Relativistic Quantum Mechanics
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Lecture - 34
Dyson's analysis of perturbation series, singularities of the S-matrix, Elementary QED Processes

At the end of the previous lecture, I mentioned that the perturbation series of quantum electrodynamics is only an asymptotic series. And in particular it has zero radius of convergence, because the theory is unstable for any value of the fine structure constant less than 0. So, let me give an argument to support the statement which was formulated by Dyson.

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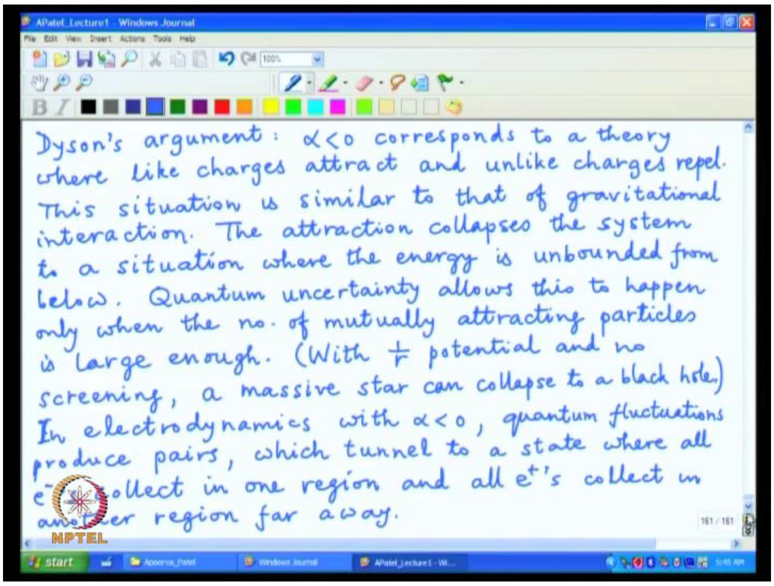

So the argument considers, the theory with alpha less than 0 and that corresponds to a situation where like charges attracts; and unlike charges repel; to see this one just ahs to note the Coulomb force between 2charges and it is proportional. So, if you flip the sign of alpha the behaviour of Coulomb force between 2 charges basically reverses from what is it is in the real world. And this situation where like charges attract and unlike charges repel is unstable for any value of alpha less than 0. And the argument essentially parallels what we know about the theory of gravity; where the force is again 1 over r square structure. And like charges which are the masses in the theory of gravity they attract.

So, this situation in a practical in gravity we do not have unlike charges there are only like charges and they attract. And then there is unstability, unstable situation of collapse of the system to an energy which is unbounded from below the attraction collapses audio. And this does not occur automatically in the quantum system. So, even though the potential can be attractive there is limitation essentially coming from uncertainty relation; where you cannot localise the objects through arbitrary precision without enhancing it kinetic energy. And one has to balance the potential and kinetic energy.

We have seen this situation in the case of hydrogen atom; the potential is unbounded from below 1 over r goes to infinity when r goes to 0. But the hydrogen atom is stable the electron does not fall and sit in the centre on the top of the proton as would be expected from classical theory. And quantum theory the electron remains around the orbit. And so the situation does not arise just by saying that the potential is unbounded from below. But there is balance which was to be achieved between the quantum uncertainty; which gives rise to finite size of the system and the potential which tries to collapse the system is a point.

Now, what happens in case of gravity is that if the number of particles involved are not very large than a star does burn out its fuel. And it collapse to a situation which is known as white dwarf; but that white dwarf remains gravitationally stable. And nothing further happens to it if increases number of particles; the white dwarf can change to a neutral star. But there comes limit where the number of particles which are mutually attracting between becomes large enough the star will become a black. And that is actually singular situation corresponding to an unbounded from below total energy. And that is the collapse which indicates that the whole theory has become unstable; there must be something else if you want to survive the theory. And the real solution in case of gravity is not known probably would involve quantum theory of gravity,

But for our purpose it is sufficient that when there are mutually attracting particles. And the sufficiently large number of them; they can collapse even in a quantum theory to singular unbounded from below in energy state. And that is what we would like to avoid in case of electrodynamics with alpha less than 0. Now, and this mutually attracting particles I should say can lead to this situation. Because the force involved is long range it is about 1 over r potential which does not allow any screening and it can act very large distances.

