Relativistic Quantum Mechanics
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Lecture - 32
Charge quantization, Photon propagator, Current conservation and polarizations

In the previous lecture, I discussed geometrical picture of local symmetries and associated gauge potentials which come out of the theory of electrodynamics together with covariant derivatives and its associated gauge invariant observables. There is one more thing I would like to point about which is easy to see in that picture is the feature of the symmetric group, and this has to do with what is the available values for the gauge function lambda.

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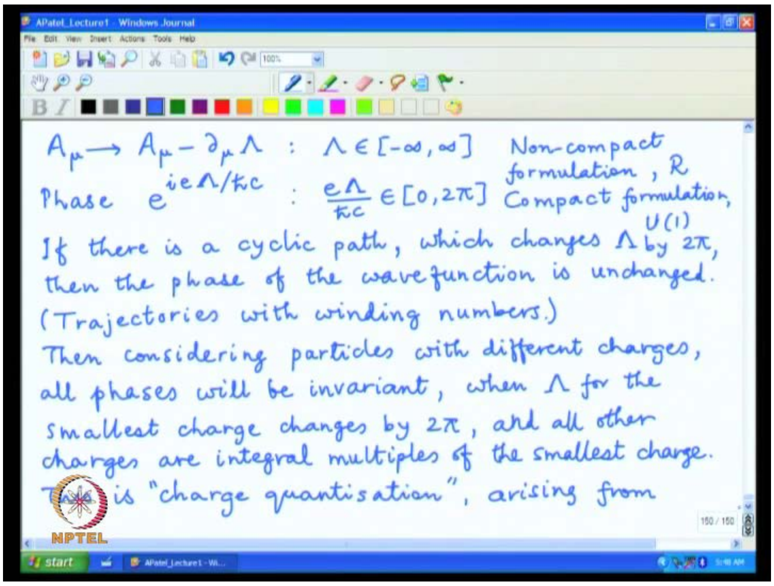

So, in the classical analysis of Maxwell's equation when we just took the variation as A mu going to A mu minus d mu lambda in this particular case lambda was just a real function, and one can take them values anywhere between minus infinity and infinity, and this is many times referred to as a non-compact formulation. On the other hand, the local symmetry picture was derived entirely from the fundamental structure inherent in the phase of the wave function, and the phase is a periodic function in the sense that the phase value can change uniquely only between the interval say 0 to 2 pi, and afterward everything basically repeats itself. So, if one looks at the phase which we use the definition with all this normalization constant.

In this particular case one can say that this combination e lambda by h cross c belongs to, say, some interval 0 to 2 pi, and this is called a compact formulation in particular the space of variable shear is, say, the real numbers are, and in this particular case it is the variable or the symmetric group is U 1, and this difference between the real line and a circle makes a topological feature appear in the theory. In this particular case it is an important feature because the function lambda on the compact space U 1 is only defined modular 2 pi, and if there is a specific structure in the problem so that when ones goes around a cyclic path. So, when if there is a cyclic path which changes lambda by 2 pi or it is a multiple then the phase of the wave function is unchanged, and in that particular case one cannot distinguish whether one went around the circuit or did not, and this kind of structures in the space of paths they are also called sometimes trajectories with winding numbers.

And the implication which follows from this kind of trajectories with winding numbers that if one can go around the trajectories come back to the starting point as in a Wilson loop, and closed flux is such that there is no change in a wave function of some particular charge, then we will not be able to make out any difference whatsoever between this path with a winding number and path without a winding number. And now one can start looking at particles which have different charges; what will happen to those different phases for various different particles? Each one will have its own value of charge which is appearing here as a variable E, and if the gauge phase background is such that it is not going to influence these trajectories under wave functions at all by going around for one particle, we might as well have a situation where none of the particles are affected by the gauge field.

