Relativistic Quantum Mechanics
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Lecture - 30
Photons and the gauge symmetry

So far in course we have seen how relativistic charge particles respond to a background electromagnetic field. And we use the well known minimal coupling prescription to incorporate that where t mu was substituted by p mu minus e times a mu to carry out all the manipulations as well as detail calculations in presence of electromagnetic field. Now, will look at the opposite system where the electromagnetic field will be considered as evolving in presence of background of charge densities and currents.

And this charge densities and currents are provided by various particles and their distributions in space time. And we will see how the electromagnetic field responds to this distribution of currents; and after having done that will be in a position to put both things together. So, that charge particles and electromagnetic field mutually evolved; according to their interaction both at the same time in a dynamical system which will be the complete theory of quantum electro dynamics.

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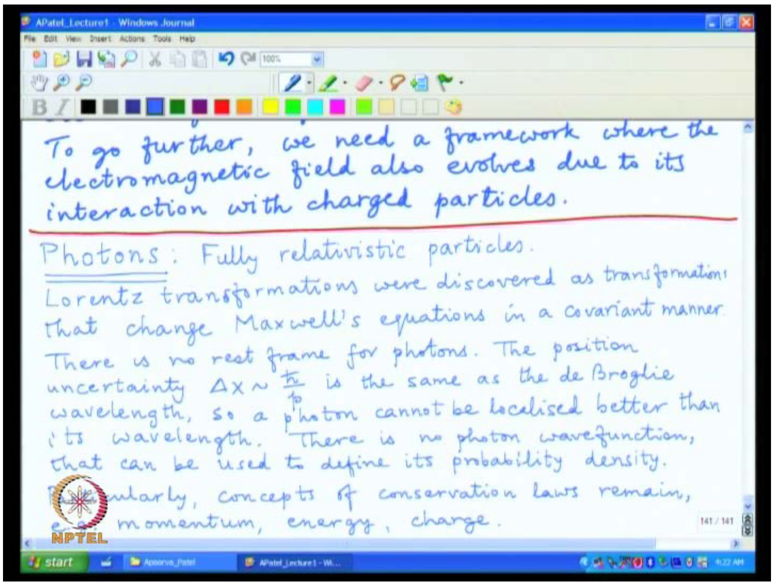

So, the first is right now is to discuss the systems of photons which are the units of electromagnetic field. And how they respond to the distribution of charges provided by various particles. Now, one thing is very obvious right at the outside; that this photons are relativistic particles to begin with there is no such thing as non relativistic photons. So, all the consequences of Lorentz symmetry which we discussed in case of particles such that electrons or neutrinos or any other similar objects; those are automatically satisfies by when construction starts discussing with photons.

So, these are objects which will automatically described the Lorentz properties from its definitions; we do not have to worry about making the theory Lorentz covariant. And the matter of fact that the starting point of relativity was the structure of Maxwell's equations; in the sense the Lorentz transformations where discovered or constructed which ever you want to look at it. So, that they described Maxwell's equations in any arbitrary frame in a covariant fashion.

So, if you use the neutrino or Galion transformations they do not leave Maxwell's equations in variant. And so one had to invent new transformations has a correct description of the symmetry of Maxwell's equations. A Lorentz indeed wrote them down and it was later that Einstein reformulated Lorentz transformation from a different perspective of what structure of base time is. So, we have the Lorentz symmetry built in the Maxwell's equation themselves. And now we have to look at the consequences of what all comes out of it; and that is a atom to convert Maxwell's equation to a quantum language. And various things are rather straight forward; so I can just mention them. So, there is no rest frame for photons the position uncertainty; which is delta x is the same as the de Broglie.

So, photon cannot be localized better than its wavelength also there is no photon wave function that can be used. So, all this features are actually common in describing Schrodinger equations and non relativistic atomic physics. But they do not make any sense; in case of photons because photons do not have a non relativistic description. So, we have to forget those words in some sense directly go to a language of what is known as a field theory; where the concept of particles is taken over by concepts of fields; where various things are not necessarily conserved in the same sense as they are in the case of particles we saw some of those features while dealing with Dirac equation in case of uncertainties. And creation of particle, anti particle pairs similar concepts have to be used in describing the structure of what electromagnetic field is and how it is decomposed in case of photons; there are various objects which are conserved.

