Relativistic Quantum Mechanics

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Lecture - 22
Classification of One Particle States, The Little Group, Mass, Spin and Helicity

In the previous lecture, I described how one can obtain the Galilean group algebra from the Poincare group. One by taking the limit that the magnitude of velocity is much smaller than the speed of light; and that gives rise to certain scaling of the various generators and one can keep the leading order terms in all the commutation rules to get the simplified algebra. This is the technique known as Inonu-Wigner contraction. And I will write down the Galilean algebra rather soon.

And before that, I want to correct signs in the equations, which I wrote down in the previous lecture; the commutators with K actually have opposite signs compared to what I wrote. And actually, it is a good exercise in all these algebra to check these factors of plus or minus 1 or indices going up and down, etcetera to make sure that the relations remain correct. The only non-trivial relations, which arise from this algebra are from the generators, which scale differently compared to the generators, which do not scale at all. And in this particular case, they happen to be the boost generators as well as the zeroth component of the 4-momentum.

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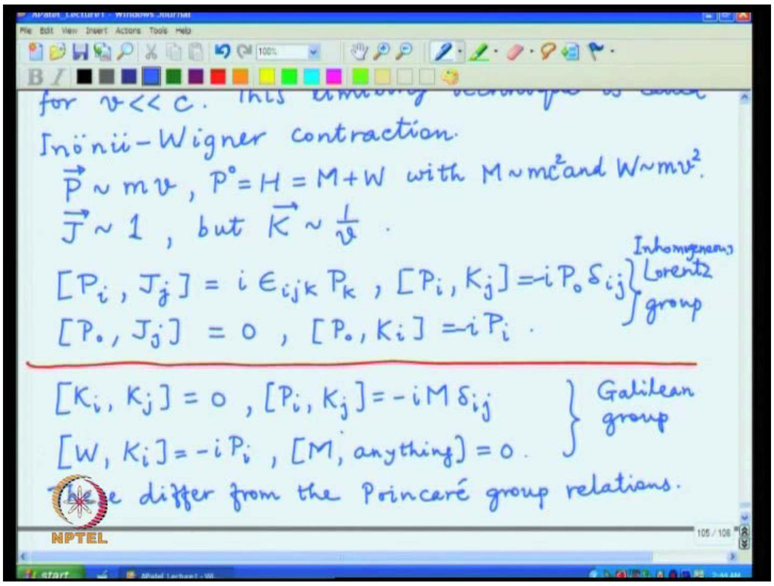

And so one can take the Lorentz algebra, and the only one which are different are the once involving this K and P 0, and I will only write down those ones, which now behave differently in the case of Galilean group. One particular relation which arises is the boost generators; they commute with each other, because they scale as 1 over v and the right-hand side does not scale. So, one can take the leading term and reduce all the sub-leading terms to 0. So, this is what comes out. The other nontrivial commutations rules are those involving momenta. So, P i and K j – again keeping only the leading term, it produces minus i M delta i j. The third one is the relation involving P 0 and K i. Now, this survives, but in a different form that the leading term of P 0 does not give any contribution to the commutator, it is a number – the rest mass, so that relation reduces to W, K i is equal to minus i P i.

And, all the rest essentially remains the same. One just has to replace the non-relativistic expressions instead of the relativistic ones. And in particular, the commutator with the rest mass term is always 0. So, these are the nontrivial relations for the Galilean group. These are the ones which are different; the ones which are same I have not bothered to write down. And in particular, one can replace the P 0 by the operator W. In all the other cases, J and P remains the same. So, this is essentially the simplified Galilean algebra. And one see the peculiar feature in this particular case that, the boost operators produce a commutator, which is essentially a constant – the mass of the particle.