And that is kind of straight forward to see if you can think about building particles with larger charge from smaller ingredients with particles with a smaller value of the charge, one can just put them together and take them around one by one, and in that particular case then considering particles with different charges all phases will be invariant when lambda for the 2 pi, and all other charges are integral multiples of the smallest charge because if e raise to I 2 pi is one e raise to I 2 pi times an integer is also one, and then the gauge field will literally have no effect on the phases of any wave function.

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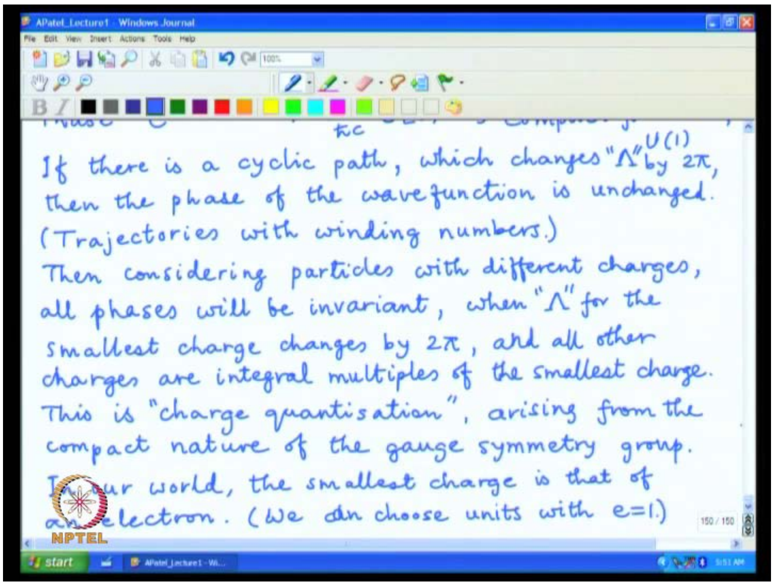

And this is the principle of charge quantization arising from the compact nature of the gauge symmetry group. It is crucial that the gauge group variable was defined only modular 2 pi, and if that is the case then the phase will be invariant not for just one particular charge but all its integer multiples; without this compact nature the condition would not have been satisfied, and so the compact nature produces winding numbers and the winding numbers lead to charge quantization. This is a generic pattern which holds for many other compact groups as well, and in this particular case the phase symmetry leads to charge quantization when that particular group becomes compact. So, this is a useful feature which can be easily seen in this geometric description of gauge theory. It is not at all obvious or rather cannot be deduced from Maxwell's equation themselves.

I should say that when I have referred lambda in this places it is actually e lambda by h cross c, because that is the value which is quantized in our world. The smallest charge is that of electron, and often we can choose simple units where this number can be just taken out from the whole algebra if we have put e equal to 1 you can insert it back later by just simple power counting when necessary. So, then all the charges observed in nature are multiples of the charge of the electron which is very much true as seen experimentally, and this compact gauge group formulation allows us reason of how that integer quantization can arise from the theoretical perspective, okay. So, now we have discussed enough of this gauge theory part, and let us go now to consider the quantum dynamics of the photons.

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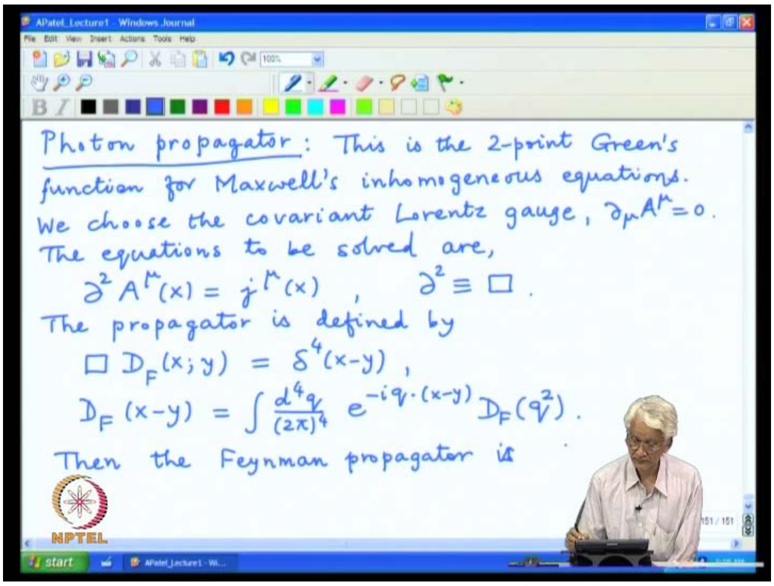