So, particularly concepts of any conservation law remain as there are even though the language changes. For example, we can talk about momentum conservation, we can talk about energy conservation, we can talk about charge conservation all those things are fine. But you cannot relate it to wave functions and probability density in the same sense as is common in case of Schrodinger equation. So, this is the feature one has to keep in mind; there is no non relativistic limit in this case. But there is a different limit which is usually taken in describing electromagnetic field.

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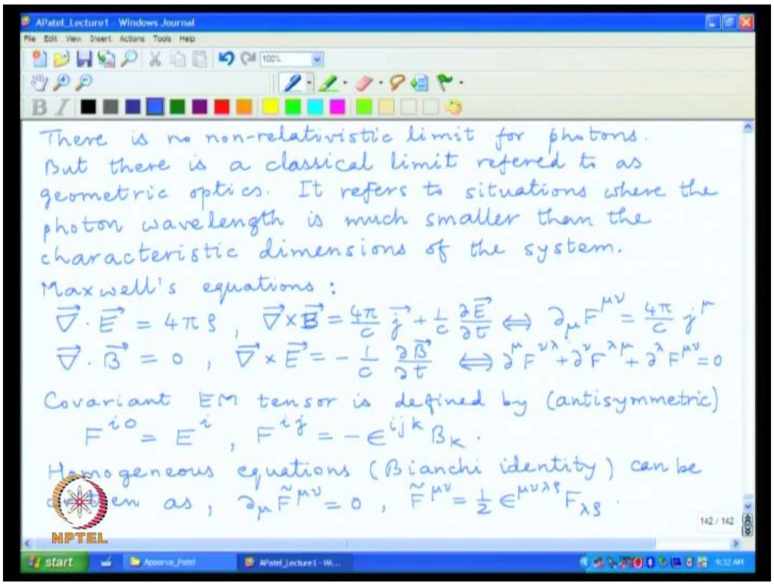

And, that limit is a called a classical limit which has been given another name as well which is geometric optics. It refers to situation where the photon wave length is much smaller than the characteristics dimension of the system. And in that case one can forget about all the quantum uncertainties inherent in the descriptions of photons. Because they are not of any significant magnitude and one gets a classical picture; it sometimes refer to as geometric optics, sometime it is also referred to as ray optics. And this is description which is heavily used in all the phenomenon one learns first in a optics such as interference, diffraction, double slit experiments and so on and so forth.

So, this limit does F exist; and it is basically specified by criterion that the photon wavelength mush be smaller than the dimensions over which the propagation or evaluation of electromagnetic field is taken into account. So, that is a possible and it is useful also in many situations. But we are not going to talk much about this geometric optics; we are really interested in the quantum features of the electromagnetic field. So, now let us go back and write down the Maxwell's equations explicitly. And I will use the Gaussian unit as they called not the SI units; which are another set of popular unit; it just different change of how normalizations are taken in various quantities in particular this symbols of epsilon 0 and mu 0. And one can choose one convention or other as long as it has been done consistently.

So, there are 4 Maxwell's equations; 2 of them are in homogeneous and in homogeneity comes from the background distribution of charges and currents. And the other 2 are homogenous mistake this has to be curl of p. And this equations are already in a form which can be written in a Lorentz covariant structure very easily; and that requires definition of electromagnetic field tensor. So, it is an anti symmetry to index object and i and 0 are space and time indices in the Lorentz structure. And we will define F i 0 as E i and F i j is equal to minus epsilon i j k B k by construction this tensor is anti symmetric in its 2 Lorentz indices. And with this definition this equations are converted into a simpler looking form; where the space and time derivatives and it can combined and in terms of covariant derivative. And the first equation becomes d mu F mu nu is equal to 4 pi c times j mu; where j mu as the time component as the density rho and space component as the current vector j.

So, that is much simpler looking form and its automatically in a Lorentz covariant structure; while the second equation looks little more complicated. But it also has a simpler form in terms of derivative acting on F. But now the 3 indices and 3 indices are permuted in a cyclic fashion and once that is done the total result is 0. So, that is the covariant form of Maxwell's equation where we make use of a very specific construction. And that simplifies the structure; the homogenous equations are also sometimes referred to as a Bianchi identities. And structure then has a form very similar to the in homogenous part.

