has kind become a bench mark for describing a production of new particles in accelerators, where beams of electrons are collided with each other. And out of the whole mess, which comes out, you try to detect what are the new objects, which may have been created. So, this is a bench mark process for colliders to produce new types of particles.

And, the reason it is popular; one thing is it is clean and that arises just for the fact that, there are just two particles initial state and two particles final states. The objects are particle-antiparticle pairs. So, they satisfy all the conservation laws. When things are added together, the only thing, which survives as nonzero is the energy. Charges cancel out, momenta cancel out all those kind of things. And so the signature is very simple in terms of detection. And that is why it is clean. And the conservation laws are very simple. In terms of analysis, you can take any two particle-antiparticle state; and all the quantum numbers basically become the same as that of vacuum, which can be obtained by another particle-antiparticle pair. So, there is essentially no restriction on what are the kind of particles or antiparticles that may be produced by this particular process. Any pair, which can be produced will be produced in this collision.

The only thing which is required that, electromagnetic interaction is needed, because the process is mediated by a photon. So, the process can also produce bosons. There is nothing wrong in it. For example, e plus e minus going to w plus w minus, which are the weak bosons and they have indeed been studied in this particular manner. The objects which cannot be easily produced are the final state particle-antiparticle pair. If they happen to be neutral, there might be a different process, which can couple this neutral particles to the initial state. For example, instead of a photon, it could be the z boson and then can the final state have nutrino-antinutrino pair. But, all these calculations actually are related by just very simple overall normalization factors. So, this process is illustrative of many useful calculations. And so now, let us try to work out the various formula relavant for this particular process.

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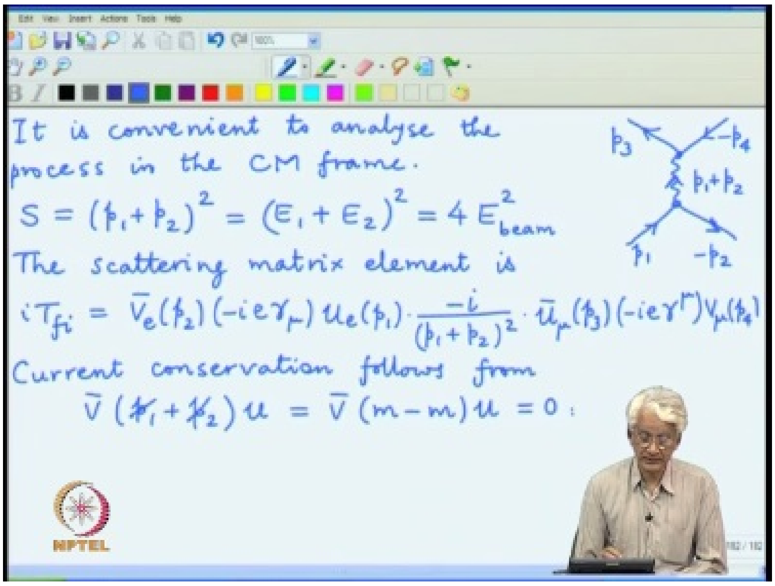

So, first, let me draw the Feynman diagram, which is relavent. And I am going to assume that, this pair production is those of fermions and not of bosons. And the particle-antiparticle pair is now denoted by these arrows pointing in opposite directions; p 1 is coming in, minus p 2 is going out, which is equivalent to saying that, p 2 is coming in. And then same way the final state has p 3 and minus p 4. And then there are these two interaction vertices coupling this initial particle and the final particle to the photon. And the photon momentum can be written as p 1 plus p 2; or, equivalently, it is the same as p 3 plus p 4 by momentum conservation. So, that is the Feynman diagram.

And, it is convenient to analyse the process in the centre of mass frame, because the calculation becomes quite simple. But, if the final result is re-expressed in terms of Lorentz invariant quantities, then the same result is true no matter what frame one works in. And we will actually do that. So, the useful parameter in this particular process is the total energy in the center of mass frame, because of everything else is 0; then in the center of mass frame, the total charge is 0; the total momentum is also 0; the only nonzero quantity is the energy.

