Relativistic Quantum Mechanics
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Lecture - 23
Massive and Massless One Particle States

In the previous lecture, I described classification of states under the restriction of Lorentz group transformations and the set of commuting generators, which we chose for assigning specific quantum numbers. They were the four momenta, which mutually commute; and components of the Pauli Lubanski vector, which commute with the momenta as well. And that kind of completely exhausts the scheme for the simple reason that there were totally 10 generators of the group.

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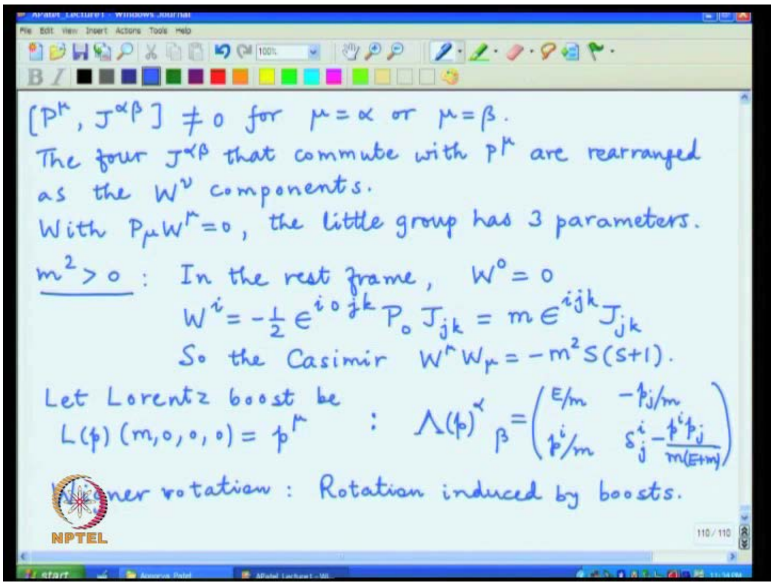

There were four momenta and six of the boost and rotation operators. Then the commutation rule showed that the components of P and J will not commute whenever the indices coincide; that means out of the six J operators, two of them will not commute with the given momentum. And so there were four remaining ones, these four are rearranged as the four components of the Pauli-Lubanski vector. And that covers essentially all the generators. We only have to identify now between the P mu and W nu, the completely commuting set of operators. And that classification was done through the construction of the little group, which basically describes which of the generators W nu leave the given momentum 4-vector invariant. And that led to a little group described by three parameters.

And, we saw the two important cases of this little group. One – when the mass is nonzero, then the little group is just the rotation group SO 3. And when the mass is equal to 0; in which case, the little group is the Euclidean two-dimensional group. And we will discuss some of the properties of these two groups now in little more detail; which describes how the remaining quantum numbers and states are labelled in a convenient convention. First, the case when mass is nonzero.

In this case, we chose the reference momentum as the rest frame. And then automatically, by definition, the zeroth component of W vanishes. And the remaining three components can be written as now i 0; and I can put j and k for the remaining two indices; then P 0, which happens to be the mass and the special index J vector. And this now can be re-expressed as mass times epsilon ijk. One anticommutation removes the 0 and we pick the suitable convention; then J j k. And this now can be written as 2 index or 1 index operators depending on the convenience. But, W is essentially the same as mass times the rotation generators.

And then we clearly have the Casimir value – minus m square times the expectation value of J square. And since we are dealing with a massive particle in its rest frame, J is nothing but the spin. So, the eigenvalues of J square are nothing but S times S plus 1, which we know from non-relativistic mechanics. So, this essentially characterizes the value of the spin. We can calculate this Casimir eigenvalue of W square. And from that, after removing the m square, you can extract what is the corresponding value of the spin. The negative sign here by the way comes from the Minkowski metric. So, this allows you to define the spin and now it gives a corresponding eigenvalue of W.

Now, it is conventional to not write the eigenvalues of W though that is the original generator coming from the Lorentz group. But, label everything according to the value of the spin, which is either integer or half integer as is well-known from the analysis of the rotation group. So, this is all one has to do to get the eigenvalue. You have to work a little more to get the complete labelling of the states. But, we will come to that later. We need a little concept to do that; and that is to define actually the transformation from the little group frame to any arbitrary frame. And that can be done by so-called Lorentz boost.

