Relativistic Quantum Mechanics
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Lecture - 35
The T-matrix, Coulomb Scattering

Towards the end of the previous lecture, I described some examples of simple QED processes, which can be calculated in the leading order using the tree level diagrams. And I will continue down that list adding some more well-known processes, which can be calculated in a straightforward manner. So, I described Coulomb scattering, Compton scattering and Bremsstrahlung.

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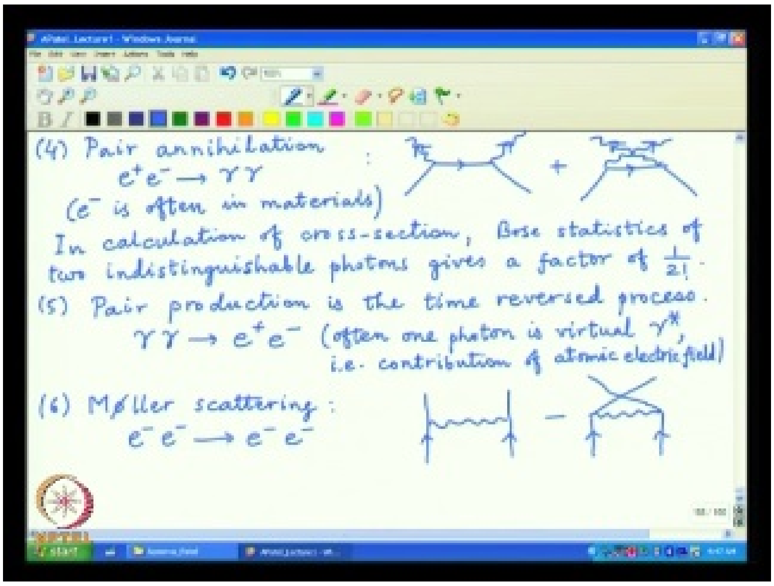

Let us go further. There is a process of pair annihilation, which corresponds to e plus e minus going to two photons. And the diagrams, which correspond to it, are of two different types with different orderings. And the two orderings correspond to exchange of the two photons. And so the relative sign between the two diagrams is positive, photons being bosons. So, this often occurs. Many times it occurs in materials, where positron is coming from outside, but the electron is taken up from an atom. So, e minus is often in some materials, but it is a well-known process, which can be described by this two different orderings.

And, when the cross-section is calculated; now, we have two identical particles in the final state. And to avoid double counting, one must include a statistical factor. So, one calculates first… The process is as if the two photons are distinguishable and then corrects the over counting in that particular calculation by putting the factor of 1 over 2 factorial and the total result for the cross-section. So, one must remember that, this statistical factor is in the cross-section, which is obtained after squaring the amplitude. It is not a factor in the amplitude itself. So, that is pair annihilation.

A related process – essentially, it is the inverse of this, is the time reversed process, where two photons will change over to e plus e minus. This again has the same two kind of diagrams corresponding to two orders of the photon interaction vertices. But, in this case, the two photons are the initial state objects; and so there is a no 1 over 2 factorial in the cross-section e plus and e minus. Though described both by the same Dirac equation, they are distinguishable, because they carry opposite charges. And so this process has a calculation, which differs by 1 over 2 factorial in the cross-section compared to the time reversed of the first annihilation process.

But, the difficulty in this particular process is this hitting one photon with another is quite unlikely; photons do not stand still to be hit as a target. So, one of the photons is often as a virtual photon, which we have been denoting by gamma star. And that can be a contribution of atomic electric field. So, that is how the process is actually often observed. One does not hit one photon in another, but a high energy photon when passing through a material picks up the virtual photon from the atomic electric field and produces pairs. So, this is how this process is experimentally seen in a radiation passing through matter. So, that is the pair production process.

Another important process is something known as Moller scattering, which is a scattering of one electron from another. And that can be again described by two diagrams with different ordering of the vertices. But, now the different orderings correspond to interchange of two Fermion lines and that gives a relative negative sign due to Fermi statistics. So, one does sum over different orderings; but always keeping the appropriate sign, which arises when interchanging either Bose or Fermion lines in the diagram. So, this can be seen just by high energy electrons scattering from another electron, which can be bound inside an atom. And it can also be seen in accelerators, where electron beams collide with each other.

