Relativistic Quantum Mechanics

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Lecture - 31

Abelian local gauge symmetry, the covariant derivative and invariants

In the previous lecture I discussed the theory of electromagnetic field based on the classical equations that is the Maxwell's equations; we identified the correct degrees of freedom and also identified an important property of gauge invariants of those equations. So, when you use the vector potential as a dynamical degree of freedom there are unphysical components which are used for convenience in writing down the equations but which can be eliminated later by imposing constraints.

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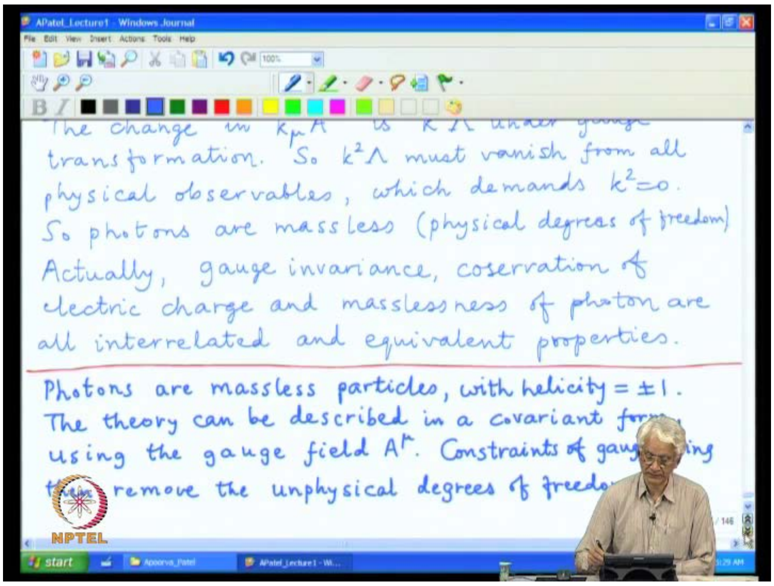

So, summary of all this can be said that the photons are massless particles with helicity equal to plus or minus 1, and that is a language which gives the appropriate Lorentz group representation to which photons belong; the both helicities together make a theory which is symmetric under parity, and in describing the theories we found it convenient to use the covariant language to make the Lorentz symmetry manifest. But in order to do that we have to use the vector potential or which is also referred to as a gauge field A mu. This has four degrees of freedom while that actual degree of physical field are only two and so we saw how the imposition of gauge fixing constraint can be removed or can be used to remove the unphysical components.

So, constraint of gauge fixing, then remove them, and that is the basis of the symmetry which is referred to as gauge symmetry. The directions specified by those symmetric transformations are not physical; they do not change any observable values, but they are convenient in order to write down formulation which looks algebraically simpler. So, this was all based on the equations of motions, and we need to give a little more general framework to go to quantum theory which goes beyond the classical equation of motion, and for that reason different language of describing this same theory is appropriate. And that language is now the standard formulation for describing a general system of fields which are known as gauge fields, and that has a direct connection with symmetries as well as the accompanied features like conservation loss.

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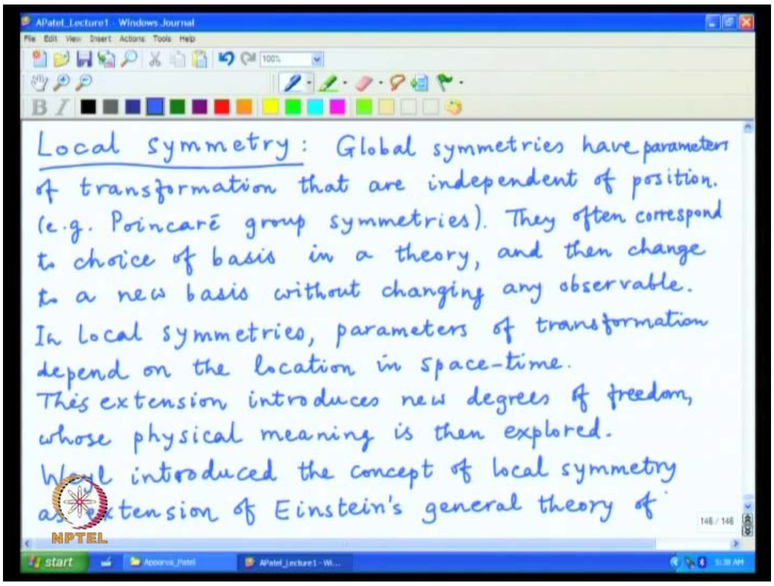