And soon that will be extended to include the interaction with electrons as well. So, just in the case of electron to solve the inhomogeneous equation the half of the Maxwell's equation which we have not solved we need to construct the machinery, and that machinery again is the Green's function method, and that produces the photon propagator which is nothing but the 2-point Green's function for the Maxwell's inhomogeneous equations. So, this is a 2-point Green's function. So, we will work in the covariant Lorentz gauge where the vector potential satisfies the condition that d mu m u equal to 0; remember that this still does not fix the gauge degree of freedom completely, and so part of the gauge symmetry will still be manifest in the results which will be derived from this equations.

But some of the gauge degree of freedom does get eliminated and then that helps in keeping the algebra little simpler. So, the Maxwell's equations are in this Lorentz gauge become and many times this operator del square is also denoted by this symbol box. It is a covariant expression the wave operator in four dimensions, and we want to solve this equation for an arbitrary j mu x; we defined the propagator. So, by the same operator now acting on a two point function which I am going to denote by d with subscript f; f is again in Feynman prescription for choosing the i epsilon extension for the causality of the propagator, and this object the 2-point Green function satisfies the equation which is the right hand side is a delta function in four dimensions.

So, it is a same equation, but the current is now used as a point object, and it is very clear that because of the translation invariance of the problem the 2-point function does not depend on x and y independently. It only depends on the relative separation of the two coordinates, and in principle you can also describe it is a Fourier transformed version which will depend explicitly on the momentum conjugate to the separation x minus y, and the notation is just the Fourier phase factor, and on the right hand side I am going to write D f of q square generically it would have been just F f of q, but will quickly see why it is q square that appears. And now this equation can be easily be solved, because the Fourier transformation converts these derivatives to just polynomials in a momenta and the box operator with this Fourier decomposition just becomes minus q square.

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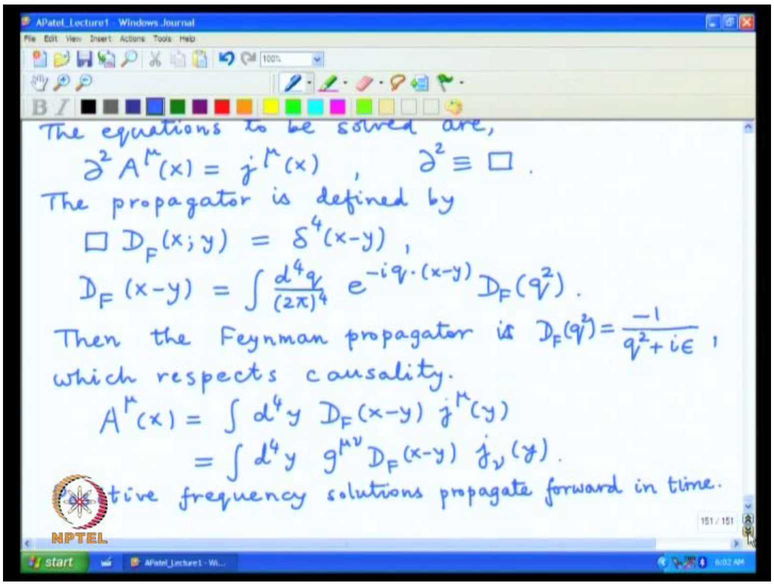

So, then rather straight forward equation box becomes minus q square, the delta function in Fourier space is one. So, this object is just minus 1 divided by q square, but we will use the i epsilon analysis as done earlier in the case of non-relativistic field as well as Feynman field and stick in the plus i epsilon in this definition. This expression clearly shows that the propagator is a function of q square on one hand, and i epsilon prescription ensures that the positive energy part will propagate forward in time; the negative energy part will propagate backward in time. So, which respects causality; so this turns out to be very easy.

