Relativistic Quantum Mechanics
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Lecture - 29
Trace theorems for products of Dirac matrices

In this particular lecture, I am going to describe algebraic techniques to simplify structures involving products of gamma matrices; this is an extremely useful part of peturbative calculations in field theories, because the setting which we saw in the previous lecture naturally produces this kind of products of gamma matrices. And the efficient way to simplify this products give raised quick way of doing those calculations.

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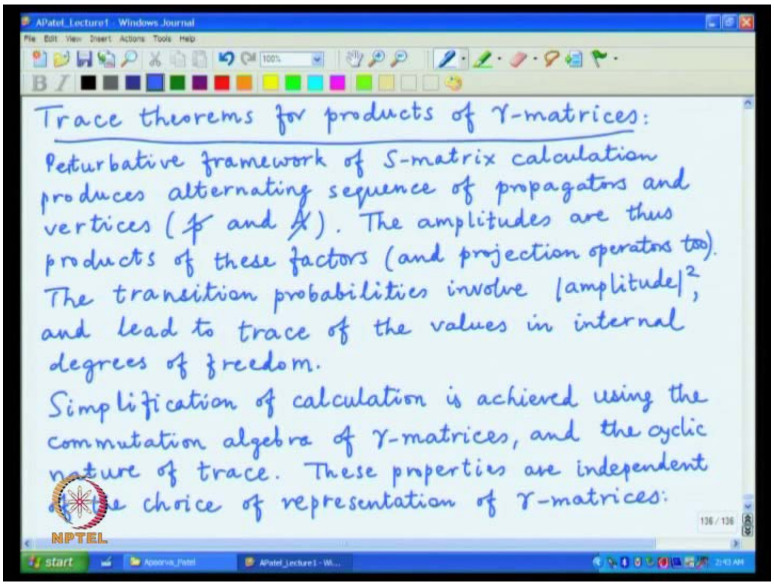

This objects I will just call as trace theorems will see by trace is appear; the S matrix frame work in the peturbative calculation produces alternating sequence of propagators. And vertices typically these are the factors involving p slash or propagators. And A slash for the vertices; this whole string of factors coming in this gamma matrix language needs to be simplified to get these answers reduced to numbers. The amplitudes are thus products this factors together with certain projections operators; which might comes from specific boundary conditions or measurement operators. But we have seen how to express the projections operators also in this gamma matrix language; the probabilities will involve the modulus square of these particular amplitudes. And then when we multiplied these matrices and take modulus square that lead to trace of the values in internal degree.

So, this is how will end up getting traces of these products of gamma matrices; we will see explicit examples of these things in several cases. But this is the generic reason behind it. And so we need techniques to multiply gamma matrices with each other. And then evaluate the traces of these products. Now, this can be done in a rather general way based on various identities satisfied by gamma matrices; the simplification of calculation is achieved using the commutation algebra of the gamma matrices. And the cyclic nature of trace by that I mean that the trace of product of matrices is in variant under cyclic permutation.

So, we can evaluate the products in many different ways depending on the convenience. One should note that these properties are independent of choice of representation gamma matrices in particular to calculate. This various transition probabilities we do not have to make a specific choice of the Dirac matrices a specific choice is only useful when you are using certain projections of the amplitudes. And that is use most of the time when you want to write down the wave function itself and how it evolves. But in calculation of the transition probabilities all those wave functions are contracted with initial and final states and everything is reduced to numbers. And it is not a particular element of this internal Dirac space.

So, the calculation can be done rather straight forwardly without using the specific basis that is extremely helpful. Because you can ultimately check the Lorentz covariant nature of these expressions or other easily if you choice a particular representation. Then, the Lorentz covariant properties are not so manifest. So, the results are automatically Lorentz covariant.