So, this formulation starts out with constructing a local symmetry for a particular sort of fields; there is no mention of what are the equations of motion at this stage. But we first construct certain operations for the fields which appear in the theory and then worry about in which way these fields are going to evolve, and what is the corresponding Hamiltonian or the Lagrangian, and then the equation of motions will come out from that particular analysis. So, the concept of local symmetry is to actually go beyond the concept of symmetries which we have seen earlier in the course, where we have essentially discussed the symmetries which can be called global symmetries.

So, these global symmetries have parameters of whatever the symmetry transformation may be, and these are independent of the position or location in space time. For example, the symmetries which we have heavily discussed is Poincare group symmetries which included translation rotation boosts and even going beyond parity time reversal things like that, and those symmetry at parameters for example, the rotation angle and the angle was the same over the whole space time; it was independent of the particular location. Now this symmetry is common in many theories Poincare group just happens one particular example, but it can be often associated with specific choice of a basis, and one can perform a transformation of bases which changes the values of various components.

But the overall observables are independent of the basis choice, and so they will be independent of those particular parameters specifying the orientation of the axis, and they all correspond to particular choice, and that is the concept of invariants that observables do not change when you change bases. Now local symmetries extend this concept to a next level by saying that not only there are going to be transformations, but the transformations can be dependent on the location of the particular point. So, in local symmetries; so, this extra freedom of choosing arbitrary transformation at different points certainly enhances the degrees of freedom of the theory, and we have to find a interpretation of what this extra degrees of freedom do, and also ask whether such exact degree of freedom have any physical meaning in the problem which we are dealing with, and if that is so then we have a new framework to study an extended theory.

And it turns out that taking a global symmetry and constructing this local symmetry out of it by changing these parameters as a function of space time location produces a theory which is a gauge theory, and we will now see explicit construction of it in a specific case of the quantum mechanics which we are dealing with. Now historically this concept of space time dependent transformation was introduced by Weyl in the context of Einstein’s general theory of relativity and many of the terminology for that reason which has entered this concept of local symmetry comes from the phrases defined and used in general theory of relativity.

So, Einstein constructed the general theory of relativity based on the principles of equivalence and constancy of speed of light, and Weyl then asked whether one can go beyond this general theory of relativity in a certain manner. In particular the postulated of general theory of relativity where such that it left the space time invariant interval proper time or proper distance unchanged when one performs an arbitrary transformations.

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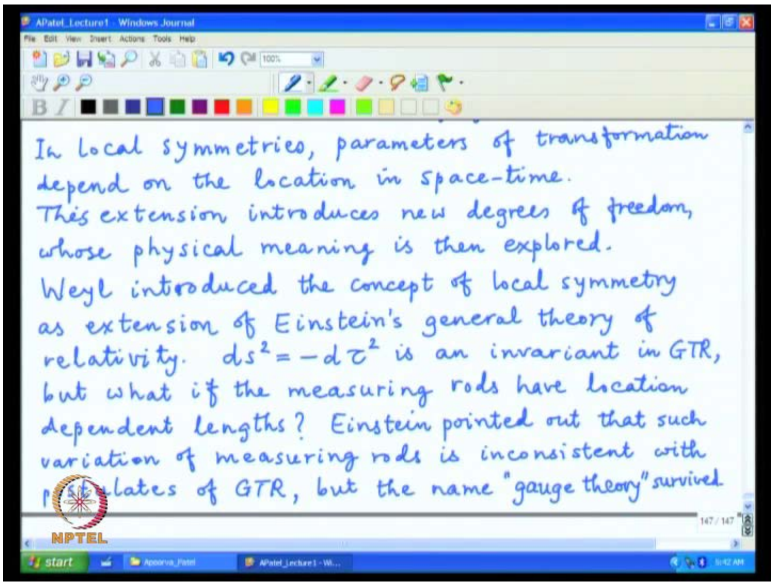

So, the question was that d s square can also be written as minus d tau square is an invariant. The question of Weyl was that this is an object with dimension of length, and what if the standard measuring rod which is used to specify the length had different size at different points in space time? And that is the reason for the nomenclature gauge theory or gauge degree of freedom; gauge literally refers to a measuring rod, and this extension turned out to be meaningless as far as general theory of relativity was concerned.

