Relativistic Quantum Mechanics
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Lecture - 33
Feynman rules for Quantum Electrodynamics, Nature of the perturbative expansion

So, we have put together all the machinery to describe the interactive theory of electrons and photons, and there are quite simple rules which can summarize everything, and they are actually heavily used in doing the calculations; they were invented by Feynman. So, obliviously they carry his name.

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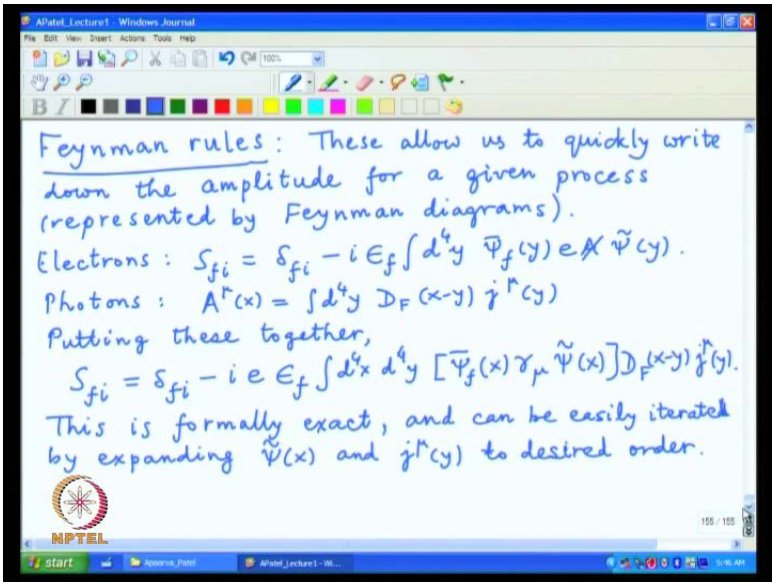

And what they allow us to do is quickly write down the amplitude for a given process, and this process will be represented by Feynman diagrams which are nothing but space time picture of what is going on. Now, while doing this we quickly forget about all the quantum dynamics in the sense of having a wave functions and distributions, etcetera. We will just look at the whole thing as in a particle language, some particle going from one place to another and interacting with some object and then scattering from it, etcetera, and so on and so forth.

And we have seen two different descriptions for these; first for the electrons we have written down the element of the scattering matrix as the part where nothing happens and

then part where the electron wave function responds to the electromagnetic field represented by A slash in this process. Now the electromagnetic field can intern arise from distributions of charges and currents, and we had that relation as well which were summarized by this equation. So, putting those two things together we can quickly iterate the procedure that electron response to photon, and photon is consequences of electromagnetic charges distributed in a particular way as well as the moments. And so net result can be written in a simpler form where now we have only various combinations appear in terms of the wave functions of one type or the other.

So, this quantity in the bracket is nothing but what was there in the electron equation, but the A part is now written explicitly in terms of the source, and this is a combined equation which is in a form which can be iterated as many times as necessary. So, if the photon is an external field, well we do not write it in terms of it sources, but if it is produce by another electron then it can be written by j mu. J mu again is represented by psi bar gamma mu psi of some other source which in turn may have been interacting with its own electromagnetic field, and one can just iterate this procedure as many times as you want to produce the necessary description which goes along with the space time picture provided by Feynman diagram.

This equation as I have written is a formally exact, because what appears on the right hand psi tilda which is the exact solutions of the Greens problem; when trying to use perturbation theory we start out with the simplest answer, where psi tilda is replaced by the initial plane wave distribution you get the result from this equation. Then iterate it; next time you put in the answer inside it. It was found at the first order, and that gives rise to all the description which we can now easily summarize in terms of this so called Feynman rules.