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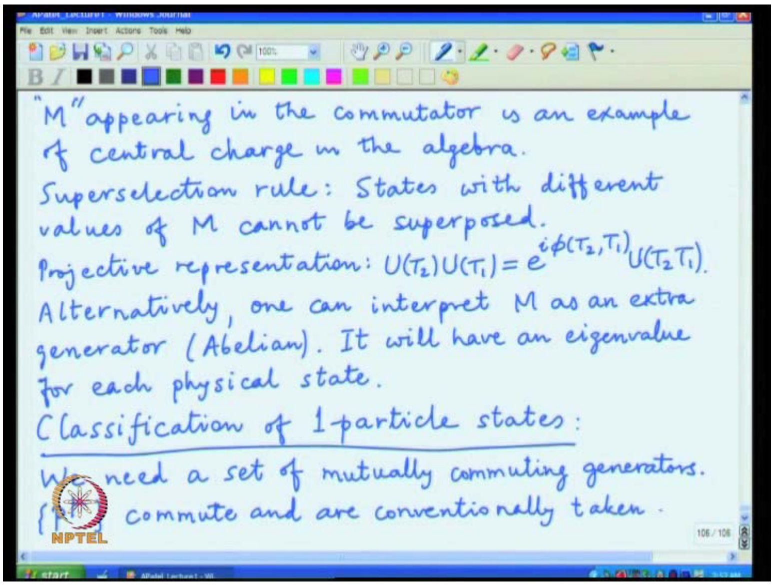

If that were not the case, we could have simultaneously taken eigenstates of both the momentum operators as well as the boost operators. But, they leave behind this mass term. This M appearing in the commutator – it is an example of what is called central charge in the algebra. There are different ways of dealing with it since it is only a number. When the algebra gets exponentiated to the group transformation, it will produce phases. And whenever such phases come in, we have to worry about the possibility of changing the rules for superposition, because objects, which transformed with different phases – they cannot be superposed. So, there is a super selection rule. And that is states with different values of M cannot be superposed.

Sometimes this thing is also referred to as projective representations. This gives relations between transformations, which are true until a phase is ignored. If the phase is there, then one has to take care of it a little differently. So, a generic structure will look like… If I take a transformation T 1 first and then the transformation T 2, it is equivalent to the transformation T 2, T 1, which will be required by the group composition law, but only up to an overall phase, which is a function of T 2 and T 1.

These phases can be conveniently handled in quantum theory, because they do not have any absolute meaning; and one can then still constructs this so-called projective representations, which are useful in defining quantum states. The only thing one has to watch out for is one cannot take two different objects having two different values of phi and superpose them. And that is exactly the superposition rule. And that will appear when you try to combine the operations of boost with a translation in the case of Galilean group. So, this is one way of stating the constraint of super-selection rule and the value of mass in Galilean group.

The alternative is one can interpret M as an extra generator. And since it is just going to produce a phase, it will be an Abelian generator. And then it will have some eigenvalue for each physical state. And again it leads to the same conclusion that, value cannot superpose two different states with different eigenvalues. So, again you cannot combine states with different values of mass. So, this choice of whether to count this central charge as a projective representation or interpreted as an extra generator in the algebra – it is a matter of language. The conclusion is that, one has to treat objects with different values of mass in Galilean algebra as different physical states and they cannot be superposed with each other. So, this is the essence of reducing Lorentz group to Galilean group and see what kind of things emerges. And I will leave it at that. There are other techniques of how to deal with projective representation and central charge, but it is a matter of group theory.