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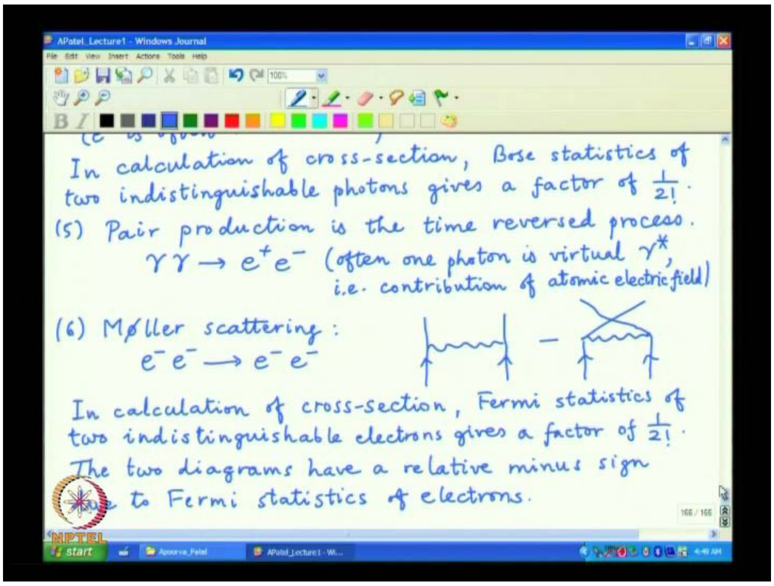

In addition to this minus sign, one also has to deal with the same double counting problem, because there are two indistinguishable particles; they happen to be electrons this time; and they again give a negative sign in the amplitude and 1 over 2 factorial in the cross-section. And I should write down the other... So, that is called Moller scattering.

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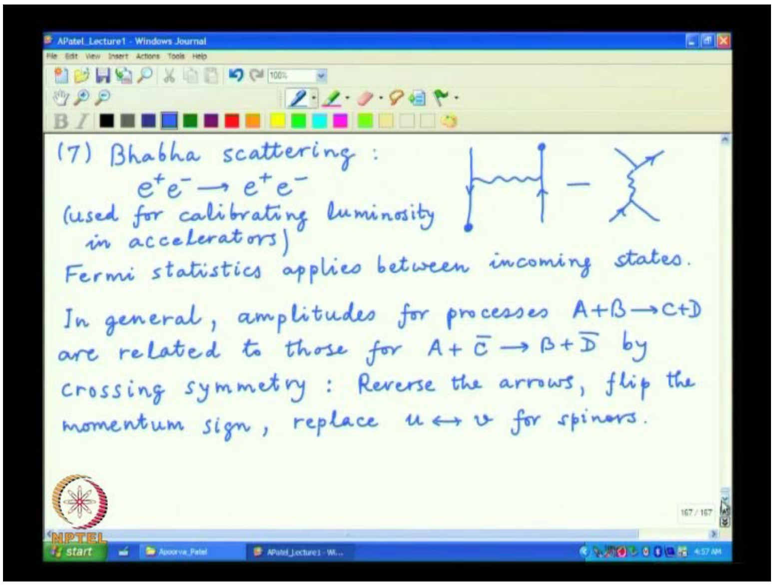

Let me mention one more process named after Homi Bhabha, who calculated first correct answer for this cross-section at tree level. And this is a scattering of electron from a positron. And again this can happen when the electron is bound inside atoms; a positron beam can scatter from some material. Or, this also can occur in accelerators, where electron and positron beams collide. And often it is a useful process for calibrating what is known as luminosity; or, that thus means that, the overall strength of the cross-section in the accelerator. And one needs that, overall normalization. And this process is the standard in case of e plus e minus accelerators. It provides a convenient normalization point and one can calculate it to a desired accuracy by including higher and higher orders in perturbation theory.

So, here again there are two diagrams; but now, we must put the arrows pointing in opposite sense, because one is a positron beam, the other one is a electron beam. And the second diagram does not look quite like the first one, but it indeed is allowed by all the rules, which we have used. And to see how it is related to the first one, one must interchange identical lines. And in this case, the identical lines would be the two incoming lines into the interaction region, but one incoming line is from the past and the other one is from the future, because of one object being electron, the other one is a positron.