So, with 1 over r potential and no screening massive star can collapse to black hole. And let us see this argument little more explicitly that in case of electrodynamics; what we expect from quantum fluctuations produce pairs of e plus and e minus which tunnel to a state where all electrons say collect in one region. And all positrons collect in another region far away.

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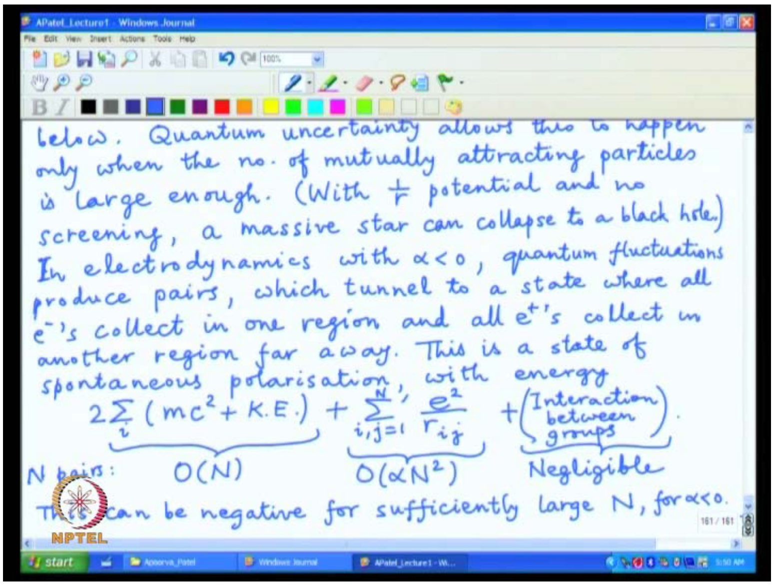

And this is the state of what can be called spontaneous polarisation; with energy of the order of sum over i where i is the particle number m c square plus kinetic energy; and this is for each group. And since there are 2 groups I will put factor of 2. And plus for interaction energy which is summed over all possible pairs prime just means, that the situation with i equal to j is not counted. And i will assume that each group has N particles. And the 2 groups are quite far away from each other. So, in can ignore the interaction between the group which is tiny.

So, now we can see the magnitude of each of these three facts. So, let us say that there are N pairs produced which spontaneously polarised in this way; the first term is order N, the last term if you take the groups to be far away can be made negligible. But the interaction within the group is of order alpha N square. And that is the crucial point that the interaction energy goes as quadratic function of the number of pairs; while kinetic energy and the rest mass energy only grow linearly, and so this can be negative for sufficiently large and for alpha less than 0. And that is the crucial point that one may require a large enough number of particles to collapse the state.

But since the function is N square for the attraction energy no matter how small; but negative alpha there may be one can always find large enough value of m when this happens. So, for those many number of pairs produced from the vacuum the theory will suddenly spontaneously polarise and collapse to a state with energy bounded from below. And this is signal that there is that instability occurs in spite of the quantum instability principle just as in the case of black hole.

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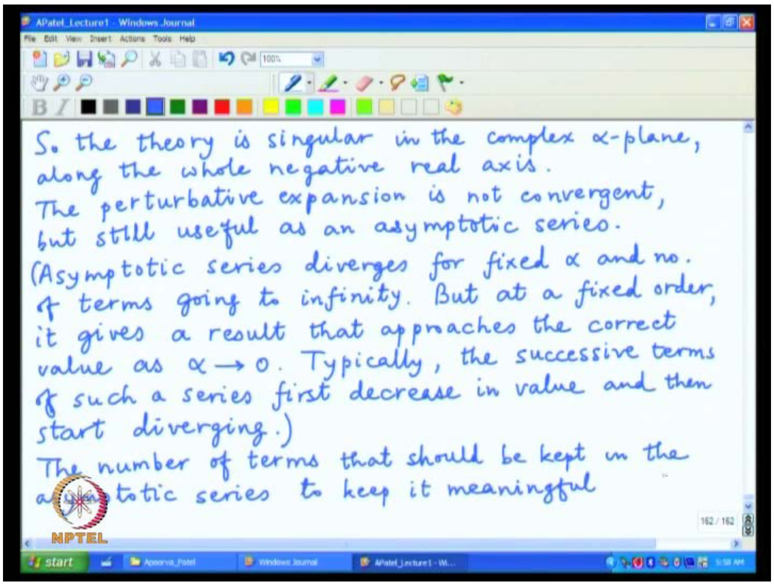

And, so we have a theory which is singular. And we have a branch cut along the negative axis in the complex alpha plane. So, that was the statement which I made at the beginning and this is argument which supports the statement. Now, we can ask well the theory is such that the perturbation expansion is not going to be convergent series but can it still be useful? And the answer is that the expansion is useful in the sense of an asymptotic series. And here one has to recall what is the property or characteristic behaviour of an asymptotic series; the asymptotic series diverges for fixed alpha and number of terms going to infinity.