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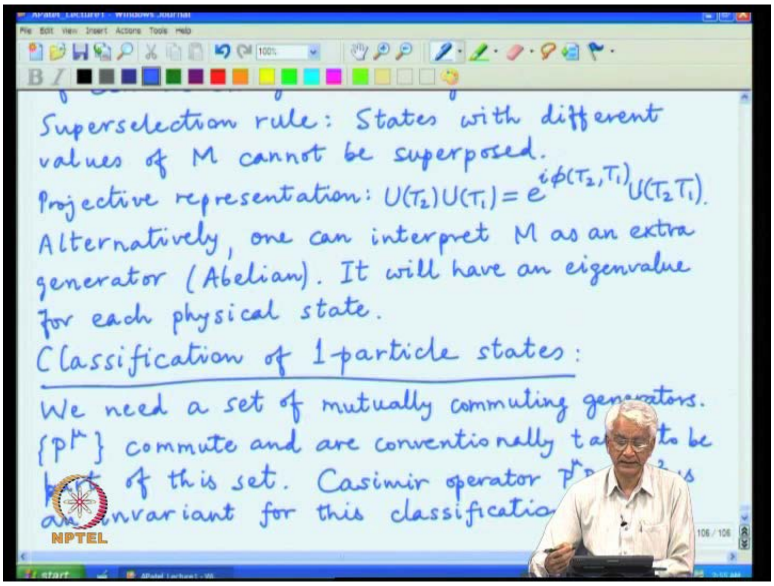

So, now, let me go to a new topic, which is to completely classify the states. I will stick to the simplest situation, which is talking about single particle. When one talks about many particles, there will be other questions involving statistics and exchange; that I will not go into any detail. But, to do this, we need to specify a set of mutually commuting generators of the symmetry group. We have already seen that, this set of P mu commute and are conventionally taken to be part of the sets. Every state will be labeled by its 4-momentum vector. And the obvious Casimir operator, which can be created from this quadratic contraction P mu P mu, which happens to be equal to m square, is an invariant for this classification.

We automatically have one important number coming from this, which is the mass of the particle. Now, we want to find out, are there any other objects, which we can combine with P mu to specify the states more explicitly. And for that, we have to find some other components out of the Lorentz group generators, which commute with P mu. And this requires some amount of algebra, because there are 6 of the homogenous Lorentz group generators; and which one commute with P mu or not will have to be deduced by doing the various commutations.

And, we already have seen that, the commutators of P with j in general will produce some combination of momentum on the right-hand side. And so we have to take some combination of J's, so that the right-hand commutator vanishes. This was worked out in full detail by Wigner. He provided a complete classification of states and their quantum numbers for the Poincare group. I am only going to give you the answer instead of going through the whole algebra. But, it is easy to see how the answer works.

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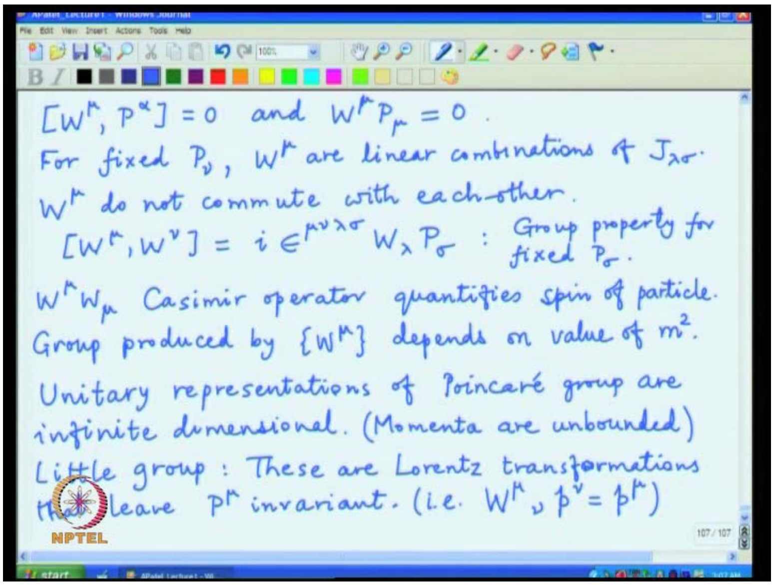

The answer is given in terms of an object, which is constructed from this generator, which is called Pauli-Lubanski vector. And it is defined by the relation W mu is equal to minus half epsilon mu nu lambda sigma – the completely antisymmetric symbol contracted with one factor of momentum and one factor of angular momentum. So, this is an object, which transform as a vector, because everything is indexed according to the Lorentz symmetry. And it has four different components. Now, it is very easy to see that, these objects commute with the momentum vector for a very simple reason that, if I take a commutator of P alpha with W mu, the nontrivial commutator arises between P alpha and J lambda sigma. But, that commutator is proportional to momentum with one of the index of lambda or sigma sitting on that momentum term. And then the complete antisymmetry of the symbol with two factors of momenta contracted with it produces 0.