And now we can solve the generic equation for an arbitrary source by the simple superposition rule that A mu x is equal to integral d 4 y D f x minus y j mu y. Sometimes this connection between the Lorentz indices of mu and nu is not written as it is here with same index on both side, but that can be quickly fixed by putting different indices and the metric tensors. So, the same thing can also be written as g mu times D f x y times j nu of y. Then one can use this extra factor of metric to say that a particular source j with nu produced a particular electromagnetic field with index A mu. And so this is essentially the solution for any arbitrary distribution of charges we know what the electromagnetic field is going to be the relativity is respected and so is the principle of causality. So, again positive frequency solutions propagate. Now one can see some extensions of this analysis which has to do with the fact that the current which appears in Maxwell's equation is automatically conserved.

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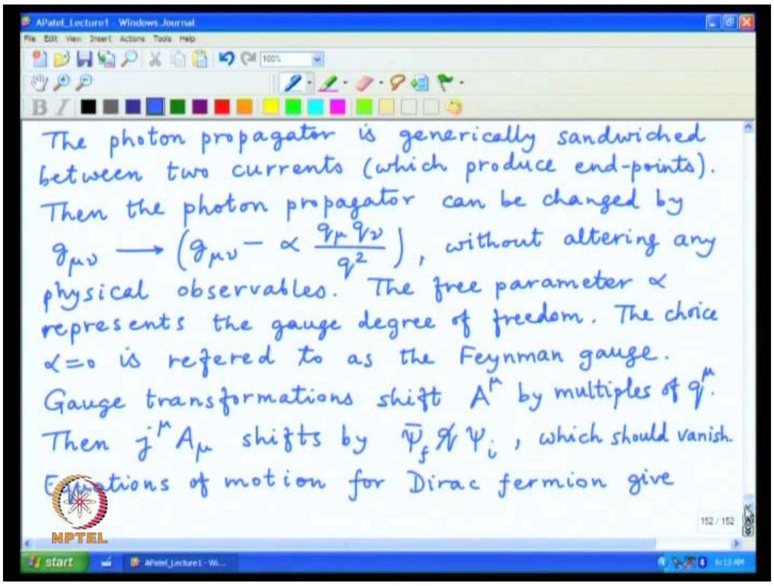

So, we have the identity that d mu j mu is equal to 0, and in momentum space that becomes just whatever the momentum variable is contracted with the current equal to 0. The momentum variable is the momentum corresponding to the photon which couples to this particular current. So, this leads to many identities which are automatically obeyed by the photons which we have introduced. So, one of the important features is that the photon propagator is generically sandwiched between two currents which produce the so called end points of the propagator, and so the photon propagator is left uncertained because the current itself obeys this extra identity.

So, that can be seen by where ever we have the factor g mu nu as seen on the previous description; you can easily put factors of q mu nu where q mu is of the momentum carried by the photon, and the two ends of the photon propagator will have the indices mu and nu. One will be contracted with current j mu at one position; the other will be contracted with current j nu at the other location, and the current conservation guarantees that q mu j mu will always be zero. So, this extra term which I have inserted as modification of j mu nu always produces 0, and so the parameter alpha is kind of arbitrary. It has no physical consequences, and that parameter represents the gauge degree of freedom, and this extra degree of freedom is indeed tied to the current conservation as we have seen before, and it is left undetermined. One can make specific choices to suit the calculation; the factor of one by q square in the denominator is just inserted to maintain the power counting because g mu nu has no dimensions.