Because Einstein quickly pointed out that such an arbitrariness in measuring rod is inconsistent with the principles of general relativity, the postulates of equivalence and constancy of speed of light do not allow such a degree of freedom, and so the concept had to be put aside, but the name the gauge degree of freedom got stuck, and it was resurrected when the concept of local symmetry came back in quantum mechanics. So, that is a little bit of history of how the main gauge theory came about, and the concept was resurrected when quantum mechanics came along.

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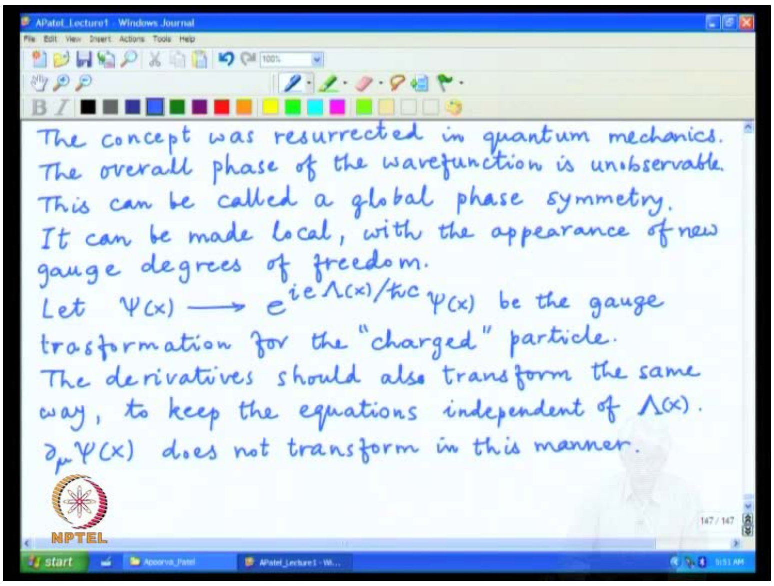

So, and here indeed there was an object which was unobservable, and that is the overall phase of the wave function; one can measure relative phases between two different wave function by doing an interference experiment, but the common phase was unobservable, and so it was a phase symmetry which can be called a global symmetry. And then Weyl's idea was to convert this global phase symmetry to local phase symmetry and see what happens to the theory; what are extra component which appear, and that is what I am going to construct explicitly, but we do not call this whole structure a local phase theory; we still call it a local gauge theory, and that has to do with the way history gives names to various ideas as things develop and get revised.

So, let us give some basic definitions that we want to construct a description where the phases can change from one point to another, but the overall theory does not see this particular change in the phase. So, the basic object is wave function for a particle; let us call it electron or any other charge particle will do, and let the transformation which we want to implement is this local phase transformation be specified with the object. We will need this particle to be charged because as we have seen earlier in the discussion of Klein-Gordon equation as well as Majorana fermion that the wave function become real if we have neutral particles. So, to get the phase we need a charged object and the various constant inside here are stuck up with a little bit of foresight so that the equations turn out to have the same form as we have used them before, and they are all the fundamental constants of charge speed of light and Planck’s constant, and lambda x is actually a real object which varies from one location to another.

So, we want to develop a set of equations and Hamiltonians, etcetera for this particular object. So, that will have derivatives both space and time derivatives of psi also appearing in the equations, and we would want a prescription which does not depend on this value of lambda, and to do that the derivatives also need to appear in such a way that they transform the same way as the wave function does. So, then the whole thing can be just factorized out as an overall proportionality constant, and then the theory does not care about what that proportionality constant is, because the equation of motion the right hand side will be 0, and you can multiply by any nonzero constant, and it will have the same solutions.