So, we automatically have this simple consequence that, these objects commute with momenta. It is also true that, because of the antisymmetry, if you contract this thing with momenta because of the same antisymmetric symbol, that produces 0 as well. So, literally speaking, these are vectors, which in some sense are orthogonal to the momentum vectors. They are legitimate candidates for choosing as new operators to label the states, because for fixed P nu, because we had already chosen momentum to be the eigenoperators for the states. So, P will have some value. And in that particular case, these are linear combinations of the generators of the homogeneous Lorentz group. So, they can be treated as valid candidates on which we will impose the eigenvalue condition. In particular, we can now choose some components of the W's to be simultaneous eigenvectors with the 4-momenta, which we classified.

And, one can construct a new property in addition to the 4-momentum and the mass, which we used. And that now becomes an additional label. And that label actually quantifies the value of the spin. This W mu as it turns out, do not commute with each other. So, in that sense, they are similar to the angular momentum operators, where various components do not commute. The momentum components commute with each other, but this one is more like the angular momentum. And as I said, it will be identified with spin rather soon. And one can actually prove; which I am not going to go into detail about what these consequences of these commutation rules are. But, this follows from the same algebra that, the commutators of W produces again a W and P if you again take P to be some eigenvalue. So, this is a group property.

For fixed P sigma, the algebra closes; P sigma reduces to a number; and then commutators of W is another W. So, the W actually do form a group for the states with fixed eigenvalues of the 4-momentum. And much of the classification of the single particle states now becomes a question of identifying this group and labeling its representations and eigenvalues, etcetera. So, this is a very useful quantity to handle.

And, the answer, which arises from this thing, is the Casimir operator with a square of this W. It actually quantifies spin of the whatever particle or state, which we are talking about. And this is a important label for finding out what are the eigenoperators or quantum numbers for Lorentz group. It turns out that, after this P mu and W mu, there is nothing more; in the sense that, there is no further operator, which commutes with all of these and give a new label. And we have exhausted the list of all possible quantum numbers that we can construct.

One can now look at various possibilities of single particle states one by one. The group produced by the set of W mu – it depends on the value of m square. And that is the complete classification, which Wigner had to work out in order to find an appropriate label for all possible quantum states following the symmetry of the Lorentz group. Let me also state at this state that, the unitary representations, which are needed for specifying any quantum states – they are not finite dimensional as far as the Poincare group is concerned.

And so the labels will correspond to infinite number of possibilities; one way to look at it in the scheme, which we are following that, the momenta, which we have chosen to label the states are unbounded. And that is the part of infinite dimensionality, which shows up in labeling the quantum states. On the other hand, the labels produced by this W mu sort of generators – they turn out to have finite dimensional representations and they are useful. And those are the finite dimensional representations corresponding to the spin of the particle.