And, that is parameterized in terms of this quantity denoted by the symbol S. It is nothing but the square of the total momentum in the center of mass frame. But, since it is a Lorentz invariant quantity, S has the same value in any frame. And then this is equal to just the object E 1 plus E 2 whole square, because the center of mass momentum adds to

0; only the energy survives. And since we are talking about particles with identical mass, this can be converted that E 1 and E 2 are actually also equal. And it can be written as 2E whole thing square. And this object is sometime referred to as 4 E square; where, E is the energy of the beam – single particle beam, whatever it may be. So, this is a very useful variable in which the whole calculation is expressed and we will use it to our advantage.

So, now, let us write down the matrix element by following the Feynman rules. And the same thing we start at the end of the fermion line and work backwards. So, there is a spinner for the positron with momentum p 2. Then there is a vertex for the photon and the electron spinner with momentum p 1. Now, there is a photon propagator, which is minus i divided by p 1 plus p 2 whole square. And then a similar object for the muon, But, now, the end of the muon line is a u spinner with momentum p 3. Then there is a vertex for the photon and then there is a antimuon spinner with momentum p 4. There is a little bit of messiness in the expression, because the mu I have used in for the Lorentz indices is also being used to denote the particle name. But, from the context, it should be clear which one corresponds to what and confusion should be avoided. So, this is the expression for the transition amplitude.

And, it obeys the usual current conservation, which follows from saying that, the photon vertex can be contracted with the corresponding momentum and that should give 0. And in this particular case, the contraction is with total momentum p 1 plus p 2, which gives this expression v bar p 1 slash plus p 2 slash times u at the vertex. And that now can be reduced using the equations for the external legs of the particle. And one term gives plus m, the other term gives minus m; and the whole result is 0. I wrote it down just to contrast with the earlier case in the Coulomb interaction, where the momentum was p f minus p i. And there both the spinners were u. And the relative minus sign between p f and p i produce the cancellation. Here the signs are positive in terms of p 1 plus p 2; but the spinner involves is v instead of u and that produces the relative minus sign; and the final result is the same. So, that is the property of this transition matrix element.

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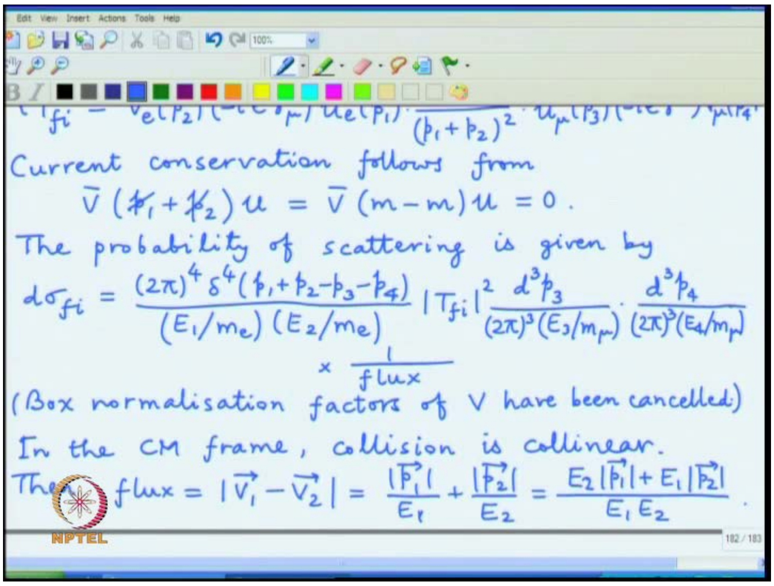

And now, we can calculate the probability. And that is obtained by taking the square of this matrix element. But, combining with it, all the phase-space factors of all the delta functions of the initial final state particles – the corresponding volume element as well as the flux. And that expression is now common place; just I have to keep track of all the normalization. There is an overall delta function for the total energy momentum conservation. One has to factor out the normalizations of the initial state particles. Then there is the square of the matrix element.