But at a fixed order; that means, you do not sum all the terms all the way to arbitrary large power; but decide on a particular order. So, that I will consider only say10 terms in the series; then it provides the answer which approaches the correct value. And here we have to assume that there must be some other formulation of the theory which gives correct valid answer that may not be series expansion; but some other kind of calculation. And this happens to a better and better accuracy with alpha becomes smaller and smaller. So, if alpha is small enough one can rely on this second property, that one does not have to consider very high order series; just consider a finite truncation of it. And that answer may be accurate enough and it converges towards the correct value as alpha is taken closer to 0.

So, typically the successive terms of such series first decrease and value. And, then so for a few terms if you look the successive terms are being smaller it seems to be approaching some value. And then suddenly you find that the term star becoming larger and larger many times oscillating in sign; and point of adding more and more terms to the series does not make sense. Because the new corrections that are coming in are much larger than the value which you would expect to be meaningful. So, this is behaviour and the point that is important is to figure out how many numbers of this asymptotic series make sense; one has to find exact solution of very simple cases.

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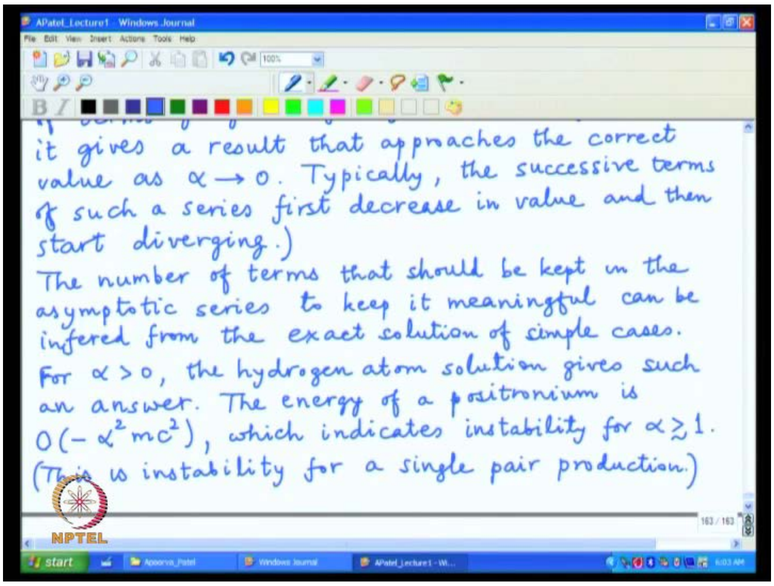

Where the same criteria apply and one can compare the asymptotic expansion versus the solution. And see how well it works in our particular question of electrodynamics this situation is 2 body problem; the same hydrogen atom situation; where we could solve and get an exact answer. And alpha greater than 0 the hydrogen atom solution gives such where one can again ask the about when situations become unstable? And in this particular case there is no spontaneous parameter, spontaneous polarisation as happened in case of alpha less than 0. But there is still the instability in the 2 body problem that one can have 1 e plus and 1 e minus; mutually attracting when they can collapse or when they can provide a potential energy that can overcome the kinetic barrier.

So, the energy a positronium is order minus alpha square m c square; we have calculated this and which indicates instability for alpha greater than approximately 1. And this is indeed true of there where charges involve which are not the charges of the electron; the alpha will be replaced by z 1 z 2 times alpha in this analysis. And one has answer that z 1 z 2 times alpha should not exceed 1 for the theory to be stable.

So, this is the criterion which we just derived by looking at alpha as the parameter; of course in the real world phi is about 1 over 137. And this instability which corresponds to again a spare production out of the vacuum; but for a single pair. So, this is instability for single pair production; different from the instability for a large number of pair production which we dealt with in that case for alpha less than 0.