And, if those two corners are interchanged, one indeed goes from the first diagram to the second diagram. And for example, I am marking the two ends, which can be interchanged to produce one diagram from the another one. And they are the ones – can be either two incoming lines or the two outgoing lines. And since these are again related in terms of exchange of indistinguishable states, the relative sign is the Fermi statistic sign, which is negative.

And, here the Fermi statistics applies between say the two incoming states, which are indistinguishable in this space-time language, which we have used. But, in this particular case, the 1 over 2 factorial, which we included for two identical particles in the cross-section – that is absent, because on either side of the calculation, the two objects, which are there – one of them is electron and a positron; and they are indeed distinguishable. So, this is an e plus e minus scattering idea. And this kind of list can go on; I am just doing the simplest situations involving only two particles, but one can go further. And in general, calculate many processes. I just want to mention a general rule that, there are amplitudes one can calculate for say process A plus B going to C plus D. And having calculated that, one can easily write down the amplitudes for the related processes. And by related, I mean interchange of particle from the initial state to the antiparticle in the final state. And one example of that kind of situation will be A plus C bar that, C bar is the antiparticle of C going to B plus D bar. And the relation is known as crossing symmetry.

And, in terms of actual calculation, what one has to do is reverse the arrows for the objects, which are shifted from the right-hand side of the equation to the left-hand side. For the same objects, flip the signs of momentum, which corresponds to changing from particle description to antiparticle description. And that also requires that, when the wave functions are described as u or v, they interchange them as well. So, this crossing symmetry is a useful and one does not have to do all the work all over again if he knows one of the processes; the other ones can be written down and calculated very easily. And so it is a useful property; it is also a useful check on some of the answers, where time reversal symmetry can be seen easily in the description of the interaction. So, this gives us several of the examples

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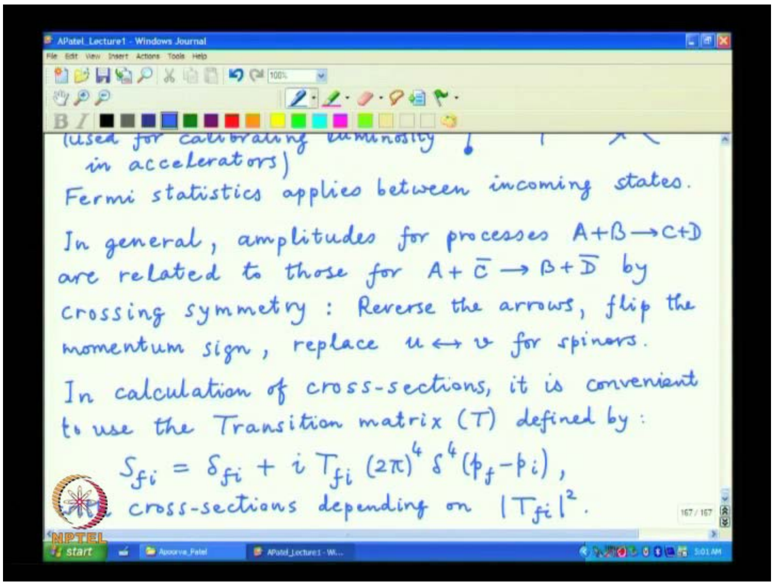

One more thing I would like to mention is that, we are often interested in various cross–sections; where, instead of dealing with the full scattering matrix, one can only deal with the what is known as the transition matrix and often denoted by the symbol T. And by defining it explicitly, one just saves time writing. And the definition is that, there is an identity part of the scattering matrix, which is not of interest. And the transition part is defined by removing the overall momentum conservation constraint from the formula. And this factor is common to all processes. It is no point in carrying it around; only the remaining part is called the transition amplitude. The factor of i is kind of inserted because of the conventions assigned to various matrices. S is unitary; and if you expand the unitary matrix about identity, then it takes the form of 1 plus i times a Hermitian matrix. So, that i is explicitly factored out. And after doing all this ((Refer Time: 22:45)) the cross-sections depend on the absolute square of this transition amplitude; and which is the general rule in quantum mechanics. So, one saves writing complicated formula and factors out all the things, which are straightforward and part of all standard formula.