So, the derivatives should also transform the same way to keep the equations independent of the phase which is parameterized by this function lambda x. Now clearly an ordinary derivative does not do the same job because if when you try transforming according to this particular rule you’ll see that there is a derivative d mu which can act on lambda as well which will produce an extra term. So, d mu psi x will not be the same phase specter times d mu psi x. So, we have to define a new kind of derivative, and the name for it is covariant derivative.

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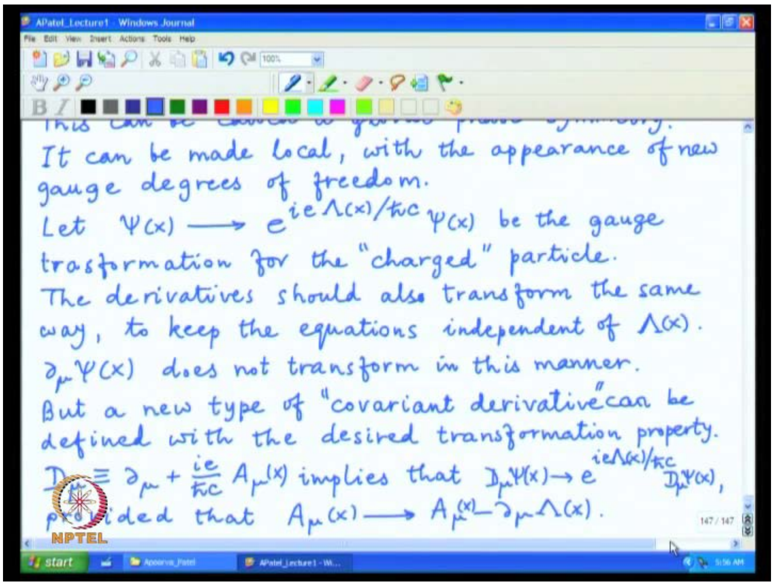

And this is rather easy to see, but I will write down the operator, and it is easier to see the consequence of it that will define a covariant derivative as an ordinary derivative plus i e by h cross c times A mu where A mu is a new object. We have stuck it in the theory at this particular stage by hand, and then we will have the property where A mu will be a function of space time as well that. So, this is a transformation property we needed. Now we can stick in the various quantity of what is going to happen to this covariant derivative, and we will see immediately that if you take psi to this particular object e raised to i then d mu will operate on it. So, there will be one term which will d mu operates on psi, but there will be a partial derivative of mu which operates on a lambda, and so this overall factor will come.

But there will be an extra term in lambda, and that term needs to be cancelled, and that can be achieved by changing this extra quantity which we have inserted also changes, and that changes exactly the one which we have seen before, and that was the reason for using all this constant in a particular way that the derivative will produce d mu lambda, and then a mu will also change by minus d mu lambda; the total result is that the whole objects transforms wide overall phase. So, once this overall phase is outside then one can now construct any complicated equations or Hamiltonians out of the wave function as well as its covariant derivates and the whole theory is invariant under this local phase symmetry.

But we have explicitly added the extra degrees of freedom; there was a phase, but now we have to add an extra gauge field A mu which also gets connected to the transformation in the phase. So, this is a whole structure of the theory; the new degrees of freedom is explicitly appearing in the theory which is this gauge field A mu, and we can now explore what all features come out of it.

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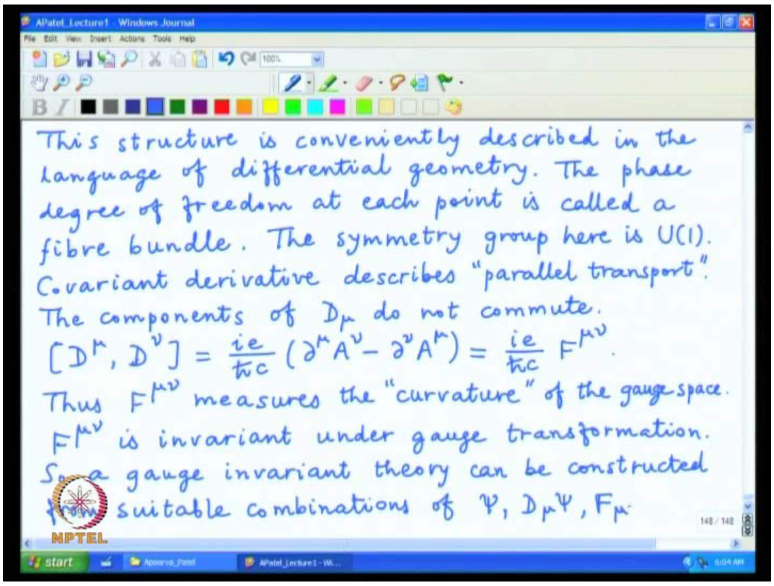