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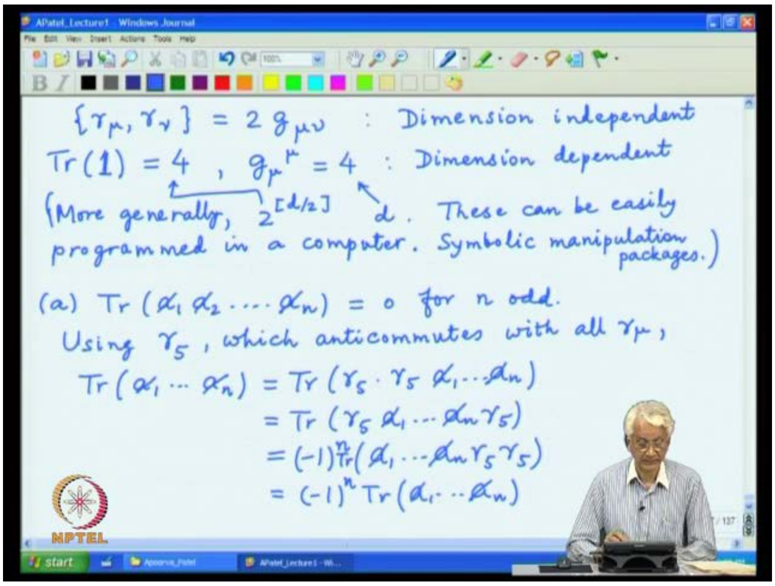

Let us write down the various rules involved one of them is this Clifford algebra relation. And I want to point out that this relation is dimension independent on the other hand there are relations involving the dimensionality of the space explicitly; in particular trace of the identity matrix in this Dirac space is equal to 4. Because there are 4 components of the Spinor also the Lorentz matrix contracted with itself is 4 as well. Because there are 4 space time dimensions. But both this quantities are dimension dependent; in general this object is a the number of the Dirac components; which is 2 rest to d by 2 in d space dimension. And this object is nothing but d; it happens that d equal to 4 this 2 numbers coincide. But in different number of dimensions they do not have to coincide.

And, then one can write down more general expressions; where this factors are 4 are replaced by dimension independent quantities. It is useful to keep in mind what kind of identities are dimension independent and what kind of identities are dimension dependent. Because one the well known techniques in doing this calculations in field theory involves analytic continuation in the number of the dimensions. Then, one has to use suitable expressions which are valid for d equal to 4, but also in d different then 4. And then has to keep track of where the Spinor traces taken where the Lorentz indices are contracted and work with the algebra appropriately, all these calculations can be easily programmed in a computer.

And, there are well known software packages which do such algebra manipulations of type. I am going to illustrate if you just type in the various products which you want to evaluate in the computer does the calculation and gives the answer; that also has become popular. Because many times algebra is lengthy and you want to get the result fast. And also make sure that you do not make any mistakes in some factors of 2 s or 4 s and etcetera. And the computer provides the good chart, but when has to keep in mind. Sometimes, the computers produce their own ambiguous result. And then one has to go back to the theory to figure out what is the part which is not well defined. And do the calculation correctly; these are all the symbolic manipulation packages.

Let me now go to the various useful expressions. The first one I want to mention is a trace of a product of operators all of them involving a 1 slash up to a n slash is equal to 0 for n odd. So, one has to worry about only the products where the number of factors is even the proof of this result is easily obtained by using the properties of the matrix called gamma 5. This trace can also be written as trace of gamma 5 gamma 5 a 1 slash to a n slash gamma 5 happens to be one the cyclic property of the trace allows it.

To rewrite this object as this then one can anti commute the gamma 5 through the whole string of a slash is and every time you anti commute you get a factor of minus 1. So, total object gives gamma 5 again collected at the end. And since gamma 5 square is one it can be easily removed I forgot that trace your trace is equal to minus 1 rest to n times the same expression. So, it can be only non 0 when n is even.