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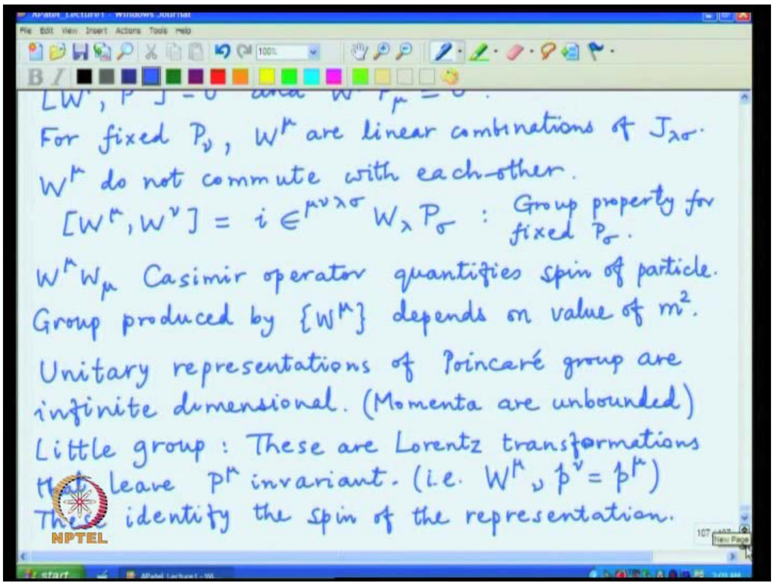

And, to do that, we now define a concept, which Wigner invented. There is an object called a little group. And this little group transformations that leave the momentum value invariant. And momentum value – we have already taken to be the eigenstates. So, these are operators, which can be reduced to numbers. So, there are four numbers and we can pick a fixed frame to describe them. And the remaining part of the Lorentz generators, which will leave those four vectors specifying P mu unchanged – they correspond to the little group. It is obviously a group, because if one of them leaves invariant, then the second one, which can be composed with it, will also leave it invariant; there is no problem with that. And it is a subset of the Lorentz group. And I will denote them by the symbol Lorentz group. I use the symbol lambda and I will now use a symbol W for this little group transformation.

And, they have the particular property that, acting on this momentum eigenvalues, the values are left as they are. So, this so-called little group – they identify the so-called spin of the representation and mu quantum label, which is different from the four momentum. So, the first job is to now take some eigenstate of these four momenta and construct this little group; and then look at this little group and its representation and try to quantify them in full detail. And that will give a label, which will immediately see corresponds to spin of the particular particle.

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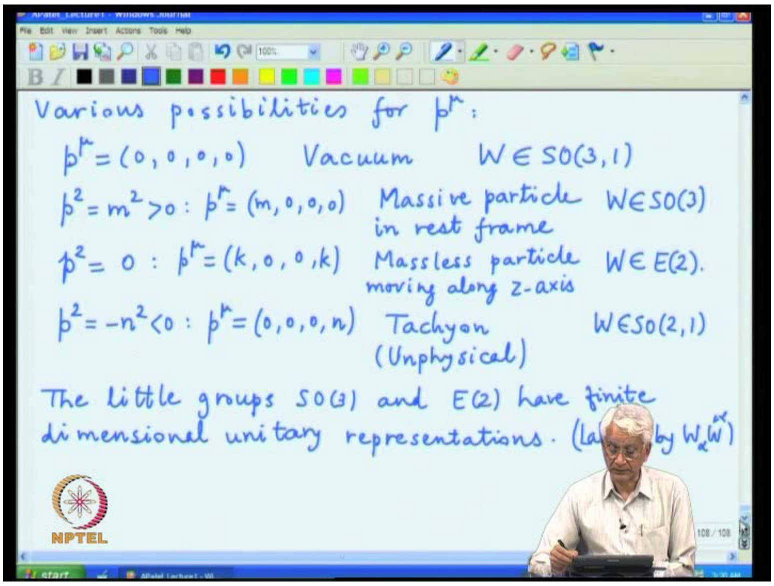

Now, I will just make a little table, which will give all possibilities of these four vectors, which are distinct from each other in a topological sense that, one cannot convert one of them to the other one by any continuous Lorentz transformation. So, let me construct that various P mu. Of course, the trivial possibility is that, this whole thing is 0. And this is actually not of single particle state, but rather what we will call the vacuum. The ground state of the whole system; mean by definition it will have 0 energy momentum value.