And so this q nu also should have no dimensions which is easily done by having this q square in the denominator, and this whole expression is still a covariant structure, and the particular choice which is simplest in this whole calculation is just not having this term at all. And that alpha equal to 0 prescriptions is referred to as the Feynman gauge, particular choice of fixing the gauge parameter and that does simplify the calculation to a substantial degree and is used very frequently in doing detailed calculations. So, this is something which is available in the gauge propagator; there is still a degree of freedom left and we are free to choose it if necessary, but one can make clever choices to keep the calculation simple, okay.

There are now other consequences as well following from the same identities that gauge transformations shift A mu by multiple of q mu. Remember the description was A mu goes to A mu minus d mu lambda will go to momentum space; the derivative becomes q mu and lambda is just a free parameter. So, A mu will change by its multiples of q mu, and now our requirement is that the theory is gauge invariants means whichever place A mu appears if you change that particular place by a multiple of q mu the result should automatically turn out to be 0. So, whenever A mu appears and this will always appear in the interactions between the fermion and the photon as part of the covariant derivative.

And that will immediately now produce a restrictions. So, we have the currents coupling to the vector field which was whatever interaction was in the case of Dirac equation as well as in case of the Maxwell’s equation. And if we use the Dirac equation structure for j mu then it will be the expression of psi bar q slash psi where the interaction was psi bar a slash psi, and this is the particular shift. And now the theory has to retain gauge invariants. This must be 0, and this is indeed a statement again of current conservation or equivalently the symmetry of the gauge degree of freedom and so one can now use the equations of motions.

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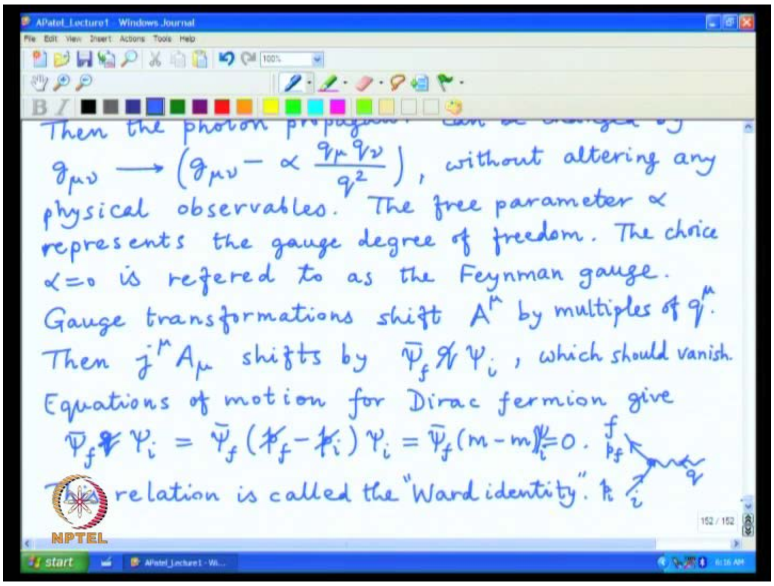

So, to simplify this object where q is the momentum of the photon and because of momentum conservation it is equivalent to the difference of momenta of the two fermion states. And now the Dirac equation says that p slash psi is equal to m and the same way for psi bar with p slash acting from the other side also gives m. So, we get m minus m is equal to 0 as required, and so this identity is indeed satisfied. It is a verification that our formulation is consistent, and many time this is represented by the property of the vertex of a fermion with a photon where there is a momentum p i p f, and then there is momentum q for the photon, and every vertex you can apply this interaction term between a current provided by the fermion and the factors of q mu provided by change in the gauge potential, and this relation automatically holds and it has a.

So, this relation is called the Ward identity. It has a very powerful connections to the principles of gauge invariants and current conservation which become manifest in the full fledged field theoretical formulation of quantum electrodynamics. But here we see the relation in a simple form, and it just follows from the type of structures we have constructed to describe the electromagnetic interaction. One more feature I can now point out.