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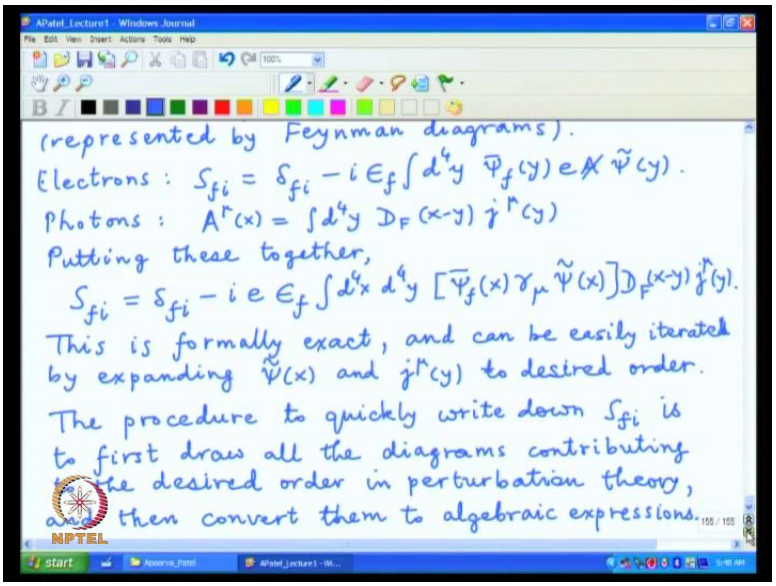

So, we now have a procedure to this to first draw all the diagrams. Then first we draw the diagram to any desired order and then convert what is represented by this diagram to algebraic expressions. And clearly in doing this perturbation theory the parameter of which powers are counted is the charge e, and we want to calculate any object, say, to the first order m e or second order a d e or third order n e. It can be easily done by just expanding psi tilda and j mu iteratively one step after the other and constructing all possible descriptions of the processes.

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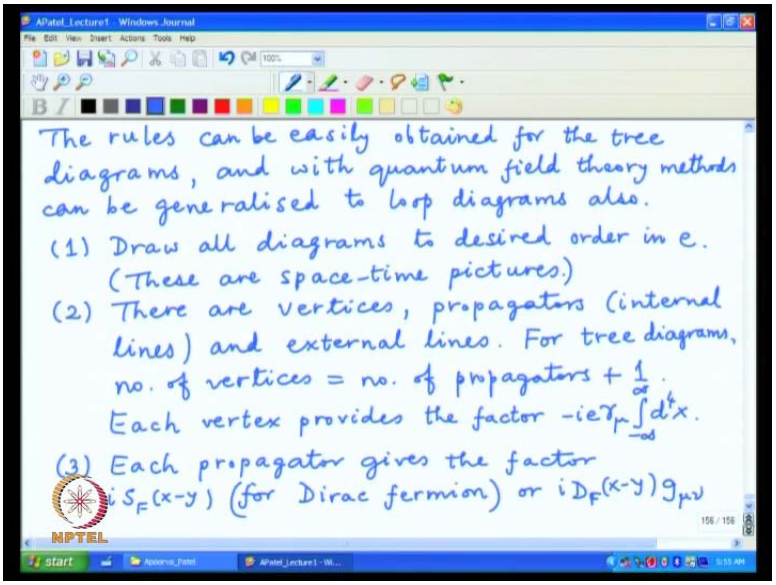

So, the rules can be easily obtained for the tree diagrams where the tree diagram refers to a topology of this space time picture. There are no closed loops, and once one learns how to go beyond that that requires some knowledge of quantum field theory. And then we have loops, but we know how to handle the loops as well in the same picture and might have to include a little bit extra stuff to deal with loops. But that is not too complicated once a complete field theory description is made; the tree level rules are easily extended to interactions of arbitrary type. So, what are these rules? One is well draw all the diagrams to desired order in e, and as we have constructed the whole thing these are space time picture.