Let the Lorentz boost be defined by this operator, where we take the rest frame 4-vector and boost it to the value p mu whatever we desire. And the corresponding matrix transformation now can be written as lambda with two indices alpha and beta. And we have seen this in terms of rapidity earlier. I can rewrite in terms of the energy and momenta value explicitly. And this is an abbreviated notation, where indices i and j run over three different values. This matrix can be used if you want to construct the four vectors explicitly. But, we will use mostly it in an implicit form for carrying out certain transformation.

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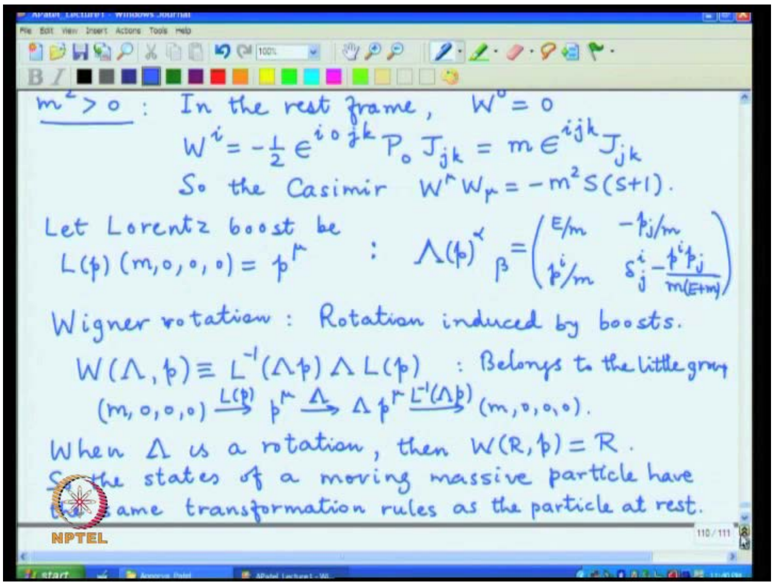

And then the key construction, which is helpful in many consideration is an object called Wigner rotation. This is a rotation induced by boost. We have seen that, the two boosts operator do not commute, rather they produce rotation generator. And the effect of that is that, if you keep on applying boost operator starting from a particular state, you might end up with a state, which is rotated with respect to the initial state. This operator – I will denote it by W lambda and p is defined as the result of three different operation; L of p takes from the rest frame to a state with momentum p mu; lambda then is an arbitrary transformation belonging to the Lorentz group, which will take this momentum vector p to lambda times p. And then you apply an inverse transformation L inverse of lambda p.

And we expect to get back to the original state, because the momentum will come back to the structure m, 0, 0, 0. What happens…

And, that is the peculiarity of Wigner rotation that, the momentum does come back to this particular value, but the state does not; state comes back in a rotated form; it will have the same momentum 4-vector, but the three spatial directions will be different than what you started with. And that is the notion of this Wigner rotation. It belongs to the little group, the transformation which leaves the momentum invariant. And one can see this thing explicitly by writing what all happens in these stages of applying these three operators starting with m, 0, 0, 0 apply L p. So, it becomes p mu apply lambda; then it becomes lambda p mu; and then apply inverse of lambda times p. And so you are back to m, 0, 0, 0. So, this is the concept of Wigner rotation that, you can construct a set of transformations, which will leave the momentum invariant, but it can have a nontrivial result in the little group.

One can work out this value of W lambda p explicitly, but only one particular result is useful. And that is that, when lambda, which is an arbitrary Lorentz transformation in the definition above, we take it to be a rotation; then one can prove through the useful exercise that W is nothing but R. So, it means that whether you apply the rotation after boosting to the frame p mu or you apply it before and then boost to p mu, the result is the same. And that is an important label for classifying the states, because we know everything about what the rotation groups does from the non-relativistic analysis that, the states of moving massive particles have the same transformation rules as the particle being at rest. And we know from non-relativistic mechanics, what the transformation rules of particles at rest under rotations are.