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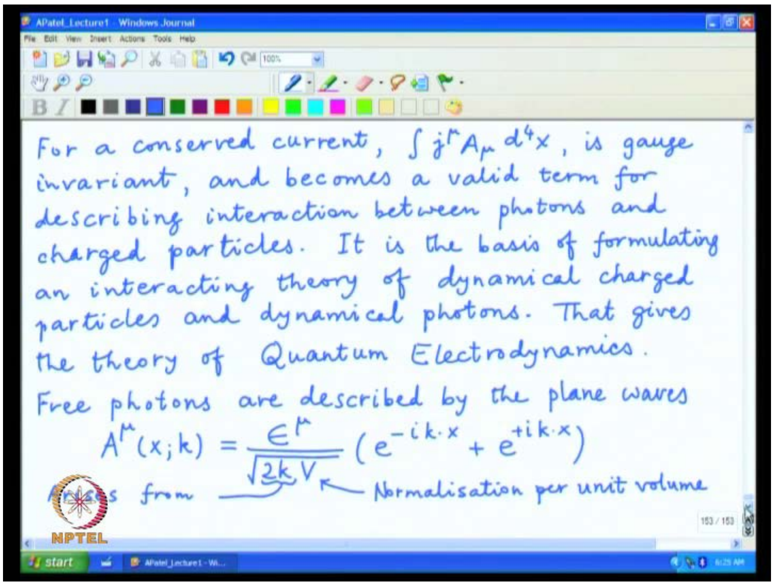

And that is consistency for this particular structure of interaction which we have used. The object j mu A mu d 4 x integrated over the whole space time is gauge invariant, and that is very easily seen when you now construct the change in this object under gauge transformation A mu will change by d mu lambda; one can do the integration by parts. So, d mu will shift from acting on lambda to acting on j mu, and d mu j mu happens to be equal to 0, and then the change in this quantity is 0. We needed the integration over the whole space time to be able to do the integration by parts, so that the boundary does not contribute by being at infinity where there are no currents and no potentials and becomes a valid term for describing interaction between photons and charge particles.

So, this is a generic statement which actually goes beyond the description of just electrons forming the particular current for any charged particle Dirac particle or Klein-Gordon particle or may be something else; one can always construct a conserved current from the equations. That current can be contracted with the electromagnetic vector potential as j mu A mu when integrated over the whole space time that is the interaction between the charged particle and the electromagnetic field, and that object automatically satisfies the principle of gauge invariance. It obeys current conservation, and so all the important features which we need in the theory are kept intact, and this becomes now the basis of describing the theory of dynamical charged particles and dynamical photons.

And that is the way we now combine the formulations which we use for the fermions in the Dirac equation where the photon was treated as a background field and the Maxwell's equation where the charges were treated as background distribution. So, in one case the fermion was a dynamical field; in the other case the photon was a dynamical field. And now if we agree to this common interaction term which appears in both this theories we can have the photons as dynamical fields together with the fermions also has a dynamical field and a common interaction term. And that is the structure of the full theory of quantum electrodynamics where all the degrees of freedom are simultaneously dynamical. They respect the necessary principles of symmetry Lorentz symmetry, gauge symmetry as well as current conservation, and that gives the theory of quantum electrodynamics.

So, this is a nice combination of various things which we have put together. There is one more feature I would like to point out, and that is the descriptions of a free photon, because now we are going to describe in terms of the vector field A mu, and we already know that they are the solutions of the wave equations. So, the easiest thing is to choose the plane wave basis, but we have to pick some notation, and so we have a photon field at some position x with a particular momentum k. And that now can be defined as some direction in space time denoted here by epsilon mu. It represents the polarization of the particular vector field divided by a particular choice of normalization, and then the two terms corresponding to the positive energy solution and the negative energy solution, and here this factors are introduced by the standard conventions that the v refers to here the so called normalization in a box, or equivalently the plane wave normalization per unit volume one needs that.