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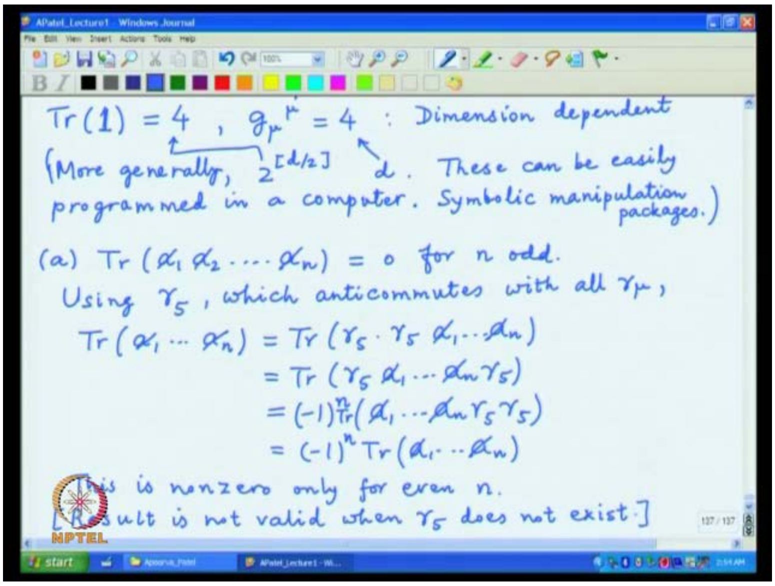

These are rather state forward manipulations, but what has gone inside this proof is the fact that we used gamma 5; once the gamma 5 does not exist which happens in dimensions not equal to 4; this proof will fail. And this kind of identities may not necessarily work. So, we should take caution that this result is not valid when gamma 5 does not exist being calculation of d not equal to 4. You cannot just throughout all the odd products of gamma matrices from trace is to evaluate them again by certain manipulations. And do the calculations with little more care the computers can be programmed to do those kinds of calculations as well. Let us look at another little example which is what do the products of even matrices.

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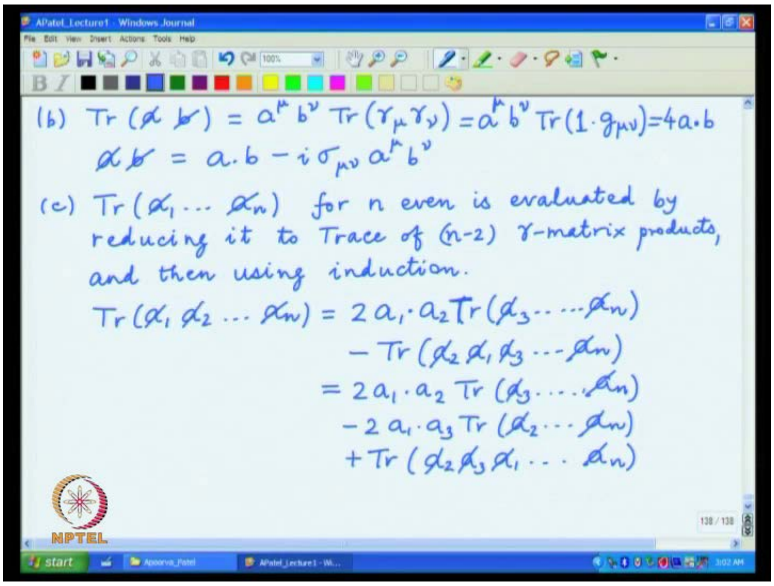

Now, look like simplest case is actually the identity, but the identity traces a 4 as we have already seen. The next one is the product 2 gamma matrices; this can be written as a mu b nu trace of gamma mu gamma nu. But this object is symmetric, because the trace is a cyclic so it can get in either order. And so the trace gamma mu gamma nu is nothing but to be precise. I should put these indices in a proper up and down location trace of identity times g mu nu. You are seeing the trace of identity is 4; and a mu b mu multiplied by with this Lorentz matrix just produces the dot product of the 2 vectors this whole object is equal to 4 times a dot b.