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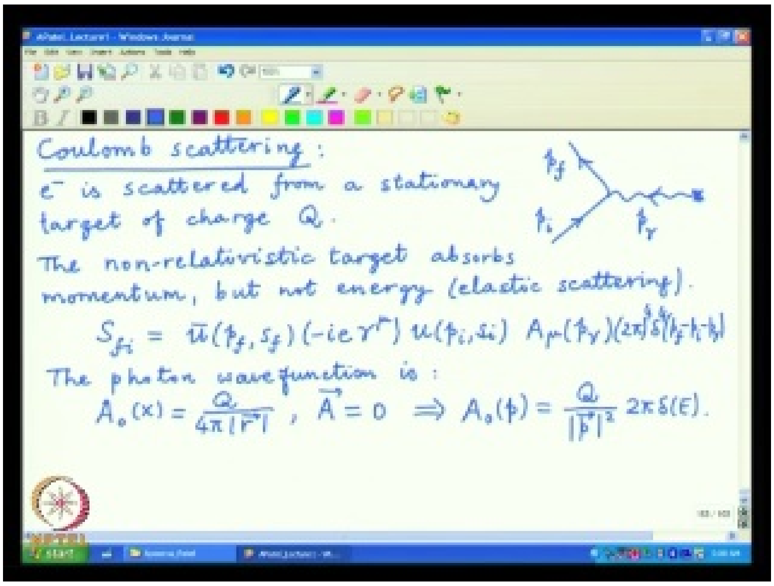

So, now let us move on from this just description of this various processes to actual details of calculations. And the first example, which I want to work out in detail is thus for a Coulomb scattering. This is a process, where an electron is scattered from a stationary target of say charge Q. This is a process, which one calculates in studying physics at several levels. First, in classical mechanics, where the description is in terms of an impact parameter and hyperbolic trajectory, one can calculate some cross-section of this process from that; then in quantum mechanics, using Schrodinger's equation and the corresponding wave functions including a partial wave decomposition and spherical basal function, which gives the same result as the classical answer.

And now, we are going to do the same calculation including the effects of relativity as well. And we will see how the answer changes and understand the reason behind it. The diagram I have already drawn before and I will kind of repeat it. So, there is an initial momentum p i final momentum p f; and the photon brings in some momentum from this static charge. And in this particular process, the target is the stationary. So, one can call it non-relativistic. And as is well-known, it absorbs momentum, which leads to changing direction of the electron, but does not do anything to the energy of the electron. So, the scattering is elastic. We will see this feature emerging out of the rules automatically.

So, what is the scattering amplitude? We have seen all the rules; we will just write it down explicitly. So, there is the matrix element S f I, which can be written as the various factors of incoming and outgoing state. There is only one vertex, which brings in a factor of minus i e gamma mu and u p i, S i. In writing this algebraic structures, one always starts with the end point of an electron line, which gives a factor of u bar or v bar and then works backwards towards the starting point including all the factors of gamma mu and the interactions until one gets to the initial state.

So, that is what we have used here. And in addition to that, there is now a wave function corresponding to this virtual photon. It is not a plane wave photon. So, we have to determine what exactly this wave function is from the description of Coulomb potential. And then there is a usual constraint, which is the momentum conservation. So, it is p f minus p i minus p gamma using the signs of the arrows, which I have drawn. So, this is the result.

Now, to simplify it, we must use the specific forms of what are the wave function and do some algebra to cancel and reduce the matrices to numbers. And so first thing we have to find out is what is the photon wave function. And that is well-known in the position space. So, A 0 of x is equal to Q by 4 pi r. Now, using again the standard units in high energy physics, where epsilon 0, c – all those things are set equal to 1. And the vector part of A is identically 0. So, we need to Fourier transform these objects; nothing happens to the vector part; it is 0 any way. But, time component does need to be transformed; and the Fourier transform of 1 over r becomes 1 over p square. And if transform is also done, not only on the space part, but also in the time direction, one will get a factor of 2 pi delta E, because the potential does not depend on time at all.