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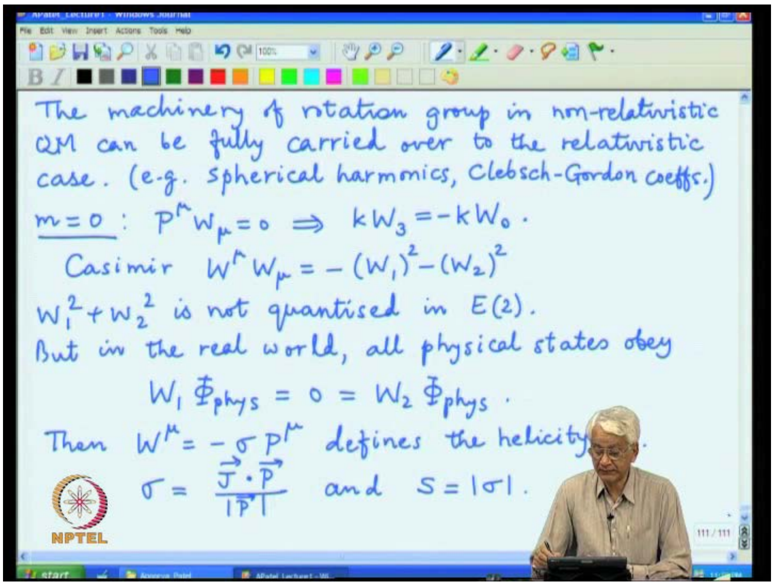

The complete machinery of rotation group in non-relativistic quantum mechanics can be fully carried over to the relativistic case. And this for example, includes all the definitions useful for constructing states such as spherical harmonics, the rules for adding angular momenta so-called Clebsch-Gordon coefficients, etcetera. And that is extremely useful, because we do not have to do any new work; everything which was known just carries over without any changes. So, this is as much as is needed for dealing with states of massive particles with one exception that, we have to fix a normalisation convention. I will soon come to that. Let me describe the similar structure applicable to massless particles. And in that case, we have the 4-momentum vector with two spatial components: 0 and the other two components being equal.

If one writes down this constraint P mu W mu equal to 0 explicitly, it leads to the result that, W 3 and W 0 are essentially proportional up to a sign. So, now, we have to construct eigenvalues of these W operators subject to this particular constraint. We can again look at the Casimir. And because W 3 and W 0 are equal in magnitude, if you just expand this thing out, it is nothing but the W 1 square plus W 2 square with a negative sign. And the W 3 and W 0 part cancel out. So, what is the eigenvalues of this particular Casimir inside the Euclidean group? W 1 and W 2 happen to be the commuting operators inside this Euclidean group E two. And so you can take arbitrary values for them and they are not constrained at all; they can have any value. In particular, any real number is valid. This square means that number will become positive. But, this object is not quantised inside this Euclidean E 2 group.

If you remember the Inonu-Wigner contraction, this W 1 and W 2 were the translation operators in this group E 2. And just like the usual momentum operators representing translations, these two things also are completely unbounded; they can take any value what so ever as far as the mathematics is concerned. But, there is a peculiar fact in the real world, which is a physical observation and not a mathematical constraint. All the physical states obey rather simple rule that, W 1 and W 2 basically have eigenvalues 0. That completely simplifies the analysis of this particular group. The Casimir basically is 0 in that particular case. And you have W 3 and W 0 having equal magnitude. And now one can just see from this particular relation that, the momenta and W have the same structure that, both of them have zeroth and third component nonzero and other two components 0.

And also, the nonzero components have equal value. So, one can quickly observe that, these two objects become proportional. And this is a peculiarity of the null vectors; both P and W turn out to be null vectors in this massless particle case. And even though W and P are orthogonal; since both of them are null vectors, they can be proportional as well. And that now leaves only one parameter completely characterising W; P is already chosen to be a quantity, which is fixed in the little group.

So, this defines the object sigma, which goes under the name of helicity. The reason for calling it helicity is rather simple that, one can now rewrite the W in terms of the vectors J and quickly see that, sigma is nothing but the component of angular momentum along the direction of motion; which quickly follows, because W and P are proportional and W is nothing but components of J. So, one can just work out what is J 3 in this particular case. So, one has these numbers; and one can define the convention that, the quantum number S is nothing but the magnitude of sigma. So, in particular, sigma is positive or negative number, but S is always defines to be positive by convention. So, what can one say about the values of sigma?