So, this is a typical examples of how various products are manipulated. Now, what we end up doing is larger calculations are nothing but repetitions of this identities involving products of gamma matrices and anti commutation relations; one can even evaluate the more general structure without taking a trace where one keeps both the symmetric and anti symmetric part. So, the symmetric part is what we evaluated over here anyway. And a anti symmetric part of gamma matrices produces this sigma mu nu matrix; which we have used many times in earlier discussions of a spin and Lorentz transformations this is again a very simple and useful identity.

Now, comes the extensions of this identity to larger products. That is the form which is most useful and most frequently included in the calculation. That is generically now just written as what happens to this string a 1 slash up to a n slash; how do you evaluate it?

The procedure is to use this identity involving 2 gamma matrices to reduce this trace of n objects to trace of n minus objects. And then use it by induction all the way down till one gets down to product of only 2 matrices where the previous identity can be used.

So, let us generically work out this induction step the first point is just to interchange this to objects and see what arises. One can now write trace of a 1 slash a 2 slash to a n slash. And the Clifford algebra can now give the result what happens; when you permits this a 1 and a 2 it has a 2 type g mu nu part with produces two a 1 dot a 2. And then trace of the remaining part which now can be a 3 slash up to a n slash. And then the term containing gamma matrices and the opposite order which gives a minus sign contribution with a 2 slash a 1 slash and then a 3 slash all the way to a n slash.

So, this procedure is just one use of the Clifford algebra relation. But now we can repeat it in a simple manner first we commutated a 1 through a 2 now a 1 is next 2 a 3 we will commutate it through a 3. So, will get 2 a 1 dot a 2 trace a 3 slash a n slash. As in the first line as in the second line will give now 2 times a 1 dotted with a 3. And then the what remains is the all the factors which are not equal to a 1 slash and a n slash inside here. And now the sign again flips there is a plus trace of a 2 slash a 3 slash a 1 slash a n slash. So, the signs alternates, because that is the anti commutation rule inside the Clifford algebra. You use it once you get a negative sign you use it twice you get back to the positive sign. So, this now can be repeated every time get in this two a 1 dot a 2 etcetera.

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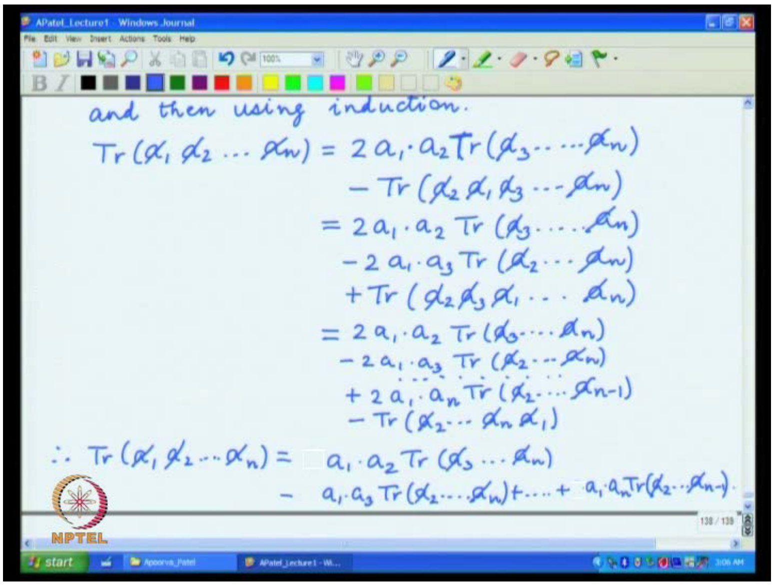

The next one will have a minus sign and then the next term which will be a 1 dot a 4 will have a positive sign. And one can continue all the way till a 1 becomes next to a n. And one should now keep in mind that this product is evaluated only when and was even. So, whenever a 1 becomes next to an even number the sign is plus the same as whatever there in a 1 dot a 2. So, plus 2 a 1 dot a n trace of a 2 slash all the way till a n minus 1 slash. And then the last object will have a minus sign trace a 2 slash all the way till a n slash and then a 1 slash at the end. This last object by cycle city trace is nothing but trace of a 1 slash a 2 slash all the way up to a n slash which is what was there on the left hand is begin with it.