And, so I will write it in almost similar language. But now instead of F mu nu have a F delta mu nu and F delta mu nu known to be the dual tensor of F mu nu defined explicitly by the relation involving the anti symmetric epsilon symbol; which is F delta is half epsilon F with the indices automatically contracted in appropriate fashion. So, with this notations Maxwell's equations rather look very simple in form it is just divergence of F in one case where it is a electromagnetic tensor is j. And when it is a dual tensor gives 0; there are identities automatically buried inside this structure by the property that F is anti symmetric.

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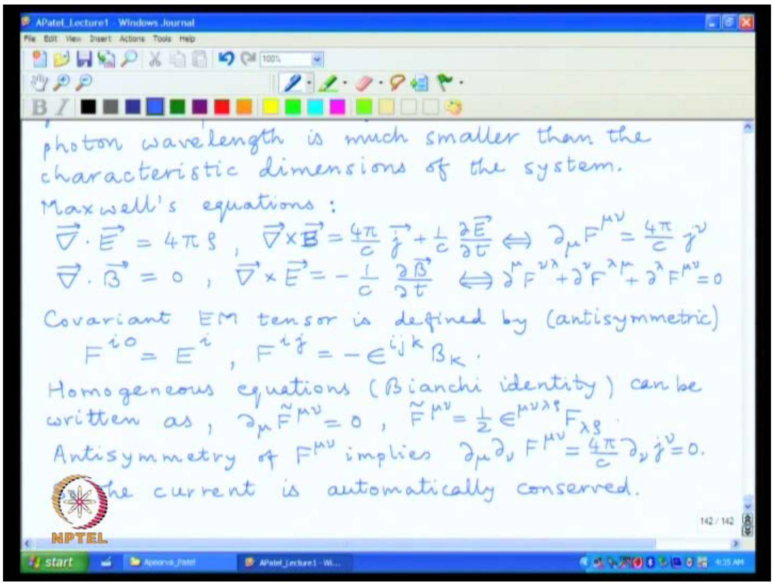

So, if I take one more derivative of this in homogenous Maxwell's equations. So, it is now instead of one derivative over there. Now, I will take derivative with respect both the indices the equations says that this is again let me correct this mistake it is equal to divergence of j. But since the 2 derivates acting are always symmetric they commute to each other F is anti symmetric; and so this divergences as to vanish. So, this is conservation law the current which couples to electromagnetic field described by Maxwell's equations is automatically conserved. And so we have various nice relations already built in from the basic structure of the whole electromagnetic anti symmetric tensor. And that turns out to be very useful both in discussing Lorentz property as well as other symmetries property as well as conversations laws. So, this is a structure and now let us proceed a little further and try to see what are the solutions that emerge from this equations.

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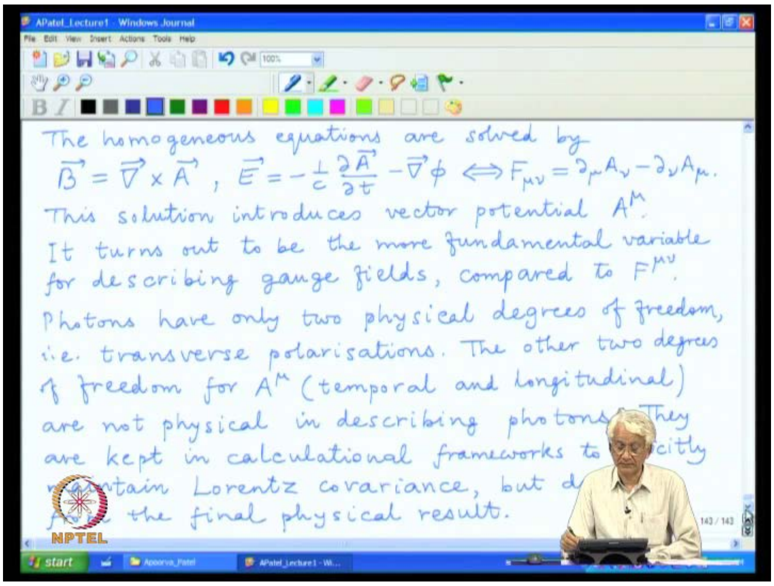