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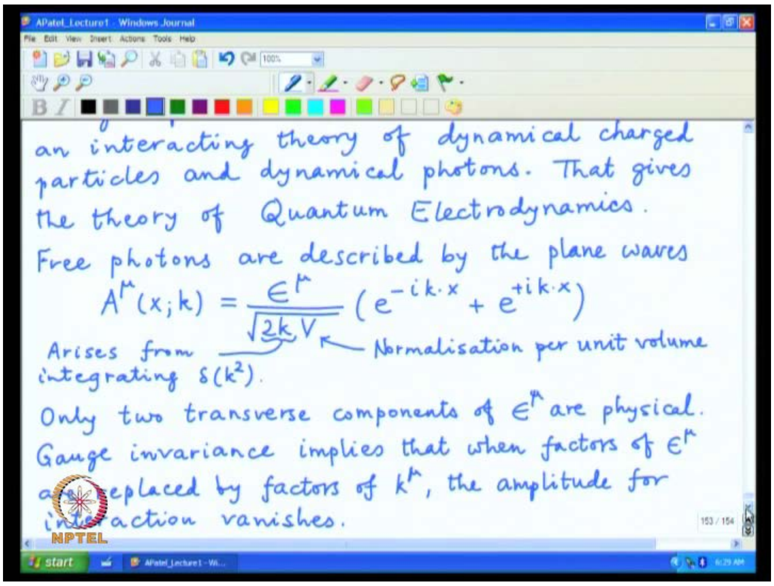

And this factor of 2 k arises from the Lorentz invariant constraint of integrating the delta function corresponding to a wave solution with 0 mass so that delta k square when you integrate over a time to produce a appropriate frequency solution it gives a one over two k from the invariant integration measure which respects a Lorentz symmetry. So, this is again a feature essentially it is a feature of Lorentz contraction showing up here again, but this is a convention which we will use in describing the various photons. The important point is the description of these polarizations. There are four components; only two transverse components of epsilon mu are physical and the other two are there, but they are in a sense present because we wanted a Lorentz invariant prescription; they must cancel out in the final description, and we will have to see the actual expression for particular amplitude to obey the gauge invariance prescription.

So, implies that when factors of epsilon mu are replaced by factors of the amplitude or interaction vanishes, and this is just restating it, because when we saw that when A mu was replaced by the gauge transform variable, it will be shifted by something proportional to k mu, and this epsilon mu is just representing A mu in this particular basis. So, when you shift it by something proportional to k mu the change which arises must vanish, and so this becomes a check of the calculation that to derive certain interaction amplitude in this particular basis, and whenever you replace epsilon mu by k mu the result must vanish. So, this is a particular description of free photons.

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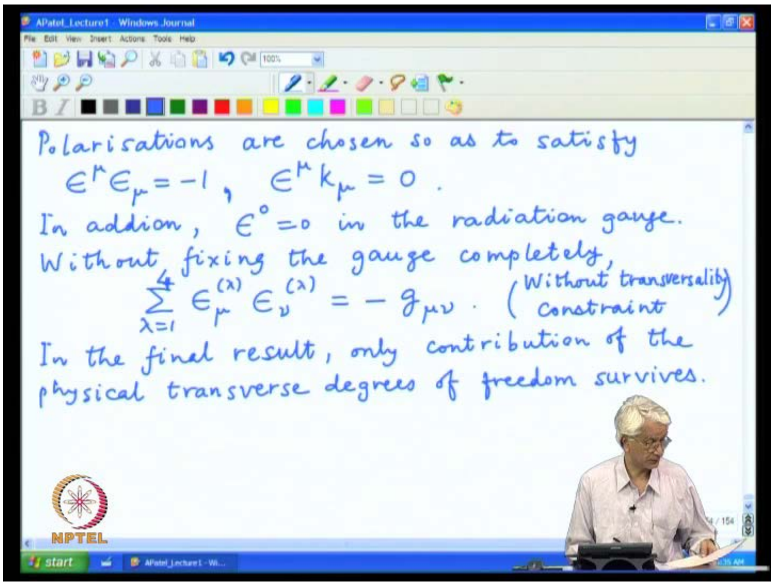