And everything, which leaves it invariant, is now the complete Lorentz group SO 3 comma 1, because 0 is left invariant by any of the homogenous Lorentz group transformation. So, this is a case, which is actually not a one particle state, but it is a quantum state. Nonetheless, it is a 0 particle state.

What we are more interested in are the single particle states. And for that, we have to now separate different sectors of the invariant P square. So, there is this so-called time like vector P square equal to m square positive, which corresponds to a vector of the form m, 0, 0, 0. Of course, this particular form corresponds to a massive particle and in the rest frame. And this particular form of the P mu is convenient, because one can work out all the algebra in this particular frame. And since the Lorentz index is explicit, can easily see what all will happen in an arbitrary inertial frame. So, this particular case describes a 4-vector with only a nonzero time component.

And of course, the little group for that now will correspond to leaving this component invariant. And it is a group, which will mix these three space components with each other without any restriction. And we know what that particular group is from our general experience. This is nothing but the group of rotations, which mix the space components while leaving the time component as it is. So, for these massive particles, the little group is nothing but the rotation group and we know it is a complete algebra; what it is as well as its representation. These are all the famous angular momentum representations, which we have dealt with.

And, since now we can use them as labels, we have the angular momentum of the particle in the rest frame, which is nothing but the spin. There is no orbital angular momentum in the rest frame. And so it will produce now a label, which we will use for a generic particle may be moving, may be not moving. But, the particle must have a mass. So, there will be some particular frame in which it is at rest. So, this is one possibility.

Now, the next possibility is little nontrivial. And that corresponds to this value of P square being equal to 0. In this particular case, there is no rest frame possible. And the best one can do in simplifying the 4-vector is write this thing as one time and one space component of equal magnitudes. So, the P square essentially becomes 0. So, this is a massless particle moving along z-axis. And the little group for this now will be all transformations, which leave this particular form invariant. Clearly, that will involve rotation of one and two components with each other. But, the group is actually larger than that; one can also mix the time and the z component in a particular way to arrive at that particular combination, which will still leave it invariant. And just as in the case of massive particle, out of the four dimensions, we have selected one dimension to label the P mu and the other three dimensions can be mixed appropriately to create a group.

Here also, there will be other three dimensions which can be mixed. Those dimensions are little nontrivial because of the specific four vectors. But, the group can be obtained from them and it is a mixing of three different directions. And that can be done; just is in the rotation group with three different generators. So, it is a group, which will have three generators, but the group is different. It is called the Euclidean group in 2-dimension, which have basically the transformations of a flat plane, which leave the geometry invariant. And those are one rotation about an axis orthogonal to the plane as well as true translations, which correspond to movement in the plane, and these 3 form a group. And that actually is the group arising in the case of this P square equal to 0 or massless particles. We will soon describe that in a more detail. But, it is a nontrivial result; not as easy to see as in the case of massive particles.

And, the third possibility left by the mathematics, but ruled out by any physical state is this vector P mu is space like. And it will correspond to a frame conveniently chosen that, one of the space component is nonzero and all the other three components are 0. This will correspond to a particle, which has been labeled as a Tachyon – a fictional particle, because we have never seen any one of these things in the real world. Mathematically, it is allowed. And just as in analogy of the rotation group, now, we have three components, which we can mix with each other and it will produce is a little group, which is SO 2 comma 1, because the one of the component is time-like and the other two components are space-like. This one is unphysical. So, essentially, we are reduced to studying these two cases: P square not 0 equal to a positive value and P square is equal to 0.