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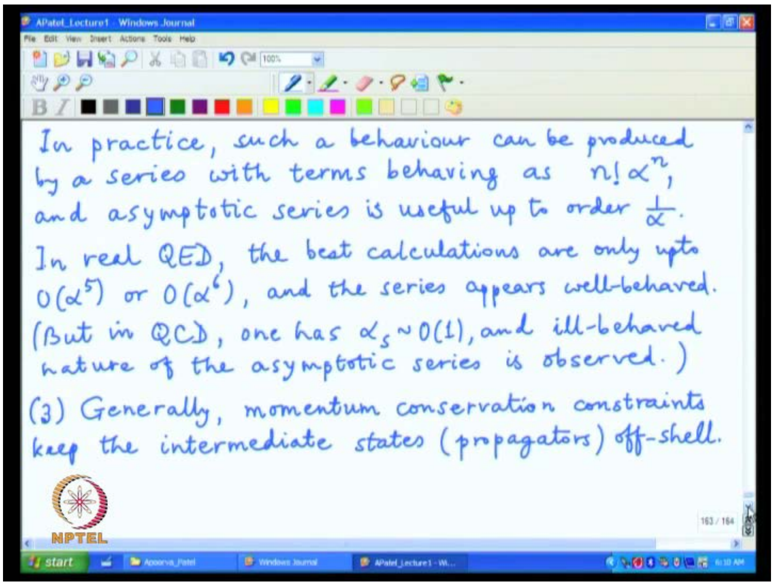

So, this is behaviour; and in real world the series of the, this particular type can be obtained by looking at the order by order behaviour and see which the whole analysis goes. And you want an asymptotic series of this particular; one can work little backwards. And figure out that an asymptotic series; whose terms are behaving as N factorial times alpha to the N it will never converge for any value of alpha; no matter how small. But its successive terms will fall and they will start diverging only 1 the number N crosses 1 over alpha. And so in case of QED the expectation is that the series will be meaningful up to order 100 and 37; we are now here in our calculations capability to reach anywhere near the order.

The base calculations are of fifth or sixth order in alpha something of that type. So, the series actually is very well behaved. And one can see that the successive terms are decreasing in the real calculations with alpha is equal to and things are work very well. But there is similar theory which is applicable to strong interaction it is also gauge theory; but with a different structure. And in particular its coupling constant alpha is order 1 in this particular case easily observed in that particular case that series terms do appear for a few first few terms to becoming smaller. But they suddenly start increasing and there is no point in calculate to higher order and some various things up.

So, this is important digration in understanding the meaning of this perturbative series; before trying to attempt a calculation and trying to compare to experiments. Now, let me add one more point in this analysis or description of nature of perturbation series and which has to do with the properties of the s-matrix. So, generally momentum conservation constraints keep the intermediate states in the Feynman diagram which we have been describing as propagators off shell. And we will see this occurring explicitly in many examples to be worked out.

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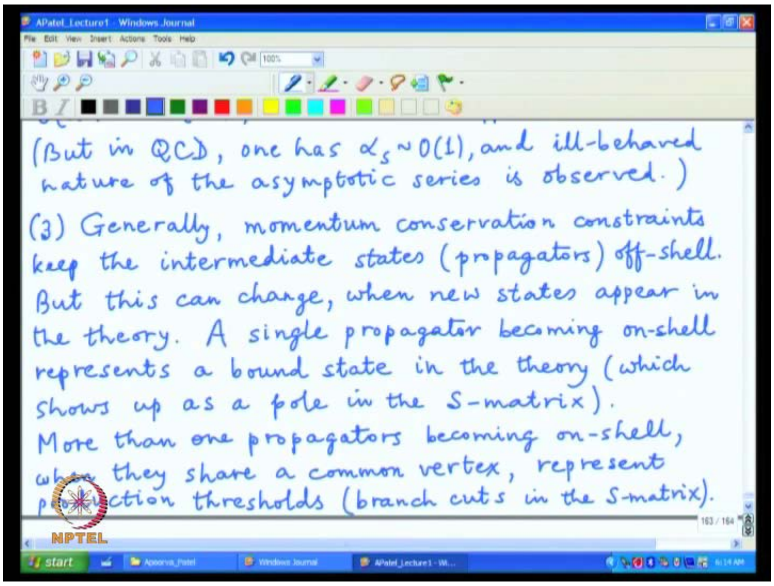

But there can occur situation; but when this can change when new states the states which are not described as asymptotic states appear in the theory. And which see this signatures of various kind of situations; which may occur in the explicit answer for the s-matrix I will mention 2 instances at a single propagator becoming on shell represents bound state in the theory. And in terms of s-matrix the propagator which was just a simple polynomial in the denominator. So, it shows up as pole in the pole in the s-matrix all the propagator terms.