Now, one way to get this Fourier transform is to remember that, the 1 over r potential is a solution to the Laplace equation. And on Fourier transform, the Laplacian operator just becomes p square. So, one can solve the Laplace equation in momentum space and easily obtain what this expression for A 0 is. But, it is an interesting exercise to directly start with 1 over r and do the Fourier integrals. To get 1 over p square, it does require little bit of care to use in presence of the singularities that r is equal to 0.

In particular, one has to use Gauss's law and the relative procedure for regularization of these integrals. But, any way the answer is uniquely defined and it is given by this particular rule. The delta function is only an indication that, the potential is time independent. So, now, one can stick these objects into the scattering amplitude. In particular, the delta function is already absorbed in the 4-momentum delta function; one does not have to count it twice. And do that counting it once is enough; or, you can count it in different ways. But, all of them will amount to the same result.

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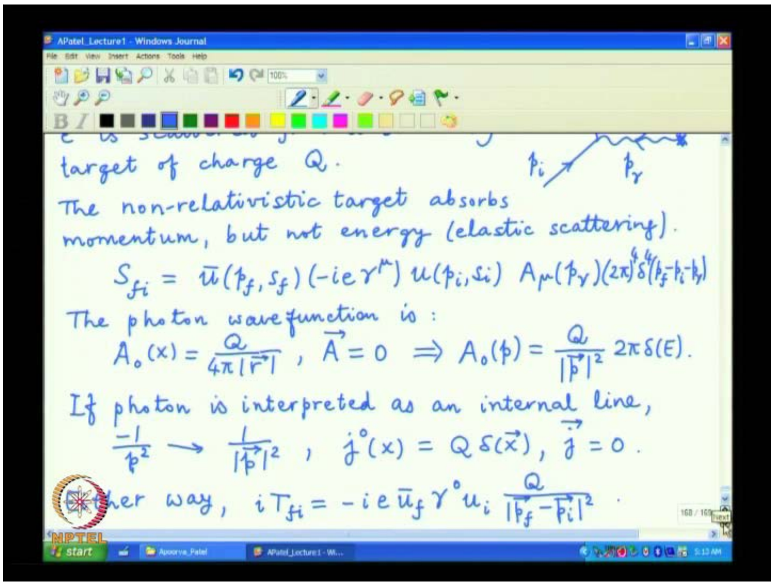

Another way of looking at it – this calculation is to interpret the photon, is interpreted as an internal line. Then one has this propagator, which is minus 1 by p square. For the photon propagator, which changes over after realizing that, the p 0 part is actually 0 for this particular coulomb potential. So, this becomes 1 over p square. And the cross sign, which have been representing as the source of charge Q just can be used to describe the source. And that source is the current. And that can be now just taken as Q times delta function at say position r is equal to 0 and the space part of the current is 0 as well. So, one can do that and then one transforms this delta function to momentum space. It just becomes 1. But, the factor of 1 over p square, which was there as Fourier transform 1 over r potential now, comes from the propagator. And the final answer is the same.

So, either way, one gets the result that, the transition amplitude, which can be written without keeping the delta function along, is a minus i e u bar f gamma 0 u i multiplied by Q. And then the photon momentum is simplified by the delta function to the difference between the initial and final momentum of the electron. So, this is an expression for the transition amplitude and which now can be converted to the cross-section. So, let us do that. For that, we need to take into account all the normalization factors necessary for defining the cross-section.

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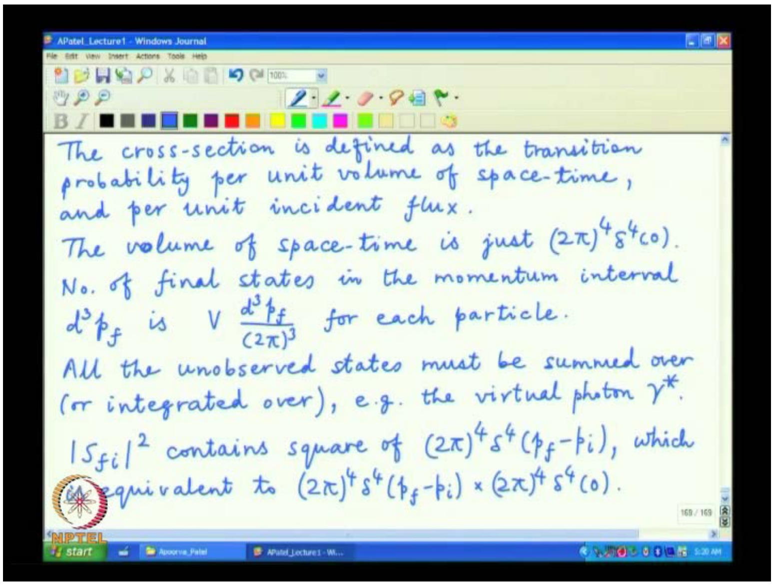