And now, one has to include all the volume elements in the phase-space of the final state particles. And they are d cube p for each of the final state object. And the last part is multiplied by 1 over flux. And again in writing this thing down, the box normalization factors of V have already cancelled; they came together with the initial state wave functions and they came with this d cube p and d cube 4 and they just exactly cancel out.

So, we have to now perform integrals, were all the states, which are constrained by these delta functions; and also, include this overall factor of flux. And so we have to first calculate what this particular value of flux is in this particular frame. So, in the CM frame, the collision is collinear. And so we will work out the cross-section or rather the flux in the particular frame, where the directions of velocity are already known. So, then flux is equal to the difference in the two velocities; V 1 is approaching from one side; V 2 is approaching from the other side.

And so V 1 minus V 2 is the relative speed of approach. And this is actually true even in the case of relativistic transformations, because one just has to count how many numbers of partilces pass through a particular section. It is V 1 from one side; V 2 from the other side. So, the total objects, which have passed through a particular section is actually magnitude of V 1 plus magnitude of V 2. And that is what this object expresses. And we can rewrite it in terms of momentum and energy in a different way, which is this. And that can be still written as a combination E 2 p 1 plus E 1 p 2 divided by E 1 E 2.

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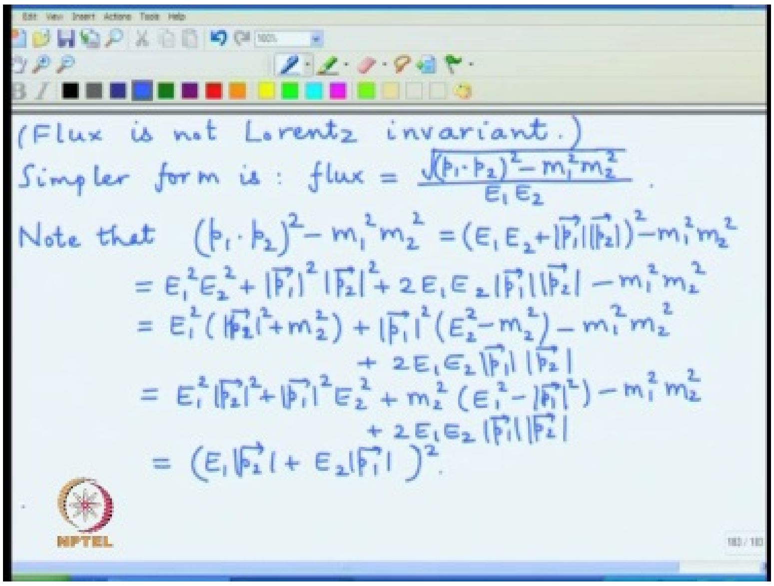

Now, this expression is not Lorentz invariant; the value of the flux depends on which frame one works with. But, one can try to convert as much of this particular part to a Lorentz invariant expression as possible. And I will first write down the final answer and then show that, it is equivalent to the expression written above, because it is partly Lorentz invariant. And that expression is a p 1 dot p 2 whole thing square minus m 1 square m 2 square divided by E 1 E 2.

And so what is appearing in the numerator actually is a Lorentz invariance quantity. And to derive that, let us just simplify this object, which is now E 1 E 2. And then the dot product of the two momenta; but momenta are collinear and they are oppositely directed. So, the dot product is actually positive sign; and the two magnitudes. And now, one can simplify this object. So, it is E 1 square E 2 square plus p 1 square p 2 square plus 2 E 1 E 2 p 1 p 2 minus m 1 square m 2 square.

Now, let us rewrite these objects in terms of the dispersion relations to cancel of a few of the terms. And so one can now rewrite this thing as E 1 square and then p 2 square plus m 2 square. And in the second term, the same kind of combination can be used, which gives p 1 square into E 2 square minus m 2 square minus m 1 square and m 2 square. Then I will keep this minus m 1 square m 2 square explicitly, and then this cross-term as it is. So, this is a form, which roughly corresponds to the square of the numerator, which is implicit over there. And so now, we can rewrite this object as E 1 square p 2 square plus p 1 square E 2 square; and then m 2 square is a common factor; and then there is E 1 square minus p 1 square minus m 1 square m 2 square; and then the cross term as it is.