We can also make a few comments about this choice of polarizations are chosen so as to satisfy two constraint; one of them is this norm of this vector is minus 1. The minus 1 sign essentially comes from the Minkowski metric; the physical polarizations correspond to spatial direction, and so that particular vector will be normalized to one. And the second constraint which we have imposed is epsilon mu k mu equal to 0, and this is a restatement of the Lorentz gauge which you are using. It is a d mu A mu equal to 0 which produces this particular constraint, and both of this signs are identically satisfied in the formulations which we have used, but still there is an extra degree of freedom left; in the sense that this constraints do not determine only the two transfers polarization. There is a feature left, and we in general will have four different values, and they can be chosen with respect to an extra constraint.

The most common constraint is the radiation gauge where one ends up choosing epsilon is equal to 0 in the radiation gauge where the degrees of freedom get fixed completely, and this is a consequence of saying that A 0 equal to 0 in the radiation gauge. Then the epsilon k mu equal to 0 means that the polarization is orthogonal to the direction of propagation of the photon, and that leaves only two transverse direction as the physical degree of freedom. But sometimes one keeps all this arbitrariness of epsilon present in the calculations. So, if one sticks just to the Lorentz covariant description then one has the summation over four different choices of epsilon mu, and the identity which is important is so called completeness relation between this four degrees of freedom, and this is equals minus the metric tensor for the two physical directions indeed this summation does produce one.

But the other two unphysical directions produce minus one for the time component and the plus 1 for the longitudinal component, and that effect basically cancels out in the final result when all the indices are contacted together and only the physical. So, in the final result only contribution of physical transverse degrees of freedom survives, and that is a convention which is often chosen that epsilon is just left arbitrary in description of the photons and only after doing the complete calculation, the final step one makes a particular choice of the physical degree of freedom of saying that the photon was polarized in some particular direction in space, and the results automatically then obey the Lorentz covariance, because all the intermediate steps have been maintaining Lorentz curvy. So, here this is without transversality constraint. So, these are the several useful things to mention, and now we can go back and put this whole structure together.

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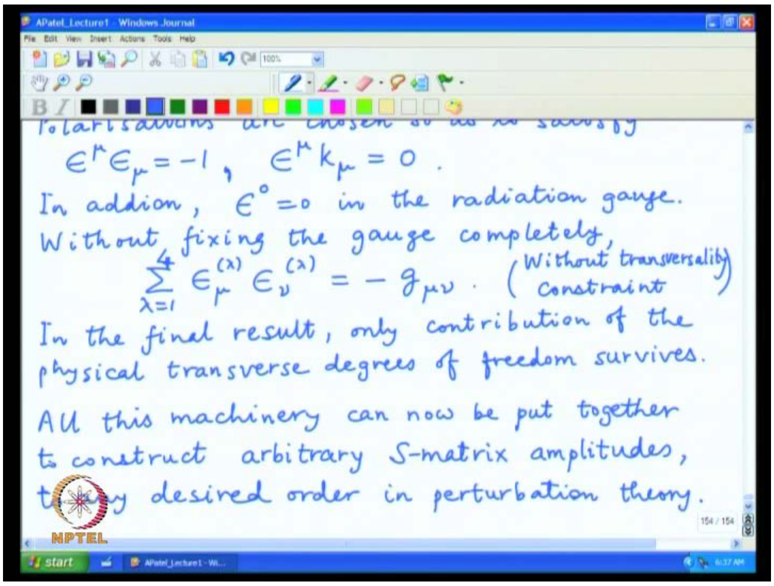

So, all the machinery can now be put together to construct arbitrary S-matrix amplitudes to any desired order in perturbation theory and which is what I will do in the next lecture. There is a very compact way of writing all those things which goes by the name of Feynman rules, and we will see how all this equations we have dealt with are summarized very effectively in terms of those rules, and you can write arbitrary scattering matrix amplitude by just drawing Feynman diagrams and counting the appropriate factors.