So, this structure actually described in a geometrical language which is called differential geometry. The phase is a called a fiber bundle and the group which specifies this phase is the group corresponding to rotation on a circle is just an angle which goes from 0 to 2 pi, and it is periodic. So, this is actually a much more formal and mathematically regress machinery; I am not going to use it per se, but this names again are commonly used. So, I would like to mention these particular words which often appear in many text books in describing gauge theories. Let me go ahead with the definitions which I have constructed. Now it turns out that this covariant derivative describes translations just like an ordinary derivative does.

But in undergoing this translations something simultaneously happens to the phase; the phase is changing as you go from one place to another and the label for that is parallel transport, and so it stand amount to carrying around a particular arrow pointing in the direction of the phase, and then as you move along see what happens to that particular arrow as you go around a trajectory. And some nontrivial features enter when this function lambda is not 0, okay, and those are the features which now can be constructed out of this covariant derivative, because it has all the information about how the phase is changing in going from one phase to another, and this feature can be again summarized in a simple way.

They do not commute with each other in particular one can explicitly workout this commutator, and that is i e by h cross c, because the derivatives can produce nontrivial effects on the A and which we know to be a definition of the electromagnetic field tensor. So, the field arises as a nontrivial commutator of covariant derivatives, and in the language of general relativity these non-commuting derivatives actually specify the curvature of the space, and so the field in some sense measures the curvature in this space of gauge degrees of freedom. And so we have a nonzero electromagnetic field which means that the corresponding structure in this sense of fiber bundle has a certain curvature, and that produces a nontrivial effect on the trajectories which we will now look at modified; they will not be the same as what happens in a flat space.

So, this is the way f mu nu naturally appears in this whole description, and now one can go ahead and construct whatever one wants, and in particular f mu nu again turns out to be gauge invariant in the sense that the same transformation which psi. So, the whole theory is. So, a gauge invariant theory can be constructed from suitable combinations of psi d mu psi and f mu nu, and this structure is exactly what appears in the theory of electrodynamics. But we have arrived now here in a much more geometric fashion about what all things go on as one changes location and then change the phase of the particular wave function, and here the wave function, change of phase, hands up getting reluctant to the strength of the electromagnetic field. There are other features which can also be seen which play an important role in looking at the gauge invariant objects which can be constructed, and there are many such instances I will just want to point out an important case.

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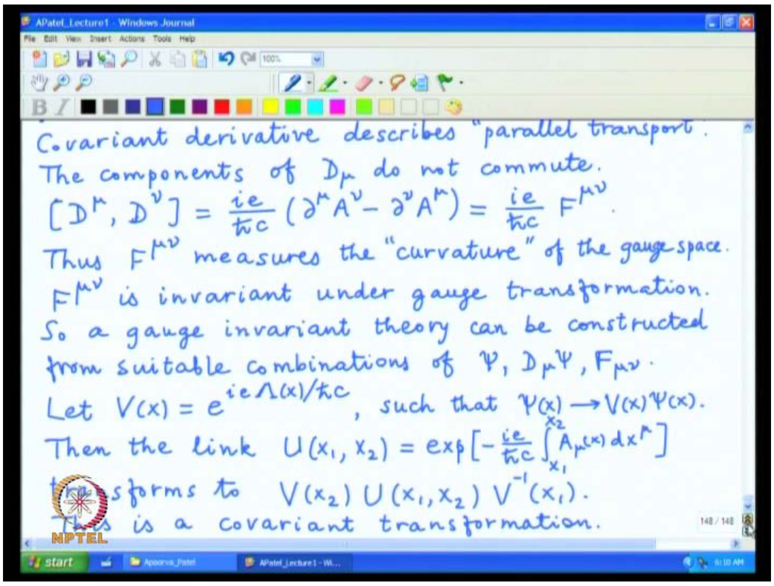