So, the cross-section is defined as the transition probability per unit volume of space-time, because we are dealing with plane wave normalization for initial and final state; and also, per unit incident flux. So, we have to take into account all these factors, which come from this unit volume, unit flux – all things like that. So, what is the volume of space time that which is buried inside the calculation. So, one way to see it is just use the same Fourier integral, but instead of putting any weight calculates without any phase factor. So, that gives the total volume. But, if you do not put the phase, it is equivalent to putting the Fourier transform at argument 0. And that just gives the normalization that, this is just the delta function with 0 argument. And this literally is infinite; but when interpreted in terms of properly regulated descriptions, this is indeed the answer; and one just has to divide by the particular factors of 2 pi raise to 4 delta 4 0 if one wants to calculate processes per unit space-time.

So, then now, all the factors have to be stuck in. This is one. One needs also the number of final states in a specific momentum interval, because the cross-section is defined for going from one specific state to another. And they are defined in terms of certain differential volumes. So, this object is v d cube p f by 2 pi cube. Again, it comes from the normalization of the states, which we have used; and factors are related to Fourier transform as well as the box volume. So, every observed object in certain interval will bring in certain volume.

And, all the unobserved states must be summed over or one can say integrated over, because they are allowed to take any arbitrary value consistent with the constraints, which are inherent in the expressions of delta functions. So, in this particular case, the unobserved object is the virtual photon, which interacts with the static charge. But, we do not really see it either in the initial state or in the final state. So, these rules now allow us to construct the cross-section. So, the probability is…. So, s f i square will contain square of the delta function. But, this has to be given proper meaning. And what we interpret it as that, one delta function is enough to impose the constraint. And after the constraint is imposed, the second one is just the delta function with argument 0. And this 2 pi raise to 4 times delta 4 0 – we will interpret as the volume of space-time.

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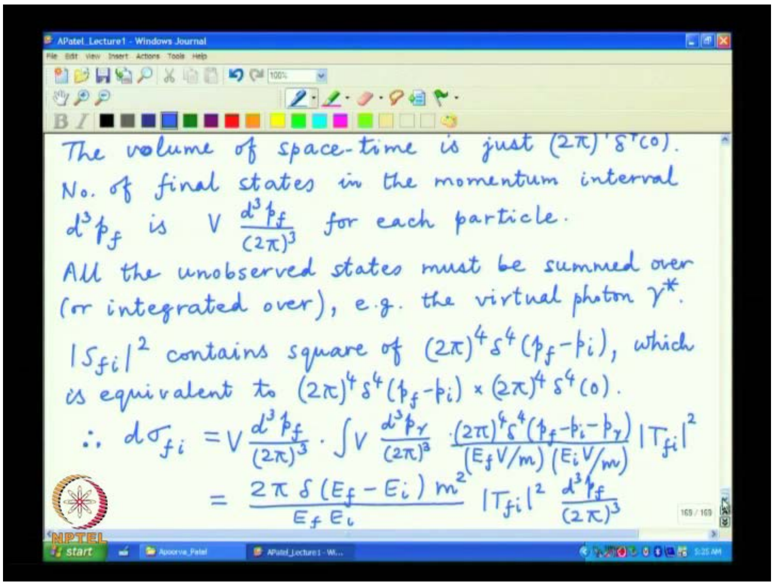

So, if one wants to calculate the probability for transition, one takes this as a pi square and divides out by this 2 pi raise to 4 times delta 4 0 and then one has a result. And so therefore, one can now write the probability of transition or the differential cross-section for sigma f I; and which now will have various elementary volumes. So, there is an elementary volume for the final state electron. I should put this V as well. Then there is a photon volume, which we must integrate over, because we do not see that photon at all. It gets captured by the static target.