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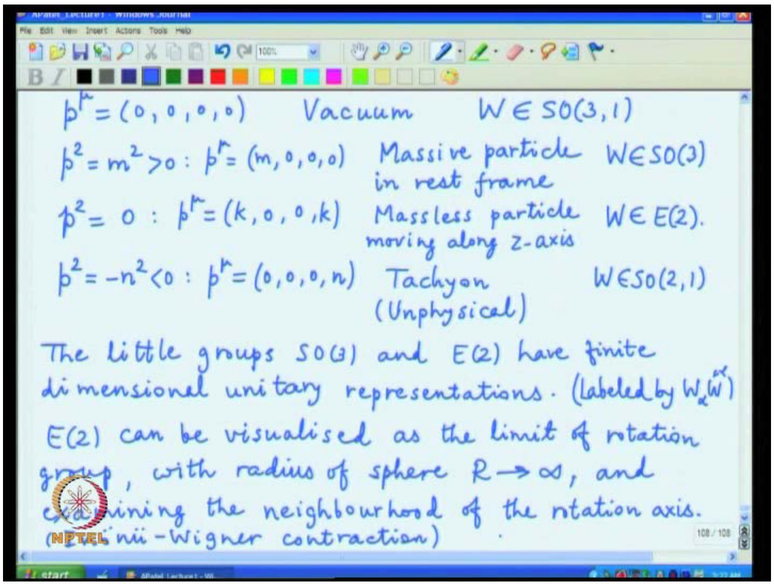

The little groups SO 3 and E 2 have finite dimensional unitary representations. And we will use those quadratic Casimir operators like square of W to label this representation in the case of rotation group. That is a familiar strategy. And we will do that in a little while. Before that, I want to explain this structure of this Euclidean group in 2-dimension in a little more detail, because it is a group, which is little bit unusual. E 2 can be visualized as the limit of rotation group with the radius of sphere on which the rotations act going to the limit infinity; and then looking not at an arbitrary point on the sphere, but examining the neighborhood of say the north pole, which defines the axis of rotation. This limiting procedure is again an example of what I mentioned in the last lecture so-called Inonu-Wigner contraction. So, let me now do that rather explicitly. And we have to now work out the scaling of the generators.

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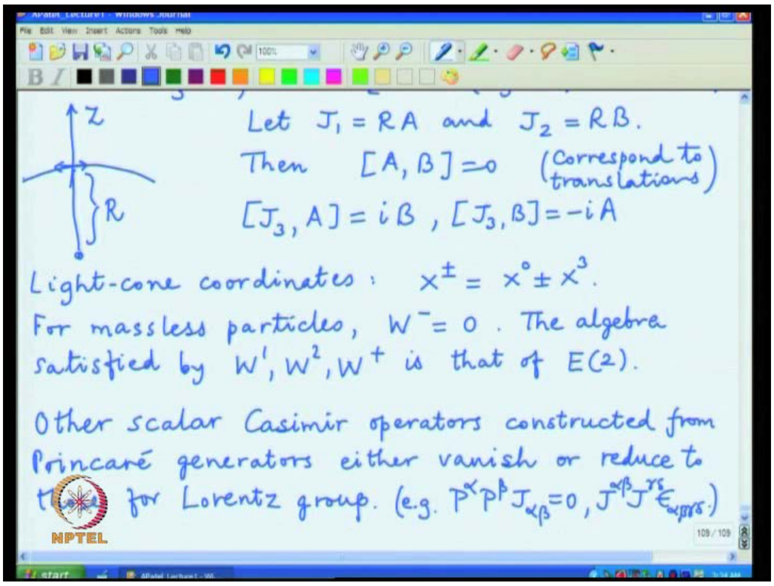

In this particular case, the generators belong to the rotation group. Now, one can write down what happens to J 1, J 2 and J 3 in this particular case. So, this is say the z-axis and there is a sphere; its radius R. And we are studying what is happening in this particular neighborhood of the north pole. Now, clearly if you define J component wise, it has a structure R cross P. If I look at the J 3; by choosing this particular scale, nothing much happens to it; it is a rotation around the third axis; and the blowing up of the radius does not do anything and we can keep J 3 of order 1.