And that is let us define short form for this space time dependent phase, psi changes to, then one can construct an object which is called a link between two points U x 1 and x 2, and it is defined as exponential of minus i e h cross by c integral of A mu x d x mu integrated from x 1 to x 2. And now one can see what happens one performs the gauge transformation. A mu changes to A mu minus d mu lambda, and so d mu lambda can be exactly integrated along this curve from x 1 to x 2, and it just produces the boundary term which is value of lambda at x 2 minus value of lambda at x 1, and that can be rewritten back in terms of v of x. So, then we only have an overall constant. So, it can be written as V x 2 U x 1 x 2 and then V inverse of x 1, and this is a very useful form, because it can be maintained quite easily while multiplying such segments together.

Whenever you take one segment and multiply by another the factors of V at either end will cancel out in combining V and V inverse and then the whole segments then transform. So, this is a covariant. So, we have a specific object single point object psi transforming at V times I, and now we have a link connecting two points which transforms at U going to V times U times V inverse. And we can now multiply this objects of links together with psi and construct nontrivial products, and show that the whole object can remain the same covariant transformation property and which can ultimately be factored out to do various calculations and form equations.

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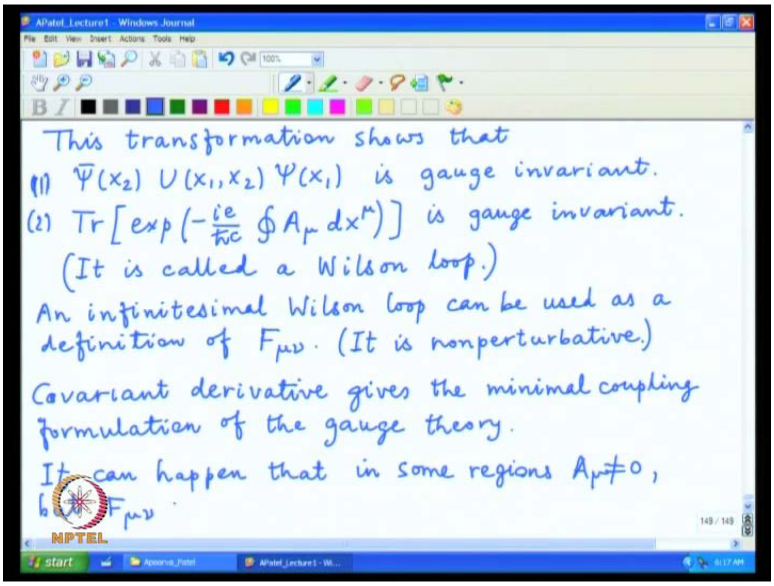

So, this transformation shows that one can have psi bar x 2 U x 1 to x 2 psi x 1 is gauge invariant. This is a useful object; it is an extended object; it is a link. There is another form which is also gauge invariant. And this is a trace of the same link but now taken around a nontrivial loop in the space, but we come back to the same starting point and so we have a V and V inverse, but both belonging to the same point. And I have written this trace as a kind of generalization of these particular analyses which is not necessary in case of electrodynamics but can be useful in other kind of gauge theory. So, when you have a trace of V U V inverse the cyclic trace basically analytes V and V inverse at either end, and you have the same object returning as trace of U, and so it is a gauge invariant object as well, and this particular object is rather heavily used in analyses of quantum field theory.

It is called a Wilson loop. In particular Wilson used this object heavily in studying a non-perturbative formulation of gauge field theories. So, this is also a useful feature of how we construct gauge invariant objects out of this various species, but this geometric language and how things transform is extremely useful in this whole context. One another things one can notice in this structure of the Wilson loop, one can use an infinite decimal Wilson loop as a definition of the field strength itself, because this is a cyclic integral one can easily apply stokes theorem to it. So, then it becomes an integral of f mu nu over the area; the area can be made infinite decimal. So, the integral will be just exponent of f mu nu times the little area f mu nu roughly constant over the infinite decimal area.