And then there are all the factors, which are part of s f i square. So, one delta function remains and the second delta function we divided out in trying to calculate cross-section per unit volume. So, that is this part. And then the other normalization factors for the wave functions of the electrons are in the normalization, which we have used. There is E V divided by m. For each of the initial and final electrons, there is actually a square root, but we are taking the absolute square. And so this factor appears.

And, after all these things, we have now the final object, which we have to calculate from the transition matrix element. There is T f i square. So, all the rest of the stuff is all the various normalization conventions. And one can easily now integrate over all the simple things cancel all the factors of volumes. And they must cancel the V, which is used in this formula is only a convention; and the final cross-section cannot depend on what we choose for the value of V. So, the delta function can be simplified by using the photon momentum integral; only the energy delta function survives. And the result is… After simplifying various objects… And a differential element of the final electron state remains. So, this is a simplification of all the normalization constant; and it gives you the probability for going into this particular state from i to f.

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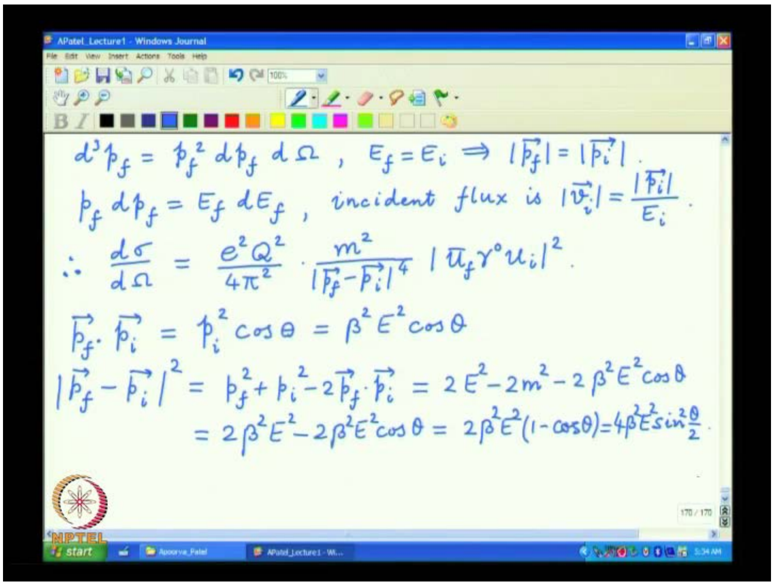

Now, we want to evaluate this object further in terms of a cross-section per angle. And to get that angle, we have to simplify this structure d cube p in terms of its magnitude as well as its direction. And let us do that. So, this d cube p f is equal to p f square d p f times d omega; where, d omega represents a differential element of the angle. The energy constraint implies that, the two momenta are equal in magnitude. The dispersion relation implies that, the differential of momentum and energy are related. So, one can rewrite the d cube p in terms of a d E d omega and get rid of the delta function by integrating over the energy direction. We can just assume that, E f will always be equal to E i in all the detection, which actually occur in practice. It is just an energy conservation constraint in this stationary target process.

And, one more thing, which needs now to convert this thing literally into a cross-section is normalize the incident flux from the plane wave convention to the number of particles, which come in – that depends on the velocity of the particle; how many of them will enter a unit volume. And in the relativistic dynamics, this is nothing but the ratio of momentum to energy. So, we have all these factors, which can now be put inside the expression for the differential cross-section. And having done that, the differential cross-section, which now is only a function of angle, all the other values or variables are eliminated by integrating them over or removing them by delta function constraints. So, the momentum delta function or momentum integral gets eliminated by the delta function in energy.

The other factors of p f square get eliminated by division of the incident flux. So, the factors of p f square actually completely disappear. And one can do the simplification of the algebra, where there is a factor of e square Q square, which comes from T f i square. The 4 pi square is all that remains from the 2 pi raise to several powers in the formula. Then all the factors of energy and momenta actually completely cancel out in this analysis, because of the conversion factors between d cube p f as well as division by the flux. And all one is left with the mass square divided by the photon propagator factor, which is now squared up inside modulus T square and then the overlap of the spinner function, which is also part of the T square. So, there is a huge simplification leading to this final formula, which is much simpler and much easier to calculate.