So, you can shift with it to the left hand side it becomes just twice the contribution and the two can be divided out from the whole calculation. So, we have a final result which is or a product of a even number; just write all the various dot products of a 1 with all the other matrices one by one. And with alternating signs and that is final result a very useful identity, because n got reduced to n minus 2, there are many such contributions on the right hand side in particular there are n minus 1 of them. But which one has them are the shorter product of the gamma matrices is and so iteration of the procedure. And cutting down the number of products of gamma matrices inside the trace is straight forward. And this is most frequently used identity in the calculation of S matrix in peturbative field theories.

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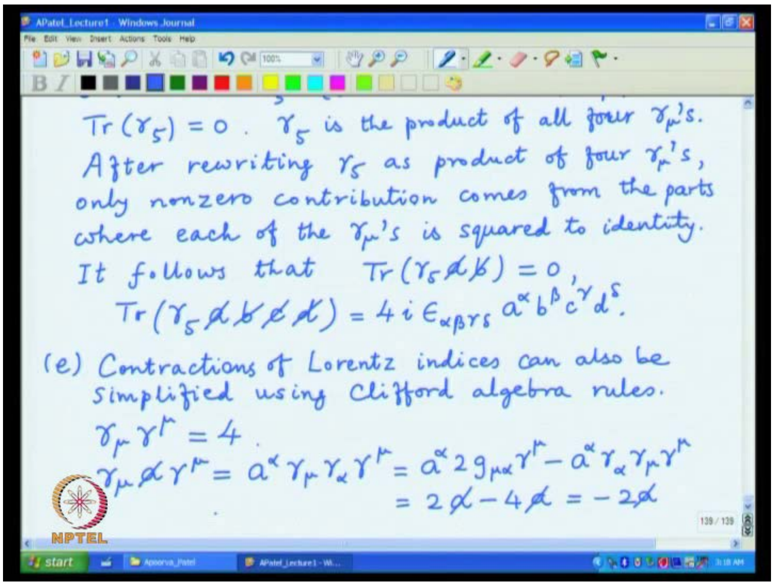

Let me now moved to the next identity which is specific to some other operators which appear inside this calculations; the propagators and vertices always have these slash operators. But suppose you are measuring some expectation value of a different object or the initial or final states are defined with respect to certain kind of wave function. One can get an gamma matrix which is not there in the set of this 4 matrices gamma mu other operators. For measurement or specific states can involve the gamma matrix gamma 5 explicitly. And that gamma 5 is not included in the previous identities which involve only the gamma mu this is outside the set of gamma mu.

So, one knows that gamma 5 anti commutes with all the other gamma mu. But one cannot use the Clifford algebra identity. Because when g mu nu does not have any meaning then the index becomes 5. So, one has to deal with this gamma 5 whenever they appear as a separated objects. And we know certain property of gamma 5 which I can write down one of them is the trace gamma 5 is equal to 0; the other one was that gamma 5 is the product of all 4 gamma mu's. Since, we can always simplify any expressions involving gamma 5 by just writing down this 4 gamma mu explicitly. One can follow the rule that in any expression of gamma 5 will first rewrite gamma 5 in terms of the gamma mu.

And, then we use the previous identity, which work very well then there only gamma mu's are involved here the string becomes larger, because gamma 5 is expanded. But everything else follows straight forwardly. After that the feature, which remain true is once that rewriting has been done. The only non 0 contribution comes from the parts where each of the gamma mu's is squared to identity. The simple consequence of the fact that out of the whole basis of gamma matrices; the only one with a non 0 trace is a identity matrix any other product of gamma matrix they all have 0 trace.