And from that one can deduce whether there is a nontrivial electromagnetic field in that particular region or not, and this is a kind of gauge invariant definition of what this field is, and one does not have to refer to any particular strength. This is actually what can be referred to a non-perturbative definition. We do not assume anything about how large or small the magnitude of the gauge field is, and so these are useful concepts which come out from this description of the local symmetry as a gauge theory concept. There are some other things which are also useful to see that; in this description the A mu becomes a fundamental variable, f mu nu is constructed as a subsidiary object or equivalently the covariant derivative is a powerful element and everything comes out from the derivative.

So, the covariant derivative actually gives what is known as the minimal coupling of the gauge theory. We have used it rather empirically earlier where p mu was just substituted by p mu minus e time A mu, but looking at it from this framework of covariant derivative we know exactly where it comes from. It is a consequence of the particular phase symmetry structure and its corresponding fiber bundle. So, that is one concept, and it can happen that this gauge field is nonzero, but the field strength is zero. This is not surprising because F is a curl of A mu nu, just means that we have a curl free gauge transformation and in a corresponding gauge field.

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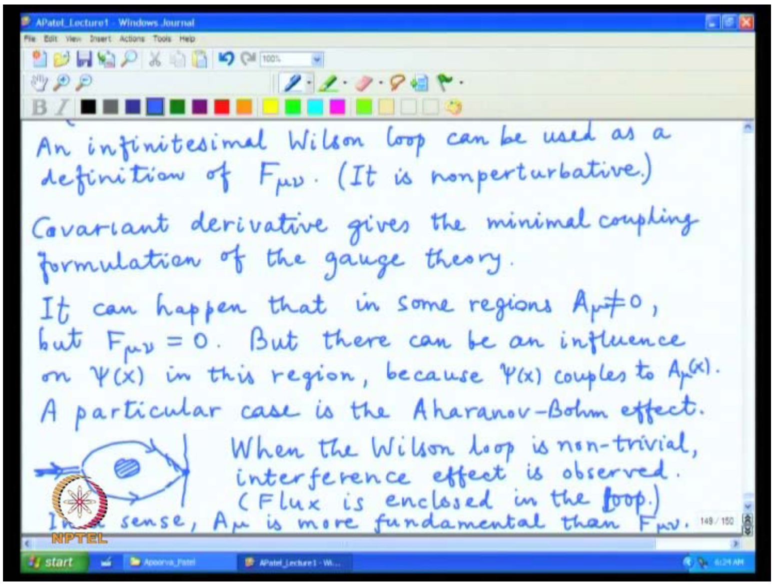

And in this case there is a question psi x directly couples to A mu through the covariant derivative, and if A mu is nonzero there can be situations where behavior of psi actually changes even though F mu nu is zero, and this peculiarity is a famous Aharanov-Bohm effect where one has a setup where the electromagnetic field F mu nu is 0. But A mu is not in this region; the charge particle passes through, and it ends up changing its phase which can be detected by an interference experiment on the other side of the region. So, one can have a situation which essentially giving accrued diagrammatic representation at one sends in a beam of particle which splits and then rejoins to produce a interference effect on the screen or observable on the either side. But the pass can be such that at nowhere in the regions of path there is a nontrivial F mu nu.

But one can cook up a situation such that the phases corresponding to the two effects are different that can happen because the Wilson loop which corresponds to this whole loop of trajectory can have a nontrivial value, and we have seen that because of stokes theorem it means that F mu nu has to be 0 at some point in the trajectory, but in the centre of the loop where the particle does not go through, it can be nonzero and then the Wilson loop can have a nontrivial value, and so one has an interference effect with a certain phase change. So, one can have a situation of two different setups; in one case there is a electromagnetic field inside the loop, and another case there is not, and comparison shows whether the introduction of the electromagnetic field in the loop changes the interference pattern or not in particular shift sit in one direction or the other if that does happen, then certainly the electromagnetic field produce a nontrivial effect.

And that can be paraphrased as flux is enclosed inside the loop, and so one can observe this peculiar feature and the various concept which we have introduced in this rather nontrivial experiment, and it is certainly experimentally observed, and in that particular sense in a sense A mu is more fundamental than F mu nu, because one sees the effect of A mu directly on the wave function even when F mu nu is absent. So, these are all the useful geometric concepts connected with gauge field and its symmetry properties, and now we will go on to the next stage trying to couple it with electromagnetic interactions of Dirac particles.