Now, one can cancel off these terms involving m 2 square, because E 1 square minus p 1 square is m 1 square. So, there is a plus m 1 square m 2 square and a minus 1, which cancels. And the remaining part is a perfect square, which is exactly the numerator above. So, this thing is E 1 p 2 plus E 2 p 1 whole thing square. So, that explains this form for a flux in terms of collinear collision. It has a simpler form, where the numerator part we reduced it to a Lorentz covariant term. It can be written in other alternative ways. For example, this p 1 dot p 2 can be obtained by p 1 plus p 2 whole square, which can be written in terms of the variable S I mentioned at the beginning; and p 1 square p 2 square just reduced to m 1 square and m 2 square by their own dispersion relation. So, if there are simpler way of writing these terms and they are all useful in some form or the other; for us, this object is a good enough or actually we can simplify in the specific case that we have at hand.

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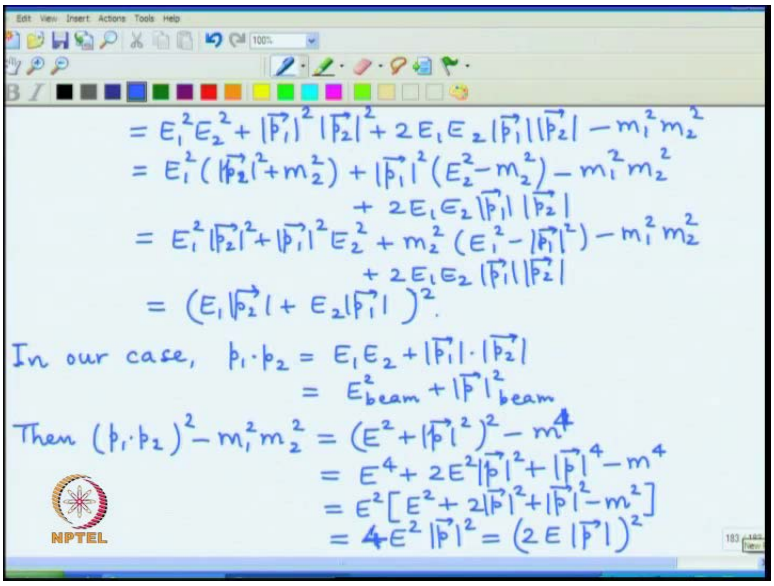

In our case, this p 1 dot p 2 is E 1 E 2 plus p 1 p 2, which can be now writen as the beam energies rather in a simple manner, because E 1 and E 2 and p 1 and p 2 are all equal in a magnitude. So, this is E square of beam and plus p square of beam. And one can write it as p square in terms of the masses as well. So, one can simplify this relation further. So, then this p 1 dot p 2 whole thing square minus m 1 square m 2 square is… I will rub the labels for the particles; they are all equal. And I will not put the subscript on it. So, there is m raise to 4. And now, one can simlify this object a little more.

So, it is E fourth plus 2 E square p square minus p fourth minus m fourth. This last term can be combined into p square minus m square and p square plus m square. p square plus m square again happens to be just E square. So, E square can be factored out. And then there is E square plus 2 times p square. And p square plus m square has been taken out. So, p square minus m square remains. And so this object is now p square minus m square is again p square; there are 2 p square and p square altogether. So, altogether, there are 4 p squares. So, this object is 4 E square p square. Or, since it is inside the square root, it is easy to rewrite it as 2 E p whole object square. So, that is what becomes of the flux. And we will now just directly insert it into the expressions and see what the result becomes.