And, so one can now write down various relevant facts. Here, trace of gamma 5 with 2 matrices is equal to 0 the in general this rewriting involves 6 gamma matrices which is an even number. But the whole product is 0, because in the product all the gamma mu's do not appears squared two of them do gets squared. But the other two do remain separate. The first non 0 contribution comes where everything gets squared that is an expression of the form gamma 5 multiplied by all the 4 gamma mu's. And this we know what it looks like you just contract each one of gamma mu with the other one inside this a b c d trace of identity after doing square gives 4. And each square basically produce identity matrix this epsilon is symbol which rewrites gamma 5 into this product of 4 gamma matrices with the correct anti symmetric property. And then each one is contracted with this a b c d.

So, one can write the result as a alpha b beta c gamma d delta. This is the first non 0 result involving strings of gamma matrices together with gamma 5 larger expressions can be work out. Similarly, once the gamma 5 is rewritten in terms of gamma matrices one can use the previous induction identities also to go from string of n matrices to string of n minus 2. So, all those results now are possible to calculate again a program can handle these calculations in a rather straight forward fashion.

Let me go to the next identity which is actually not a trace. But it is a part of the calculation which appears frequently due to the dot products arising from Lorentz indices; one can reduce again those dot products to g mu nu; and rewrite these expressions by various commutations rules and index contractions. Let see some example of it first one is rather trivial it is a product of gamma matrices which immediately can be rewritten as g mu mu, which as we know is equal to 4. The non trivial part is the next step when you put something in between. So, let us do that is one a slash stuck between these 2 gamma mu's.

And, now we can rewrite this objects as explicitly denoting the a component. Then, this now can be rewritten with a Clifford algebra identity there will be a alpha 2 times g mu alpha gamma mu that is the first part. And the second part is the same object with the opposite order. So, it is a alpha gamma alpha gamma mu gamma mu. And this is now very easy to rewrite the g here this contraction indices between a and gamma. So, we have the usual a slash here gamma mu gamma mu contracted becomes 4; and the first part is again a slash the final result is minus 2 times a slash.

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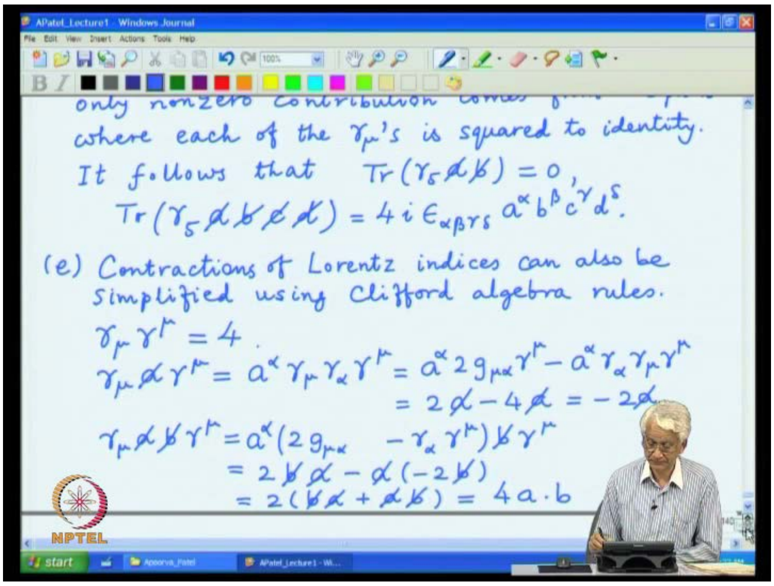

So, the procedure now can be repeated for higher string operators; let me work out one more example where there are 2 of them in between. I will rewrite using the same logic a alpha and then gamma mu gamma alpha will just rewrite as 2 g mu alpha minus the opposite order. The result is a alpha multiplied by 2 g mu alpha minus gamma alpha gamma mu; which is the Clifford algebra use on this product gamma mu a slash. And then the b slash gamma remains as it is.