And, vertices which do not have any problems as far as the singularities are concerned; one can have more than 1 propagator also becoming on shell. And that will represent many separate bound state appearing in the theory or when the share a common vertex represent; what can be called production thresholds instead of bound state. And in terms of the s-matrix this will be represented by branch cuts.

So, both these situations occur; and every time the show off this we have a new dynamical feature of the theory to understand. And one has 2 ways to go about looking at this situation; of course 1 straight forward base you just expand the theory to include all these new states. And there are decay properties as well as production property in the analysis. And then you allow much larger theory where these problems will go away or one can stick to the same theory. But try to express the singularities in terms of behaviour and constraints about where the theory is well behaved. And above which particular points one has to look for new facts and that is done.

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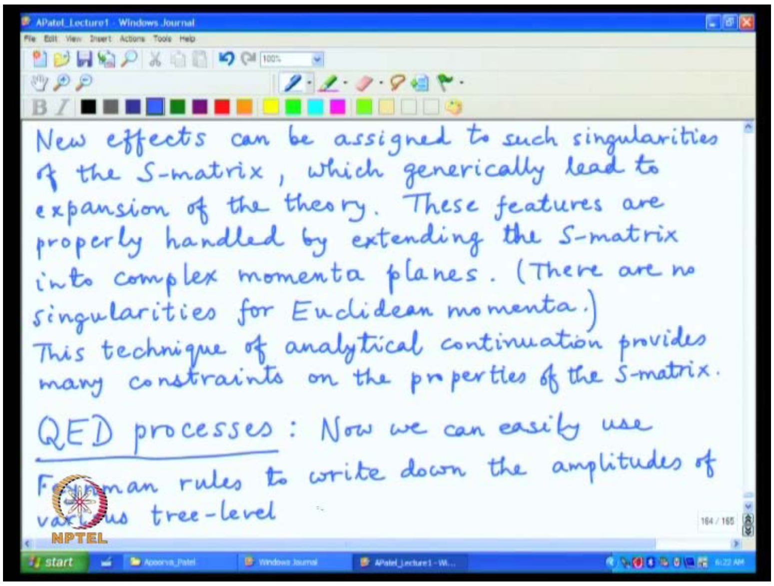

So, new effects can be assigned to such singularities of the s-matrix and which generically lead to expansion of the theory. And the tool which is most useful in this particular analysis the treatment of all these poles and branch cuts is easily done. So, these features are properly handled by extending the s-matrix into planes; where all the momentum become complex. And then one can ask the same situations and when the meaning of this poles and branch cuts become more appropriate in the sense of dealing with functions of complex numbers. And in particular one can note that there are poles and singularities appeared because the denominator could become 0 with these factors p square minus m square cancelling out because there are signs coming from the Murkowski symmetric.

And, there are no singularities for what are known as Euclidian Momenta; where instead of Murkowski symmetric we have just Euclidian matrix delta mu nu. And all the 4 components of p square add; then they are all positive numbers. And that mathematical expression can never be 0, and there are no singularity. So, one can connect these 2 features from Murkowski space to Euclidian space doing an analytic continuation in the complex Momenta plane. And then one can describe the behaviour of what is happening around the poles and the branch cuts much more accurately. And this technique of analytical continuation it provides many constraints on the properties of the s-matrix. And these properties many times are beyond the restriction of Perturbative expansions they are far more general. And they are in that particular sense powerful constraint on the whole structure of quantum field theory.

So, I will not go too much into details of the analysis of this analytical continuation. But it is useful feature to know that such things might be done; we might use a few instances of this Euclidean connection to do very simple calculations. And that will come in the later part of the course. So, with all this background we have built up the background to do explicit calculations and understand the physical Consis. And that is what I would now turn to; and that may just introduce the topic of various kinds of processes.