And, one can still look at what is the expression for this p f and p i rather straightforwardly. And that is one can write p f dotted with p i in terms of the angular variables. So, the two magnitudes are equal; one can write it as either one of them and then dot product gives cos and theta. And this p can be written as beta times E; E f is equal to E i. And so one symbol is enough to denote them instead of carrying out both the E f and E i together. And there is cosine theta.

So, this is what remains of the dot product ((Refer Time: 53:55)) everything can be converted from the momentum language to the energy language; and which is the standard way to write down the cross-section, because energies are what which are detected easily inside the experiments. And so one can actually calculate the square of this object as well, which is p f minus p i whole modulus square is equal to p f square plus p i square minus 2 p f p i. And this can be written as again in terms of the energies and magnitudes as 2 E square minus 2 m square minus 2 beta square E square cos theta. Or, other way of writing it is p is equal to beta times E in the magnitude.

So, this is also 2 beta square E square minus 2 beta square E square cosine theta. And so it can be rewritten as 2 beta square E square into 1 minus cosine theta. And again the standard convention is to rewrite this thing as half angles; and 1 minus cosine theta then gives 4 beta square E square sine square theta by 2; which is a common expression used to describe this particular cross-section. In particular, how it behaves near theta equal to 0. But, this is a standard simplification.

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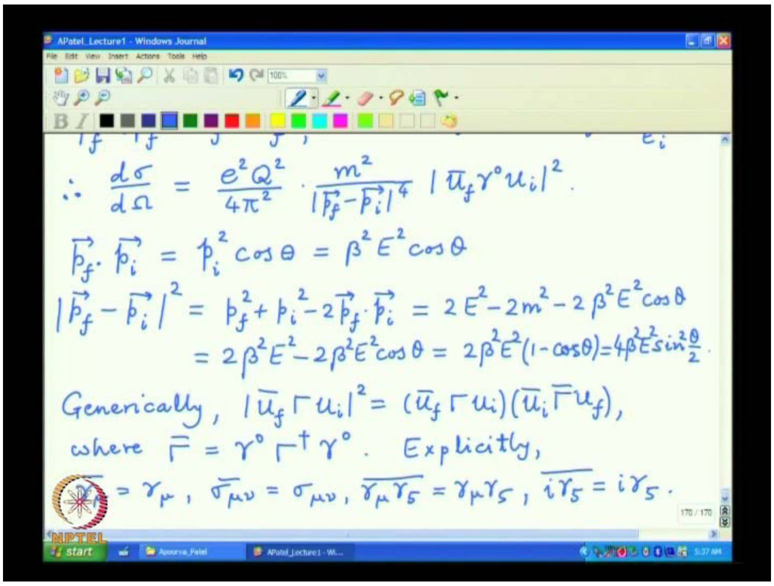

And, the other standard simplification which we need is how do you calculate the square of this particular amplitude. So, generically, an object of this type can be rewritten as… A number time is complex conjugate; but the complex conjugate is nothing but u i and u f written in the opposite order with a little matrix. And the matrix is defined by taking the gamma 0, which is part of u bar on to the other side. And so gamma 0 gives the transformation rule; that is, capital gamma bar is nothing but gamma 0 capital gamma dagger times gamma 0.

So, this is a generic rule of how one can define this transformation. And one just had to work out every time when one encounters some capital matrix gamma, what is the corresponding gamma bar; which goes through with. And explicitly, one can work out various relations that, gamma mu bar is same; sigma mu nu bar is also same; gamma mu gamma 5 bar gamma is also the same; and i gamma 5 bar is also the same; so this particular basis that is used, so that nothing happens under this bar operation. And of course, for identity, nothing happens at all as well. So, one can now use the various factors, which are buried inside this capital gamma according to this particular relation and simplify them one by one. And that procedure and all these various simplifications of the momentum space algebra will give an explicit answer to the cross-section, which I will discuss in the next lecture.