So, there is a very simple structure which is dictated by the Binachi identities; that one can find an exact solutions of those identities. It just says that the dual tensor has to be have in a sub particular way or equivalently the homogenous Maxwell's equations can be solved in a particular way. We know this result from the vector calculus the divergence be equal to 0 is solved by exactly the definition is curl of A. And the similarly the curl of E being proportional to del b by del t is exactly solved with the first relation between B and A; by the construction that E is minus 1 by c del A by del t minus gradient of a scalar phi. And this again can be converted into covariant language in terms of the field strength. And it has a very suggestive looking form that F mu nu is d mu A mu minus d nu A mu. So, it is generalization of the curl, but now will 4 dimensional space time.

And, this automatically also include the anti symmetry properties of F mu nu and with this definition we have solve half of the Maxwell's equation. And one can now go and plug this back in into the other half of the Maxwell's equations the in homogenous one; and see what comes out that particular result. So, this is a very straight forward formulation which now introduces this so called vector potential A mu; we have used this vector potential already in dealing with Dirac equation in converting ordinary derivative to covariant derivatives. But here one can see appearing from the definition of the electromagnetic field. And one can wonder a little bit whether this vector potential is standard frame work or a the electromagnetic field F mu is the standard frame work which one is more fundamental and which one is not.

And, it turns out that A mu is a more fundamental for describing what are known as gauge fields; electromagnetic field is an example of gauge fields compared to the fields F mu nu and there is a reason for it. But there is also extra thing one has to tackle with this mu feature. And that has to do with degrees of freedom involved in the descriptions of the gauge field; in the language of Maxwell's equation there were 6 components 3 for electric field and 3 for magnetic field. Now, we have rewritten those objects in terms of these vector potential as in object with 4 components. So, how many degrees of freedom are really there in describing the whole system? And one has to keep track of which degrees of freedom are physical and which are just auxiliary concepts needed for writing the mathematics in a simpler language.

And, it turns out to be that photons have only 2 physical degrees of freedom and they are known as the 2 directions of transfers polarizations. So, all the extra indices which are floating around they are all in some sense subsidiary to the fundamental degrees of freedom. And once the mathematics is through they must kind of disappear in one form or the other; F mu nu as we have seen anti symmetry tensor in 3 indices or it had 6 degrees of freedom by imposing the solutions of the homogenous equation we cut down this 6 degrees of freedom into 4 which now parameterize A mu. But even out of this 4 degree of freedom is A mu only 2 physicals which are the transpose polarizations. And the other 2 degree of freedom in particular we call them temporal and longitudinal are not physical in describing photons. But yet what we will do is keep all the 4 degrees of freedom around; so the all these temporal longitudinal components will be there in the explicit equations and calculations.

That is useful in maintaining the whole calculation in a Lorentz covariant form. Because if you projected it out directly the transpose degree of freedom the Lorentz covariant automatically gets reduced. So, we will keep the extra degrees of freedom; but at the end of the calculation whatever extra degrees of freedom produces essentially cancels out; and becomes 0. And only the contribution of corresponding to the 2 physical degrees of freedom survives.

So, and this turns out to be very important check of the whole frame work; that you calculates various things you have physical degrees of freedom and some auxiliary. And un physical variables also floating around at the end of it all the unphysical part basically goes away it contributes 0; and only the physical objects survives. But this is very deep principle it is not just mathematical tricks.

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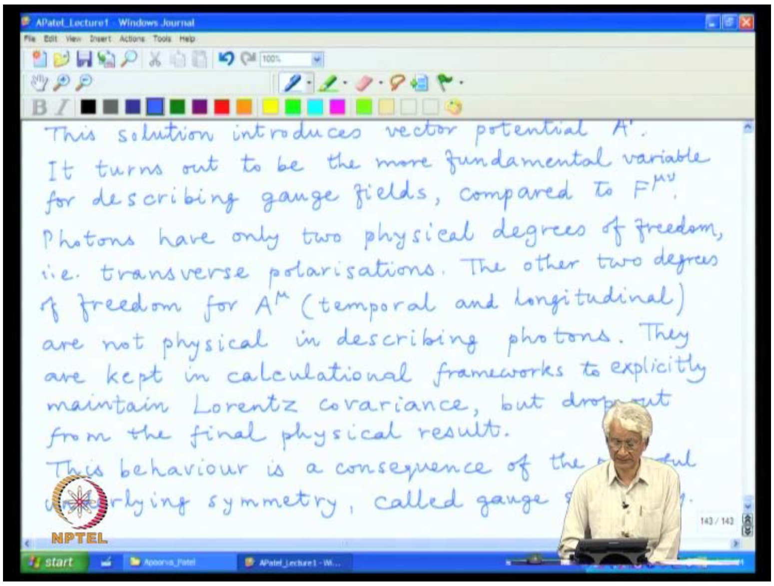