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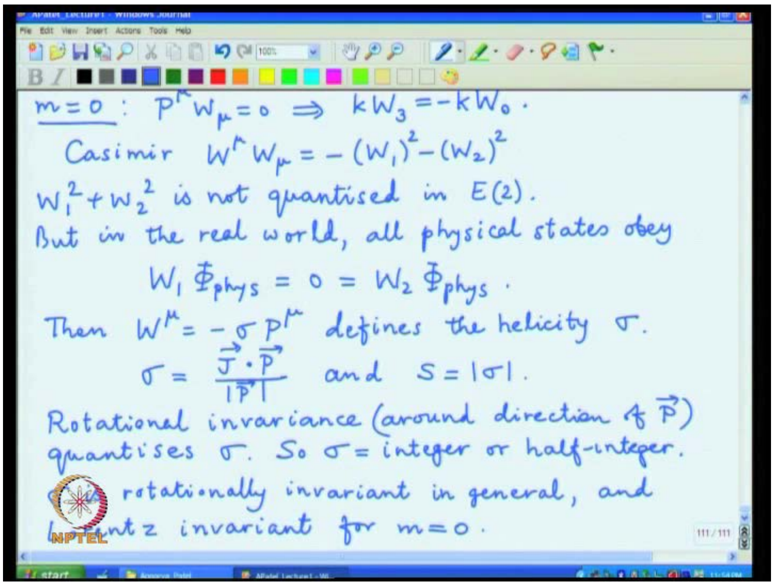

Again we can go back to the non-relativistic analysis. One can take this frame of particle moving in some particular direction or from a rotation along that direction itself. And that should be rotation around a fixed axis by a certain angle. And we know that it has to be periodic up to rotations of 2 pi or 4 pi depending on the value of the spins. This rotational invariance around direction of P ends up quantizing sigma. And we know from all the known results about rotational invariance that, sigma is integer or half integer. But, other than that, there is no further restriction. And the spin will again be referred to as a magnitude of sigma.

What this analysis tells you is there is nothing wrong with having a massless particle with just one value of sigma; it does not have to be a member of larger multiplate as this is forced on by the massive particle case, where once you choose a value of S, there are two S plus 1 components, which automatically comes with it. But, in case of sigma, which is conventionally defined as this projection, all you can apply is the rotational invariance around that particular axis and nothing more.

So, we have these results that, this sigma is rotationally invariant in general even for massive particle as long as we consider rotation around direction of P. The other way of looking at it is a dot product between J and P. So, dot product do not change their value under ordinary rotation. But, what this analysis tells you that, it is also Lorentz invariant. So, if the rotational part is extended by including all the boosts as well, and that happens for the particular case when the mass is equal to 0. And once you have a Lorentz invariant quantity, it can stand on its own; one does not have to make it member of a bigger multiplate by any necessity.

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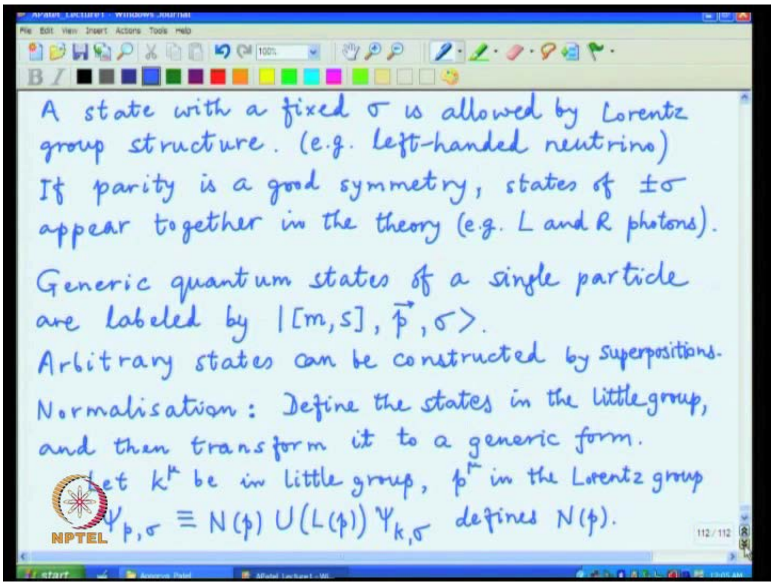