So, the rules as you have constructed will be obtained in the coordinate space. The second step is there are various segments of these diagrams. So, they will be referred to as vertices. There are propagators which can also be called internal lines and external lines which correspond to particles in the initial and final state and the numbers of various things of various types is actually related. And for tree diagrams one can easily convince oneself that the number of vertices is equal to number of propagators plus one, and that is easily obtained from the equation which I wrote down for the scattering matrix; every time one expands the right hand side to a particular order there is an extra factor of a propagator coming in, and there is an extra factor of the power of the charge e coming in.

So, the power of e actually counts the number of vertices and both of them at every order increase by one. So, one has to only identify the lowest order term where there will be a single vertex but no propagators, and that fixes this particular relation, and the power counting is a just the factor which we have put in the equation minus i e gamma mu and an integral over a location of the vertex; that is because the vertex represents the interaction and interaction could have occurred anywhere in the whole space time. So, every vertex basically counts the power of e.

So, if you want to calculate say to third order perturbation theory the theory or the diagrams will have three vertices inside them; there is no other place which contributes a power of e. So, that much is for the vertex; what about the propagators? We have seen both type of the propagators corresponding to the fermions or the photon, and so each propagator gives the factor i S F x minus y. This is for Dirac fermions or i D F x minus y together with this g mu nu if necessary to contract the indices for photon.

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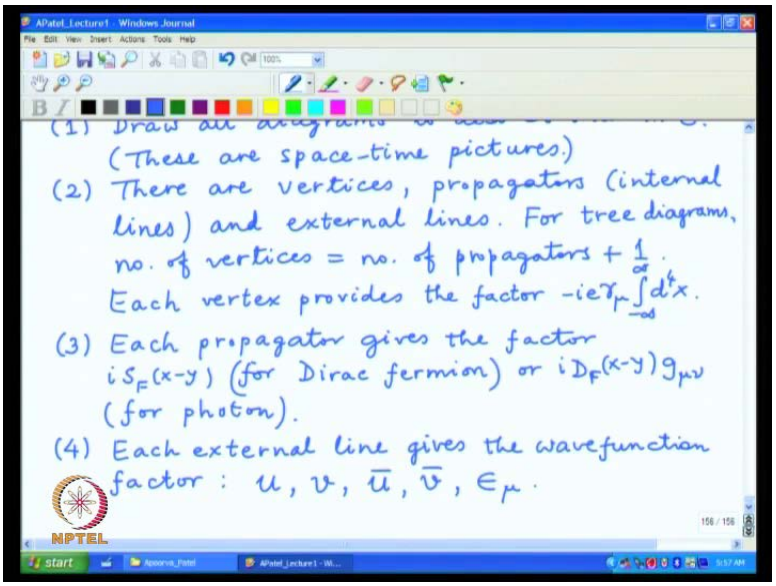

So, this essentially counts the second part that propagators each will bring in a certain weight; the vertices bring in a certain weight, and the last weight will be the external lines which will have a corresponding. Each factors line gives the so called wave function factor, and this can be of different type depending on what the particle is. If you put v for an electron in the initial state or a v for a positron it can be u bar for an electron in the final state. Similarly, v bar for a positron, and if it is a photon we have seen that the description is given by this polarization times for epsilon mu.

So, this now has completed the description of the diagram; every vertex every propagator and every external line have been accounted for, and beyond this now there are a couple of factors of the signs which appear. They have to do with our conventions as well as the statistics which has to be followed by the theory, and this may not be too obvious, but once one works out the full field theoretical formulation it becomes quite straight forward to handle.