So, now one can reduce this the object in a rather straight forward fashion this alpha will become the same index as mu because of this metric. And then it is multiplied on the other side. So, it becomes 2 b slash a slash. And this object we have already seen how to simplify in the previous results. So, it is minus a slash and gamma mu b slash gamma mu is equal to minus 2 b slash. This can be rewritten as 2 b slash a slash plus a slash b slash. And so one as the same product. But with opposite order of gamma mu inside here;

which is exactly the form that can be simplified using Clifford algebra once again. Then, the result is the g mu nu sandwich between these 2 operators b and a; and that result is 4 times a dot b. So, this is kind of illustration that one can use this identities repeatedly to simplify this contracted gamma matrices.

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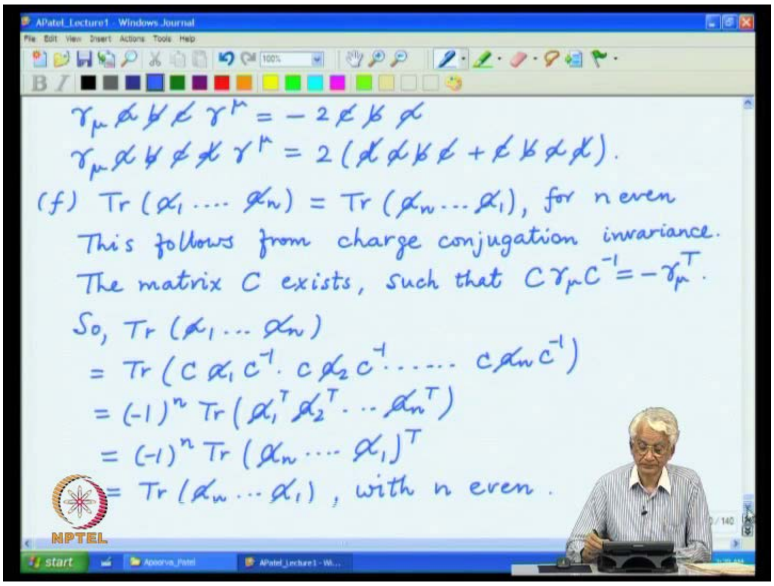

I will not work out many more cases, but just write down some of the result which can be proved in a rather straight forward fashions; what happens? When they are 3 of them sandwiching between the result is minus2. But then this product appears in opposite sequence when they are 4. One get similar result with gamma mu disappearing. And one can actually again use the iterative procedure to reduce the heat to 3 slash sandwich between gamma matrices and using the earlier result. This gives the result that looks like this; and the previous one are the basically contractions of all the Lorentz indices gamma matrices. And how they simplify? The algebra we are not included any traces inside here. But the fact that there are 4 values for the Lorentz indices has been explicitly used in this calculation.

Let me mention one more general result; which illustrates a different symmetry in doing these particular calculations. And that result is the trace of n gamma matrices is equal to the trace evaluated with exactly opposite order. And we have seen this one is non 0 only when the number of factors involved is even of this property follows from charge conjugation invariance of the electromagnetic interactions. That we have seen before in particular the matrix c exist such that c gamma mu c inverse is equal to minus gamma mu transpose. One can just insert c and c inverse between each a i slash and a i plus one slash. And rewrite this whole string and c multiplied by c inverse of course, is a trivial insertion of identity. So, the value does not change, but then the expression can be now rewritten using all this transposes is involved.

So, one can see that the expression becomes with the trivial insertion of c inverse of c together with the cyclic property of the trace. Now, we can substitute for every part this identity involving gamma mu transforming its transpose. There are so many factors which give minus 1 rest to n; and then the trace is now a 1 slash transpose a 2 slash transpose etcetera a n slash transpose. But the transpose of individual factors is equal to the transpose of the whole product written in an opposite sequence. So, this is now equal to trace of a n slash a 1 slash the transpose first. And then the trace where trace of a matrix and its transpose are equal and the number n is even.

So, that together gives the result that we wanted this is a important result; which connects the particle description of evaluating the matrix elements. And the anti particle description, because when one evaluate the Feynman diagrams the products are described according to the direction of the arrow; this sequence with opposite order corresponding flipping the direction of the arrow.