What happens is that, one might have other restrictions outside the Lorentz group. That can impose more constraint. But, let me give an example that, the particle with just one particular state of sigma, which is possible, is a left-handed neutrino. We saw that in the discussion of Weyl equation; one value of sigma is there; the other one does not have to be there for any mathematical consistency of the Lorentz group structure. But, if there is a restriction outside this Lorentz group, it can impose certain constraint. And the most common symmetry outside this Lorentz group is the parity; then states of plus or minus sigma appear in the theory. And again an example are the photons, which can be left-handed or right-handed. In this particular case, the degeneracy is 2. It is not 2 S plus 1 because a photon is S equal to one state under the same convention, but it does not have the third state corresponding to sigma equal to 0.

The convention has become now to classify generic states. They are now labeled by all the quantum numbers, which we extracted from the group. And two of them are the Casimirs – the mass and the spin, which are many times not written explicitly. But, for a fixed particle, these objects do not change their values at all under the Lorentz group; they are the invariants. And then the labels which change under Lorentz transformation in going from one frame to another – these are the commuting generators with eigenvalue P. I am not writing P 0 separately, because the mass can take care of determining what P 0 is.

And, from the analysis of the Pauli-Lubanski vector, we have one component coming out which commutes with this P n. In the conventions, we have chosen that, can be easily taken to be sigma. In the massless case, it is explicitly constructed as just discussed. And in the massive case, the sigma is nothing but the generator of rotation along a particular direction. And this can be taken to be the third direction. So, this is the complete classification of the single particle states and what its quantum numbers are.

And, now one can construct arbitrary states from this basis under the usual rule of Hilbert space, which allows you to do superpositions. One should make sure that, the superposition mixes the states, which are allowed by the corresponding selection rules of whatever the transformations are. So, that is the constraint. Once the states are there, you have corresponding transformation rules of all the generators as well as the possibility of superpositions. You have a complete framework to do all kind of calculations with these states. The only thing which is useful to be able to do the calculation quantitatively is one has to fix the normalisation of this particular state. And that normalisation is little different than in the non-relativistic case. In the non-relativistic case, it was quite easy that, the mod square of the state is equal to 1. That fixed everything.

But, in case of Lorentz group, there is Lorentz contraction as we have discussed above and one has to fix the normalisation to take care of that particular Lorentz contraction. It can also be done in the language used so far. And I will briefly point that out. We define the states in the little group and then boost it to a generic form. And that procedure allows us to fix the normalisation uniquely, because in the little group, everything is decided by the rules of the rotation group and we know how to normalise the states over there; and then it just carries over by this operation of a boost, which takes the state from the little group to arbitrary frame.

Let k be the vector specifying the little group and p mu in the full Lorentz group. An arbitrary state – let me call it psi p and sigma. I am keeping the m and S implicit, because they are not going to change under this transformation. This is nothing but some normalisation constant. And then the transformation corresponding to the operator L p applied to whatever the little group state is psi k and sigma. And this object once you take psi k and sigma to cover all the representation or the basis states of the little group, will now take a state to any arbitrary frame. And we want to find out what the normalisation factors N of p is. So, that is the problem. And now, we will work out whatever happens in the little group and then convert it into the arbitrary Lorentz frame.

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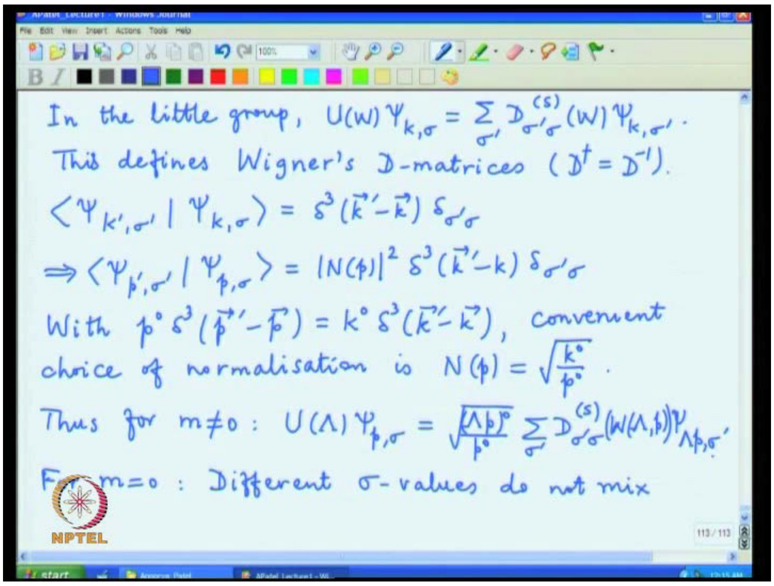