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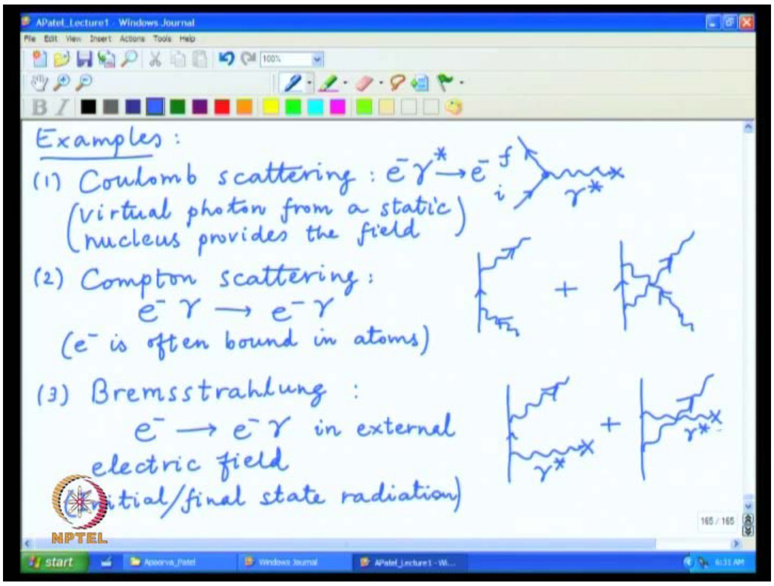

And, then calculate from the scattering amplitude. What is the rates of these procedure? which are often describe in terms of cross sections. And I will stick to 3 level process; because these are much easier to handle. Then the one which involves loop at maybe I will make comments about various loops at a later stage. So, let me give some examples the simplest example is that of coulomb scattering; there is heavy nucleus producing certain electric field. And the project I will basically scatters of from that field while the heavy nucleus hardly moves or rather it recalls what it recalls? Very little and the coulomb scattering that recalls ignored. So, in terms of Feynman diagram process can be describe as some by projectile coming in i some projectile out f after the dielectro magnetic field; that interaction is represented by this photon. And; the photon is produced by some nucleus which I am denoting by cross here and this photon is not a real propagating photon.

But its photon representing coulomb field which is virtual photon denoted by gamma with star on it. So, this is virtual photon from static nucleus provides. So, this is the process and we can calculate its amplitude and corresponding scattering. And we will do that later let me first only give related process; the second common process which I would like to mention is Compton scattering; where there is not virtual photon. But there is a real photon which actually hits electron and is absorbed and reemitted.

So, one can denote it by this particular diagram. But now there is alternative available in different ordering of the processes in the quantum world; because everything intermediate is summed over the whole space time. And all particular time orderings can occur. So, the alternative is that first final photon is emitted. And then the initial photon is absorbed and this again we will see how to calculate it. But now one has to add the 2 diagrams and the relative sign between these processes depends on the ordering of the 2 photons. And it just happens to be positive because photons are bosons and interchanging them does not give any extra sign which we have to track off.

But in real world the Compton scattering is not done on a free electron; but it is a process. So, it can be described as E minus gamma going to E minus gamma and the electron is of 1 bound in atoms; even though it may be bound in atoms or it could be there in metals the scattering can occur. And then one can again calculate the rates of these particular processes in a same symbolic language; the coulomb scattering would be a process where e minus gamma star will lead to e minus. Let me mention another common process it has the name Bremsstrahlung; which means literally breaking radiation. And what happens is the electron in particular case emits a photon and slows down. And so it is kind of reverse of this coulombs scattering.

But again it cannot emit a real photon like this; because of energy momentum conservation something else must be there to absorb the recoil. And in this can occur in external electric field and again this in real world is just initial or final state radiation. And in terms of diagrams though the reaction shows only 1 photon; the external electric field does appear as virtual photon in the process. And the diagram will appear at the photon is in final state; but and virtual gamma star from the external electric field like in the coulomb potential is absorbed. And again in the same way there are 2 possible ordering; where the photon can be emitted before or after the virtual photon from the electric field is absorbed.

And, so there are two diagrams again the ordering between the 2 bosons does not give any sign. And the 2 contributions have to be added again more correctly I could put a gamma star for the eternal field in this particular process. So, these are the simple processes involving radiation I would give some more examples of simple processes next time before going on to details of calculations one by one.