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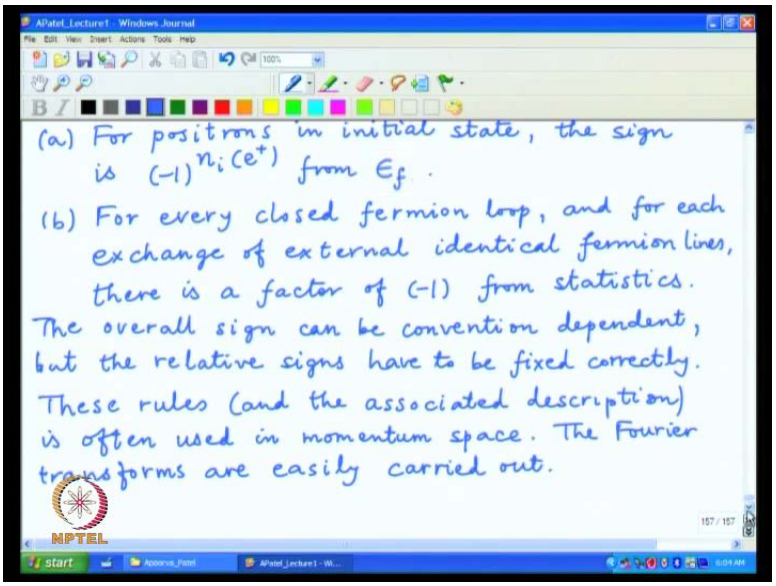

So, there are signs, and there are two sources of this particular sign. One is that for positrons in initial state; there is the sign is a minus 1 raise to the power of the number of positron, and this arises from the various rules we have used to define the wave functions or the free Dirac particles, and in particular there was an epsilon r which defined the sign of the energy of the solution whether it is positive or negative. For positrons epsilon r is equal to minus 1 and which enters in this whole analysis in some particular way. So, that is one overall sign, and for every closed fermions loop and for each exchange of external identical fermions lines, there is a factor of minus 1. And this is a consequence of statistics in dealing with the single particle theory we have not seen statistics directly, but in a field theory it has to be properly included.

And for the fermions the anti comumutators between various fields they produce this statistical sign of minus 1 when two of them are exchanged, and the close fermions loop is related to this statistical sign as I have illustrated in analysis of the Lorentz group. That it is something which can be visualized as a rotation by 2 pi, and that rotation by 2 pi produces the same sign as the sign for the statistics. So, these are overall signs picked by the various conventions which we have used, and one more thing one can keep in mind that the overall sign can be convention dependent, because quantum wave functions cannot be specified with an absolute value of phase. But the relative signs they have to be fixed correctly, because they correspond to observable interference; in fact then if you do not get it right the answer will not be correct.

So, this is the summery of how to convert a given Feynman diagram to an algebraic expression describing the scattering amplitude, and we will work out various examples of a different level of complexity to illustrate these rules and algebraic calculations. But there are some more things which can be easily added to these descriptions, and in practical calculations these rules are often used in momentum space and which means that we will Fourier transforms everything from the coordinate description to momentum space. And that gives rise to description which is actually simpler, and the reason for it that the propagators which we have written down look much simpler in a momentum space than they look in position space.

And what happens in this Fourier transforms is that you have generic factors like e raise to i p dot x for every point. So, in case of vertex it is just a single factor; in case of propagator it will be a factor like e rise to i p dot x minus y where x and y are the two ends, and then one has a description of the vertex and propagator both in a momentum space. And then one can combine these various factors of e rise to i p dot x which accumulate at every vertex from the position of the vertex itself as well as all the propagator as well as external lines that may be connected to a vertex. And one has to do an integral over the position of the vertex which can be now easily done in other trivial fashion.

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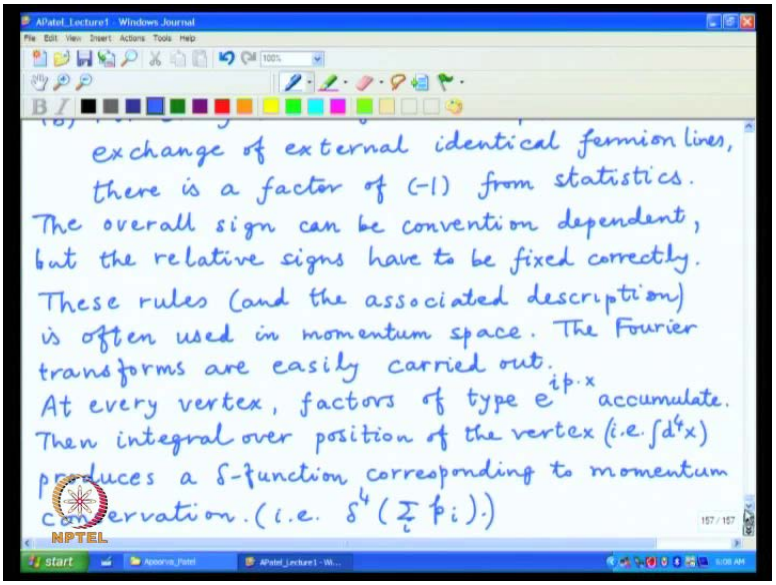