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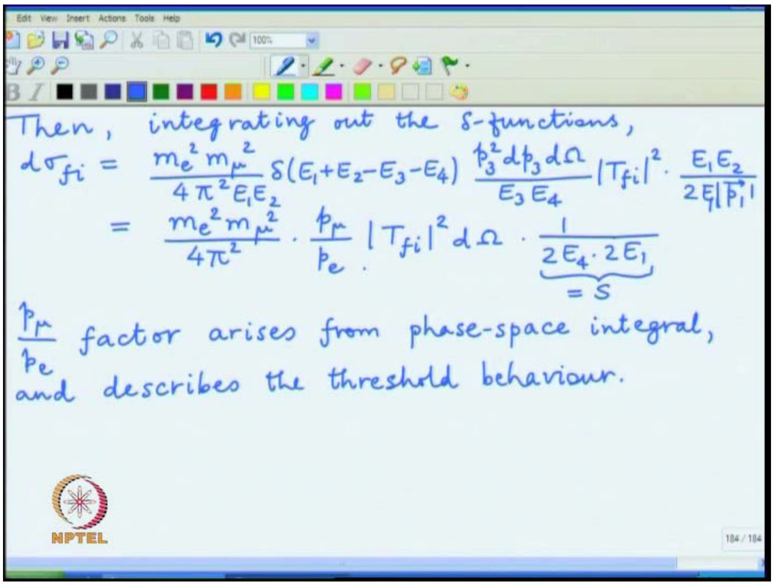

So, then we have this expression d sigma f i; where, we now remove all the delta function constraint one by one. And first, is easy to get rid of p 4, because it is a kind of straightforward. And that gives a result, where there is overall m e square m mu square coming from the various normalizations. The 2 pi's raise to various power cancels to produce this 4 pi square momentum constraint is gone. But, the energy constraint remains. The d cube p can be written in terms of various angles. So, it is p 3 square d p 3 d omega divided by the factor of E 3 E 4, which is part of the story. And then there is this matrix element t f i square. And I should include this factor of E 1 E 2 for the initial normalization as well. And now, there is a final flux, but which we simplified to the form of E 1 E 2 divided by 2 times E p. And I can take it to be the initial particle 1 in terms of the beam energy. So, this is the expression.

The delta function of p 4 is gone. We can cancel off some of these energy factors to simplify the result. And the d p 3 integral can also be converted to be a d E 3 integral, because p 3 d p 3 is the same as E 3 d E 3. And so we have a result now, which looks like m e square m mu square by 4 pi square. This E 1 E 2 can be cancelled. The momentum p 3 and p 1 – one of them is a numerator, one of them is a denominator. And that can be written as just the ratio of muon momentum derived at the electron momentum. Then there is t f i square; then there is d omega. And the part, which now survives is a factor of E 4. And a factor of E 1 or rather 2 E 1 together with the integral over the energy delta function.

Now, in the energy delta functions, we have E 1 is equal to E 2, which are fixed by the initial state. But, we have to take into account that, E 3 is equal to E 4 guaranteed by the momentum conservation. And so it is a delta function of 2 E 3 in the argument. When you integrate it, it produces half. And so the total denominator is actually 1 divided by 2 E 4; that is from the momentum integration. And then there is 2 E 1. And now, one can go and look back at the definition of the variable S – the center of mass energy square, which I defined. It is exactly this particular object 2 E the whole thing square; it does not matter whether it is an initial state or the final state. The total momentum is just 0 and 2E. Then becomes just the total energy in the center of mass frame. So, this is a final expression, which is written in terms of the Lorentz invariant variable S. The masses, which are also Lorentz invariant; the part, which is not lorentz invariant are the momenta.

And, p mu by p e factor arises from what are called these integrals over delta functions; or, otherwise, they are also known as phase-space integrals and describes the behavior near the threshold, where the momentum is close to 0. The energy is roughly equal to the mass. And that is where the cross-section will very heavily dependent on the magnitude of the momentum. And we have to include that explicitly in this particular form to estimate the actual size. Once one goes away far away from the threshold means energy is much bigger than the mass; then the momenta and energy are roughly of the same magnitude and this overall phase-space factors do not matter much.

But, this is a feature, which is experimentally easy to see as the energy of the beam goes beyond the particle-antiparticle pair threshold. At one sense, suddenly sees the cross-section rising at that particular point. And the raise is defined by this phase-space factor. And it can be fitted to the experimental data to verify details of the theory and properties of the particles involved. We will continue from this to actually calculate the cross-section next time.