But, on the other hand, J 1 and J 2, which have a transverse relation; and so if you define them, it will have R cross P with one of the radius components equal to this capital R, that is, the third component of the radius vector R, which will appear in the definition of J 1 and J 2. And so these things end up scaling as R. The third component of radius does not appear in the definition of J 3. So, now, we take the angular momentum algebra and ask what happens to this particular situation. So, this is a consequence if you want to say, so that x 3 scales as R, but x 1 and x 2 remain of order 1 when 1 is in the neighborhood of the north pole.

One can now redefine the scaling. I am explicitly factoring out the radius R. And then the relation automatically emerges; the commutator of J 1 and J 2 gives J 3. But, because of these extra factors of R, we have now the simplified relation that, these two new generators – the scale generators commute. On the other hand, the commutator of J 3 with J 1 will produce J 2. And that the factor of R just scales out. And the other two relations look like whatever is there in the rotation group. And this is now the algebra of the Euclidean group in two dimensions. We treated A and B as rotations here as well, but one can now see what happens when the radius becomes very large; the neighborhood of the north pole is almost like a flat plane; and these rotations are of J 1 and J 2, the axis which are orthogonal to the third axis, are essentially now little translations away from the north pole. And this A and B – they correspond to the two translations. So, we essentially get out translation generator from the rotation part by factoring out that R; means it is essentially by definition, because J was R cross P; R we already factored out; what is left is a translation and the directions got cleverly picked out. So, the algebra emerges. So, this is the method of obtaining the Euclidean group in 2-dimension by a contraction of the rotation group.

What happens in the case of massless particles is this is the algebra, which arises not in terms of directly the components of W, which we have seen, but in terms of what are known as light-cone coordinates. These are the coordinates, which are defined by the notation x plus or minus is equal to x 0 plus or minus x 3. So, it is a component of the zeroth or the time part and the space part along which the particle is moving. And so these are the directions in which light rays will propagate. The two extreme limits of the so-called Minkowski space-time in which physical signal can be transmitted; and for that reason, they are referred to as a light-cone coordinates. Clearly, instead of using x 0 and x 3, one can use x plus and x minus as also a valid complete description of the coordinates.

And, the relations between the components of W simplifies in light-cone coordinates in the case of massless particles. And that now can be seen; I will give you the answer. For massless particles, which means the P mu, which we are going to use, have the structure of the four components K, 0, 0, K. And one can easily deduce that, the W minus component identically vanishes. In that particular case, that is the one which separates out the direction with a specific relation to the momenta. And then we are left with the algebra satisfied by the remaining three components, which are W 1, W 2 and W plus. That algebra is that of E 2. So, in this manner, one can explicitly verify that, yes, the little group in the case of massless particles is indeed E 2; the original xyz anti-components are not very clear about demonstrating them, but one can transform to this light-cone coordinates, where the algebra is clearly seen. And then one can now ask about labeling the states by these irreducible representations of this Euclidean group in two dimensions. This essentially is a complete description of various properties of the little groups: their Casimir operators, their generators and the various quantum numbers, which they produce in all possible situations that occur in one particle physical states.

We are essentially left with two possibilities and I will describe the possibilities in more detail in a little while. But, one can now ask what happens to any other labels one can construct from looking for new generators or contracting them with each other to construct new kind of Casimir operators. It turns out that, other scalar Casimir operators constructed from this Poincare generators either vanish or reduce to those for the Lorentz group.

And, in that sense, there is nothing much left. After going as far as in this particular analysis, we have the 4-momentum vector P mu and this little group labeled by W. And they completely exhaust the various possibilities. And just for example, one can form contractors like P alpha, P beta contracted with J alpha beta. This happens to be 0 for trivial reasons of symmetry. Things like J alpha beta contracted with the completely antisymmetric epsilon symbol; it reduces to other objects also belonging to the Lorentz group. So, they are not independent objects any more. So, this is as much as I have to say about this one particle state. And next time, I will describe the explicit structure of the different components of the little group for both massless and massive particles.