And, this behavior is a consequence of a powerful under lying symmetry which is called gauge symmetry. So, this description is not just a mathematical fluke but there is a principle involving in the calculation. And that principle plays a very important role in descriptions of electromagnetic fields in particular; and many gauge theories in general. So, one now have another symmetry to worry about in addition to the Lorentz symmetry in case of photons the Lorentz symmetry part was almost automatic.

Because it started out with the fully relativistic system and Maxwell's equation already had that symmetry built in. But this gauge symmetry is a second symmetry which now shows up; and it has very specific consequences which are not. So, straight forward to see from equations themselves; but by introducing a little bit of extra frame work those features appear. And they play a very important role in the description of physics in particular how many degrees of freedom are physical? And how many degrees of freedom are kind of dummy variable just taken around the right; and ultimately not contributing to the final result.

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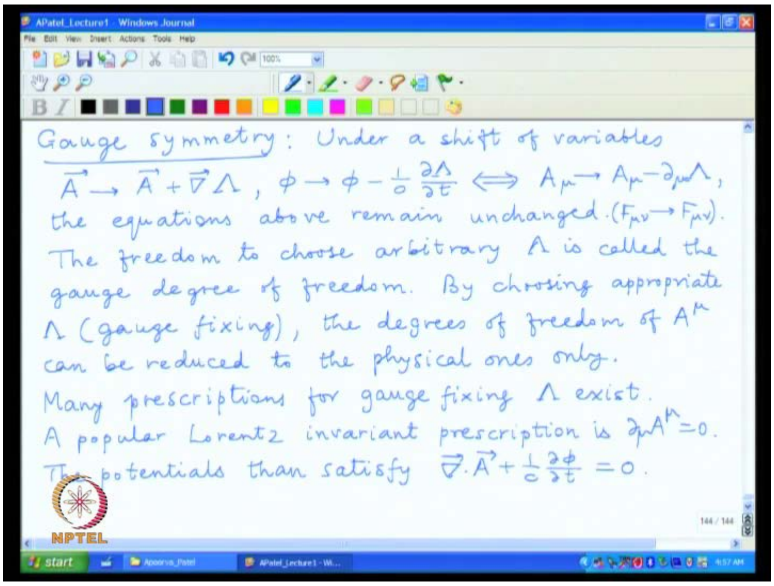

So, now let us see this gauge symmetry in a more quantitative fashion; remember we introduce this auxiliary field the vector field A mu has a solution of the homogenous Maxwell’s equation; where F was just the curl of A that was the only place where it appeared. And it is very easy to see that all that algebra remain on altered; when the variables are shifted according to the prescription that A is shifted by gradient of A arbitrary fashion of lambda. And the scalar component is simultaneously shifted by time derivative of the same function lambda; and the Lorentz covariant form of this object is A mu goes to A mu minus d mu lambda.

So, all the equations which introduce A mu remain unchanged explicitly because well we have to take a curl of A to get F. But curl of this gradient is automatically 0 it is a mathematical identity. So, one can say that under this change F just goes back to itself and so nothing happens to the Maxwell’s equations at all. So, this particular feature brings out that this vector potential which we introduced is not uniquely defined; there is a arbitrary degree of freedom here labeled just lambda which can be used to change it in many different ways without any consequences. And this freedom to choose and that tells us that component of A are not physical there are these unnecessary things introduced by lambda. And we can choose lambda cleverly to remove some of those extra degrees of freedom. And that indeed turns out to be true by choosing appropriate fashion lambda which sometimes also referred to as fixing of a gauge; the degrees of freedom of A mu can be reduced to the physical ones only.