So, at every vertex factors of type e raise to i p dot x accumulate and then integral over

position of the vertex which I wrote down produces a delta function, because integral of a factors of e rise to i p dot x will just produce delta of the summation of all the momenta corresponding to that particular. And we can see that that is delta 4 summation over all the momenta, and it have to be counted in terms of the direction described by the arrow of the various lines and so one has to now keep track of the momenta but in a direction dependent manner.

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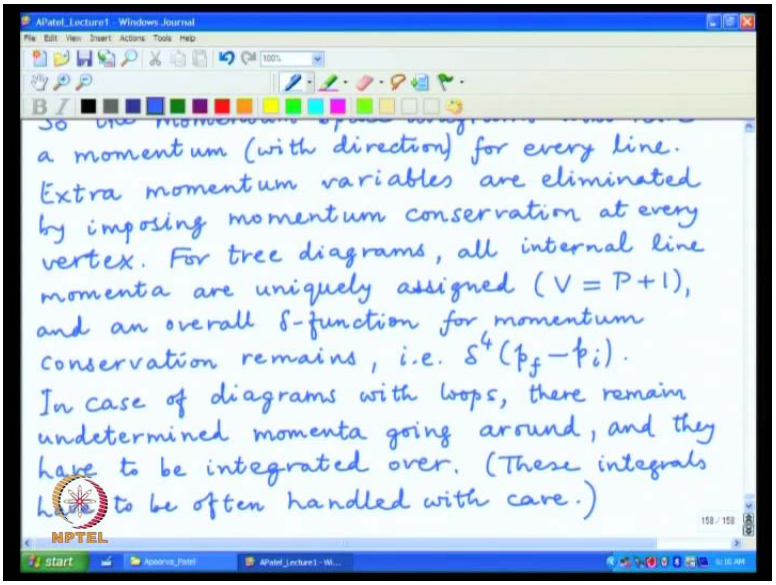

So, then now the momentum space diagram will have a momentum with direction which is represented by drawing an arrow on the line for every line and extra momenta. So, one draws a generic diagram. We have seen that the number of vertex is one plus the number of propagators. So, the external lines have an incoming momentum which is provided by the definition of the problem; we cannot do anything with it, but all the internal propagators will have their own momentum. Then we will impose all the constraints at every vertex that will keep on eliminating all the internal momenta one by one, and what happens is one gets the number of delta functions correspondingly equal to the number of vertices.

Now these number of delta function are capable of eliminating all the internal propagator momenta for the tree diagrams, because the number of vertices is one plus the number of propagators, but that extra one means that there is an one overall delta function is left. So, that has to have a meaning, and it is quite obvious what that meaning is. For tree

diagrams all internal line momenta are uniquely assigned, because this relation which I can schematically just write that v is equal to p plus 1, and overall delta function for the momentum conservation remains. And that is the total outgoing momentum must equal total incoming momentum for all the external legs of the diagram.

If that is not satisfied then there is no interaction at all, because the theory has an overall translational symmetry, and the total momentum conservation must hold. So, this one overall delta function completely factors out of the whole problem. We did the counting for tree diagrams here, but it is true as well for any arbitrary diagram it may have extra loops as well, but this overall momentum conservation is still valid. So, there will be always be a factor of delta for p f minus p I, and many times it is convenient to factor it out of the whole calculation as well, and write only the remaining part of the object which corresponds to the so called transition matrix element.