In the little group, if you apply a transformation, which we have been denoting by W, that transformation will change the components of the various basis states and can be generically written as some linear operator acting on k and sigma prime. And this rule defines the matrices, which are unitary by constructions and they go under the name of Wigner's D-matrices. In particular, one can construct it for any arbitrary value of spin and they are well-tabulated in the simplest cases of spin equal to half, 1, etcetera, which are frequently occurring. Of course, for spin equal to 0, this matrix is identity; there is only one state and nothing happens. So, this is an useful convention. And what we have is a normalisation rule, which says that, psi k prime, sigma prime and psi k, sigma is an orthonormal construction. So, it comes out as delta function. This is the standard normalisation in non-relativistic quantum mechanics.

Now, we want to find out what happens under boot transformation and we know that the boosts are unitary operators. So, this gives the constraint that one gets this mod of N p square as an extra factor; the unitary transformations basically just cancel out. So, one has a general rule. If one can relate this delta function in terms of k, somehow to the delta function in terms of p, we know how those things transform by the rule of Lorentz contractions. So, this rule of Lorentz contraction now can be conveniently exploited. p 0 is nothing but the Lorentz contraction factor times k 0. So, one has this rule and the Lorentz contraction comes because it is the integral of delta, which is normalised to 1, not just delta function itself. And that extra integration volume undergoes the contraction according to this rule.

One can now write this psi p prime psi p also as delta cube of p prime minus p provided this extra factor cancel out. And the convenient choice then is this N p is nothing but square root of k 0 by p 0. And so N p is equal to 1 if one is in the little group frame. If one is not, then it becomes the Lorentz contraction factor. And one can quickly workout this normalisation k 0 times delta cube k prime minus k can be now combined as p 0 times delta cube p prime minus p. And various things look nicer. They will look delta cube p prime minus p in any arbitrary frame. This has become the standard.

Now, an arbitrary definition of normalised states; and I have used the conventions for massive particles in terms of going to the rest frame and using this delta function, etcetera. So, all these analyses apply in that particular case. And now, one can talk about what happens to the normalisation in a generic state with a generic Lorentz transformation. And that now has two parts. One is the normalisation got fixed and it has this particular form. And the other part is one can have rotations or transformations within the little group themselves, which now are described by this matrix so-called the Wigner rotation matrix depending on p and lambda. And the final state will have momentum lambda times p; and sigma prime is the spin component. So, this now becomes a complete specification of the states in terms of how they transform from any generic quantum numbers p and sigma under any generic Lorentz transformation.

This analysis with little caveat can also be extended to the massless case. In this particular case, one does not have a generic superposition of arbitrary sigma. The different sigma values – they do not mix; in particular, we can have only one and nothing may happen. So, the only thing leftover is just this Lorentz contraction factor. The D-matrices are absent. But, one still has to include what happens under rotations under this transformation. And the rotation does leave its effect that, this D is not a matrix, but it is allowed to be a phase. The value of which will be dictated by sigma.

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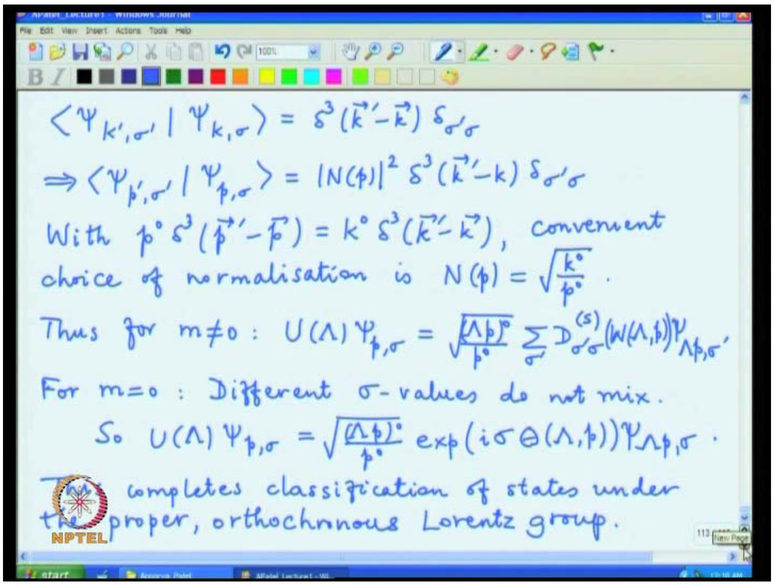