So, indeed it is true that by suitable choice of lambda all the extra degrees of freedom can be made to disappear. And then will have only two transfers polarizations of photon left which are physical. And the other staff is only a convenience certainly there is no unique way to say that I will choice one particular lambda. And it will be better in some sense or other and there are many different prescriptions for this gauge fixing lambda exist.

And, one prescription may be convenient in one particular circumference and another prescription may be convenient in another circumference that is delta case by case basis. But again there are certain prescriptions which are popular and they are popular; because of the property of Lorentz covariant which is built in inside the formulations we have discussed so far. So, it says that divergence of A is equal to 0 and that is automatically Lorentz in variant by construction. And the potentials than satisfy divergence of A plus 1 by c del phi by del t is equal to 0; which is in the space and time degrees of freedom written independently compare to the covariant language.

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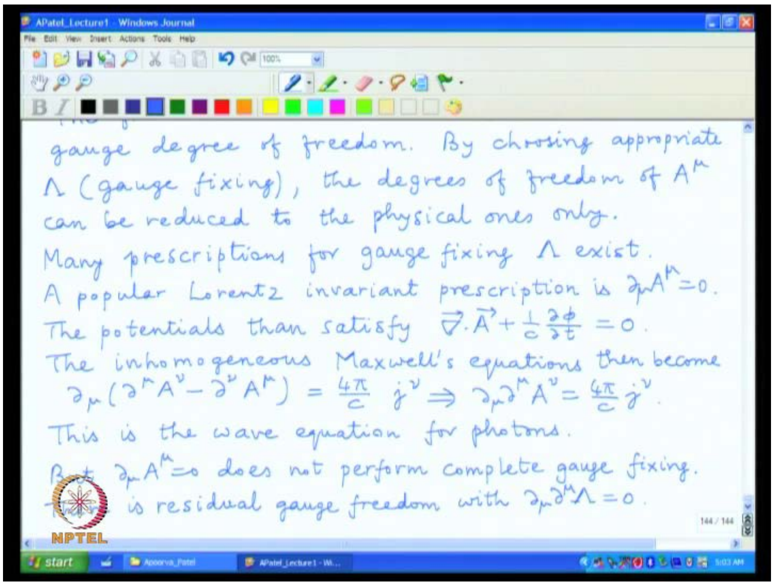

And, then the so called in homogenous Maxwell's they had 2 terms in terms of the equation. But now I will write those things explicitly d mu of F mu nu is equal to 4 pi by c j nu. But now this d mu acting on A mu the derivatives commute and that part is 0 by the so called Lorentz gauge fixing conditions. So, out of the 2 terms only the first one is left and we have result which is the standard form of wave equation; and c is the velocity of propagation for the particular wave. And that is the way we are happily looking at the electromagnetic field as wave and it has a particular propagation speed. And it automatically follows all the super position principle interference, diffraction kind of phenomena; and current basically becomes the source which produces the wave.

So, this is the so called popular form of gauge fixing; the Lorentz condition it produces this wave equation. But even inside this wave equation we have certain degrees of freedom which are not completely fixed. And that is because that condition Lorentz condition it does not fix the gauge completely. Because again one can ask what happens when a mu goes to A mu minus d mu lambda what happens to this particular condition. And one can easily see that this condition gives a extra degrees of freedom still remaining; which I will call residual which can be written as the wave operator acting on lambda a equal to 0.

So, if I take a lambda which is solution of wave equation and change A mu by gauge transformation involving that particular lambda; nothing will change in the wave equation as well. So, A is still determinant up to the solution of this homogenous wave equation. And so there is something more which can be removed as a matter of fact; when we impose one condition of the Lorentz gauge fixing out of the 4 degrees of freedom essentially removed one part which can be called the longitudinal part if you wish.

Because its divergence of A which is removed but using this residual degree of freedom; we can still remove one more component which in this particular case can be called scalar component. Because a wave operator is acting on it and it gives 0. And when one removes both these components form a mu the 4 degrees of freedom reduced to 2. And then it gets down to the only the 2 physical degrees of freedom described by the transpose polarize. So, we can now choice another solution to fix the gauge completely; this operator is already complete scalar. So, there is no way we can do the complete gauge fixing in a Lorentz covariant fashion. And one has to now break the Lorentz symmetry explicitly to do complete gauge fixing.