So, in case of diagrams with loops there remain undetermined momenta going around which are not at all constrained by what the external legs provide without running into any conflict with momentum conservation. And this have to be integrated over, because the Fourier transform always produces integrals d for x or d for p; if something is not eliminated by delta functions, well we have to perform that particular integral. And in case of loop diagrams these integrals may have to be handled with little care, because their momenta go from minus infinity to plus infinity and these integrals have to be often handled with care. We have not seen anything like the loop diagram picture so far, but when it occurs and we will see some examples we have to be careful in how you deal with these integrals over momenta.

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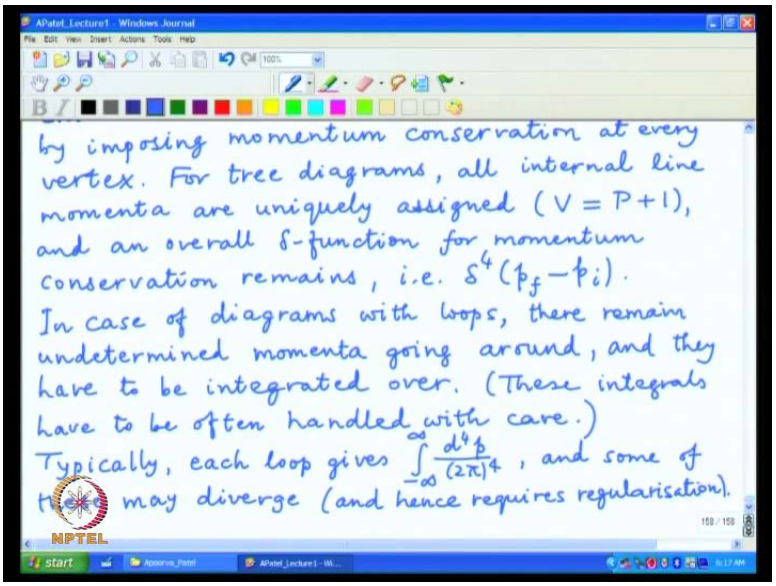

And typically, each loop gives a structure which is of integral in four dimensions, and the care is required because some of this may diverge, and so the care which I mentioned is a procedure which is called regularization. So, one has to do a little bit of jugglery to take care of these loops, but that is a necessary price to pay once you go beyond the simple equation of motion to a field theoretical language. And the normalizations of photons and electrons I already wrote down, but I will repeat them once again.

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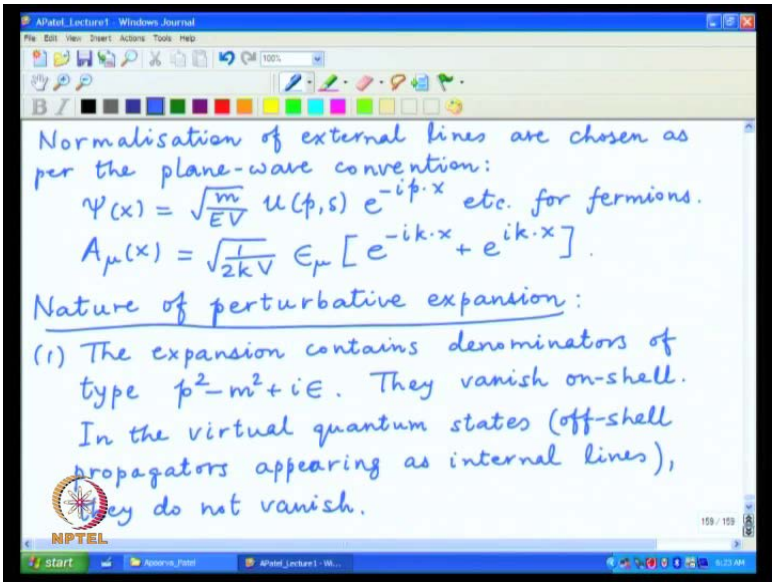

So, the normalization of external lines is chosen as per the plane wave convention. In

case of fermions it is a structure which is of the form square root of m over e which normalizes according to the Lorentz contraction factor v is just box normalization for unit valium. And the spinner is u or v depending on the positive or negative energy solution, and for the photon we have written formula which I will repeat its 2 k v epsilon mu being the polarization tensor and then the two forms either the positive energy part or the negative energy part.