And, one can write down a similar rule U lambda applied to again a generic state. The normalization factors – it is a Lorentz contraction factor and that does not depend on whether the particle is massive or not. And then the rotation phase is simply dependent on the value of sigma. There is a rotation angle, which depends on lambda and p just like the Wigner rotation does. And then there is an original state, except that there is no sigma prime. Sigma is left unchanged; it only produces a phase factor.

So, this essentially defines now the total set of features. We need for specifying a quantum state, both – its quantum numbers as well as its normalised definitions and their transformation properties. So, this completes classification of states under the sector of the Lorentz group, which we have dealt with it, is the proper orthochronous Lorentz group. But, we saw it at the beginning that, there are four such sectors. And what we did was to focus on one of them by choosing a particular label for parity and time reversal.

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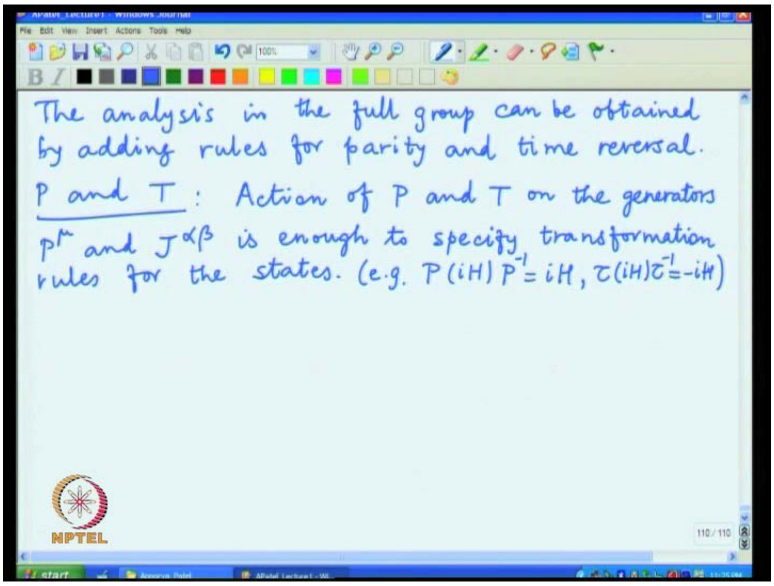

Now, we can put those symmetries back and extend the analysis to the full group. All we have to do is specify the rules for parity and time reversal, which are space-time transformations as well, but they are discrete. And in particular, there are only two possibilities for either of them. The action of P and T on the generators is enough to specify what all happens to the quantum numbers as well as the states. And these transformations are kind of straightforward to work out by these discrete properties. For example, the parity acting on Hamiltonian or the P 0 operator does not do anything. And I am deliberately writing the factor of i in here to stress the point that, the time reversal operation does something nontrivial, because it includes complex conjugation. And the i flips in sign (( )) the P 0 operator or the Hamiltonian may be invariant under time reversal.

But, one has to take these things into account and then one quickly gets eigenvalues for P and T as well for the various basis states, which we obtained by finding the quantum numbers for the 4-momentum as well as the spin labels. And that is something I will discuss next time – what are the eigenvalues for P and T for all the one particle states that we described. Keep in mind that, the continuous Lorentz group is an exact symmetry of all the fundamental interactions that we know. While parity and time reversal are violated to a small extent because of the weak interactions. So, consequences of Lorentz symmetry apply to all the particles and fields, but the results for parity and time reversal that we will discuss, apply to only those situations, where these discrete symmetries hold.