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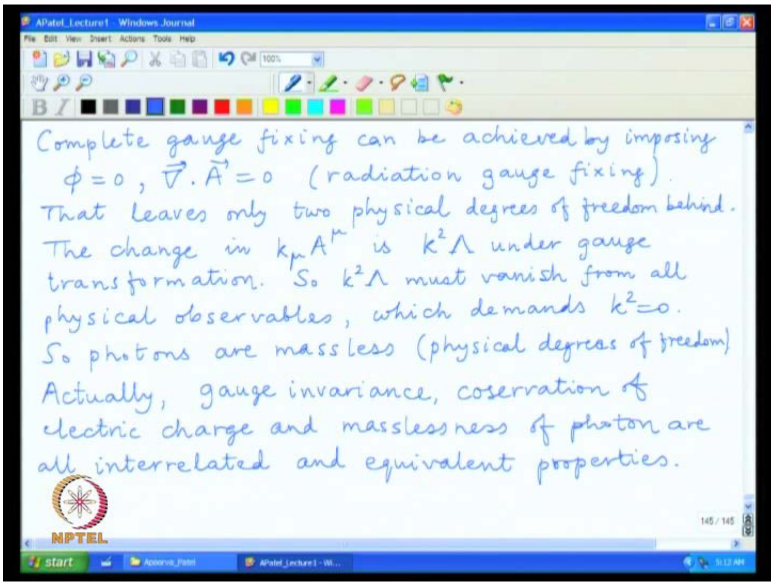

So, we will now put the 2 conditions explicitly the sum of these 2 conditions gives the Lorentz gauge fixing automatically. Because it is a 4 divergence and the time and space components mutually satisfied. But in addition to that there is a separate condition for the time and the space. And now if you choose this particular conditions you will see that you made a specific choice for a lambda there are 2 separate equations to be imposed. And sometimes this choice is referred to as a radiation gauge fixing; and it removes both the temporal and longitudinal components of A.

And, the two transfers degrees of freedom which have said at the beginning other physical ones they are the one left behind. And that is a example of complete gauge fixing to get down all the way to the physical picture; one does not do that very often all the way down to the physical degree of freedom. But the one step earlier which imposes the Lorentz gauge fixing condition that is used very often in calculation. Because it does not destroy Lorentz invariance of the analysis. And in that particular case there is automatic cancelation which comes out of the result between the temporal and longitudinal degrees of freedom.

And, that cancelation is nature of the uncultured degree freedom dropping out of the final result. One can see several other things also of in this radiation gauge which are useful in describing the physical degrees of freedom; how it behave? And one can look at them of spatial cases; if one look at the change in the object which I will right in momentum space in K mu A mu. And because A mu goes to A mu minus del mu lambda and d mu is essentially gives k mu after factor of I; the change in this particular quantity is K square times lambda under gauge transformation.

And, no we have the gauge fixing condition where d mu A mu is the same object came A mu d mu. So, not only this object is 0 but any further change in it necessitated by a further gauge transformation must also be 0. Because it is true in arbitrary frame or arbitrary gauge transformations both ways. And so K square lambda must vanish because d mu A mu can appear in lots of calculations without any trouble; and the lambda happens to be arbitrary functions. And only way when one can satisfy and that condition is satisfying that K square is equal to 0; this is a indirect way of saying that we have a wave equation which came out by this particular gauge fixing condition. And the wave equation was describing a Mass less particle it had the desperation relation such that the 4 momentum K square is equal to 0.

So, one can see that photons are Mass less degrees of freedom when reduce down to physical components; and they satisfy the Mass less equations. And this is a relation between the gauge symmetry and gauge choice or gauge degrees of the freedom and mass lessness of photons. And we have also seen in the Maxwell’s equations that the same wave equation was of structure which automatically satisfied conservation of charge. So, there is a very tight relationship between this various quantities; gauge invariance, conservation of electric charge and mass lessness of photon are all interrelated properties. And in some sense they are also equivalent in the sense that one can derive one of them are given the other; and this is a dip principles which will 3 examples of as the course progresses.