So, this is a sort of complete description quite elaborate in terms of various conventions, but it is sufficient to now do calculations to any desired degree in the perturbation expansion for any process described by the scattering matrix. I would like to make a few comments about the nature of; we will get a series working out terms one by one in the analysis, but then we want to compare it with other kind of perturbative expansions which we have seen either in mathematical physics or in non-relativistic quantum mechanics and compare what are the new features or related features which appear here. So, one fact here is that the expansion contains denominators of type p square minus m square plus i epsilon. Generically this is the feature in relativistic theories; for massless photon m is 0, and for fermions it can be mass nonzero, but these are the factors which appear, and they vanish in states which are called on shell which obey the dispersion relation exactly you mean p square equal to m square.

In the expansion which we are constructing in the virtual quantum states these are so called off shell propagators appearing as internal lines. They do not vanish, and this is related to the fact that all these internal states are transients. They are not asymptotic states. They can live for short times and in the short times you are allowed to have fluctuations in energy as well as momenta consistent with the uncertainty principle. And so the p square and m square can go little bit away from each other’s values and the denominators then do not become zero. And that is the part which actually contributes to physics, and what we do in this quantum theory calculated the contribution coming from all these internal virtual states, and they add up to giving us full result for the perturbation theory. And one can see the relation of this kind of objects to the description which appears in normalativistic theory.

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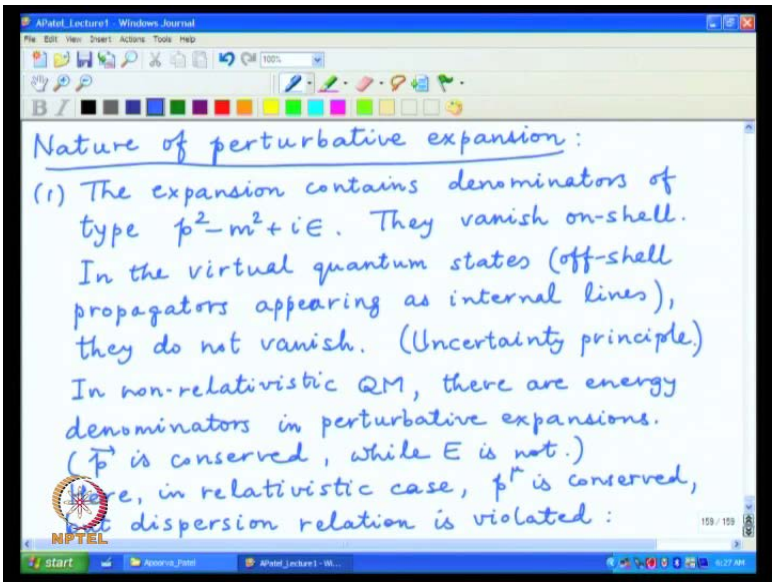

Where in non relativistic quantum mechanics there are so called energy denominators in perturbative expansions, and they correspond to virtual states which live for very short times. So, one has a transition from one state to another, and then you come back to the original states or produce another transition to another state and so on and so forth, and the momentum is explicitly conserved in the non-relativistic expansion. But energy is not that uncertainty in energy allows us to deal with this energy denominators which appear in the expansion, and there so the p is conserved while E is not.