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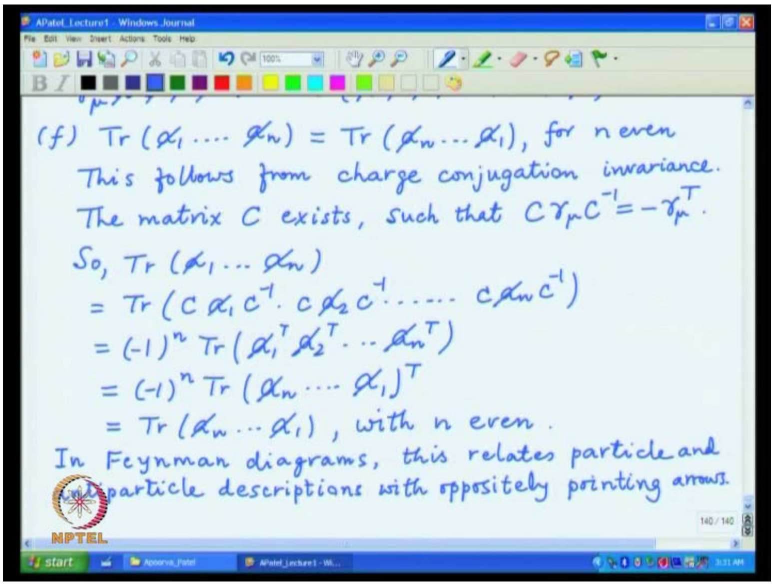

And, then evaluating trace flipping of the direction of the arrow is equivalent to going from particle description to anti particle description; that is a useful property to have and also to understand. These are some of the basic ingredients for evaluating products of gamma matrices in terms of numbers and the calculations make full use of this text which I have illustrated.

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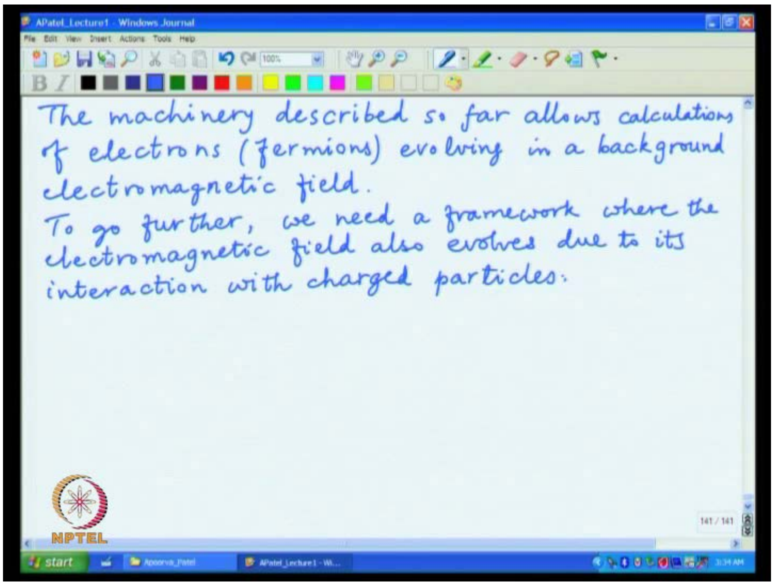

Now, we can move to the next step in the description of the theory and that is an extension of what we have done. So, far is to calculate general fermions evolving in a background electromagnetic field which is described by the term a slash in our description. So, far we use the potential model description which is common in non relativistic mechanics; where the electromagnetic field is provided as a background.

And, in which this charge particles are going to propagate and all which we have done. So, far is machinery which allows this kind of calculation to go further we need a more general frame work; where the electromagnetic field also evolves due to its interaction with charge particles; once we do that we will have the both the electrons and the photons becoming dynamical degree of freedom. And then we have complete theory of quantum electrodynamics the ingredients for making photons dynamical we already have those are the Maxwell’s equations. And will now see how to include Maxwell’s equation and its own frame work in this propagator theory in the next lecture.