Here in the relativistic case we have already conserved the momentum at every vertex; it came out from our formulation automatically, but the uncertainty is appearing as the dispersion relation between energy and momentum is not conserved for all these internal lines. So, it just got disguised in a different form, but the uncertainty principle is having a particular effect in the structure of the calculation, and how we then sum over all these internal diagrams, and one can see the relation between the two by just a simple decomposition.

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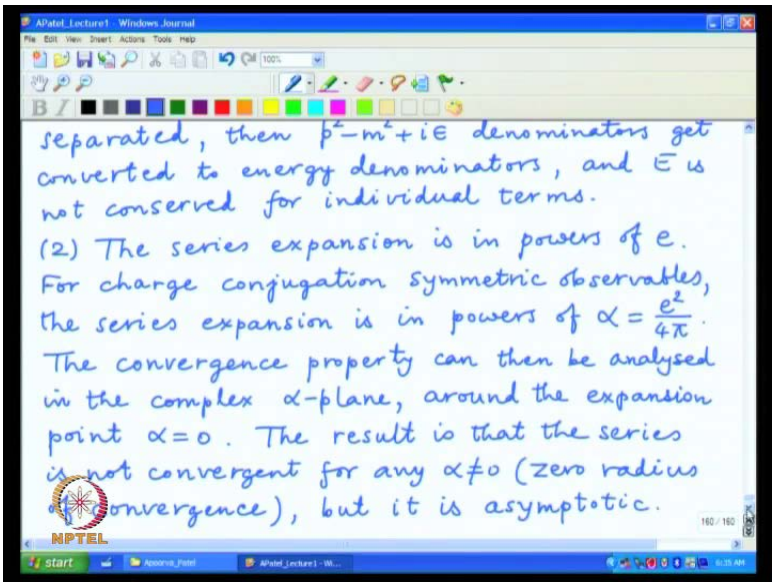

If different time orderings are separated we did the reverse in combining the stop functions into a single description with I epsilon prescription and so on, but if you now go back and rewrite this propagator as sum over two different poles one corresponding the positive energy and one corresponding to negative energy. Then all these factor of p square minus they separate out into the energy denominator of the same nature as seen in non-relativistic quantum mechanics, and in that case e is not conserved for individual terms. Of course, the total energy momentum conservation is still exact that corresponds to the overall delta function of p f minus p I. All these stuff is the internal virtual states which do not have to obey this rule, and that is exactly the place through which the quantum effects are entering if we do not allow these fluctuations of the states to official part then we will have only the classical mechanics and no quantum effects at all.

So, this is one particular feature for the organization of perturbative expansion just illustrating what it is. The other feature which is useful to note is the nature of this particular series in terms of its convergence that you can write down order by order terms in e, but it can be some of these things is any mean meaningful fashion; if not analytically then can you do it numerically. So, that question has an unusual answer that the series is actually not convergent, and so we have to put in some effort to make some sense of that this particular series.

So, the first thing to note that the series expansion is in powers of the electromagnetic

charge, but if you go to processes which are charge conjugation symmetric, then one can change e to minus e and expect not to see any difference at all in the result. And so the series expansion is in powers of the fine structure constant and which in the normalization which I have been using it is equal to e square by 4 pi. And so e goes to minus e; alpha does not change, and that is a natural parameter which appears because lots of processes which we calculate in this description are charge conjugation symmetric.

And now one can ask what is the convergence property can then be analyzed in the complex alpha plane around the expansion point alpha equal to 0? This is the standard machinery of mathematical function when expanded in a series whether the series has a analytical form, and it can be summed to any particular order or its non analytical, and then it has no meaning in terms of a expansion. And the result is that the series is not convergent for any alpha. This means more explicitly the expansion is zero radius of convergence, but it has another name which is that the series is asymptotic, and we have to understand these features in a little bit of detail; why it is so but a quick answer is that the theory is unstable when analytically continued to alpha less than 0. And there is the whole branch cut for the whole negative real axis of alpha, and we will see the argument which substantiates this result next time it was provided by Dyson.