Relativistic Quantum Mechanics
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Lecture - 21
Finite Dimensional Representations of the Lorentz Group, Euclidean and Galilean Groups

At the end of the last lecture, I stated that the 6 generators of the homogeneous Lorentz group, 3 rotations and 3 boosts can be combined into new operators, which are essentially J plus or minus i K; and I called them N such that the group factorizes in the sense that, the N and N dagger are mutually commuting, and the two factors essentially obey SU 2 or the angular momentum algebra by themselves.

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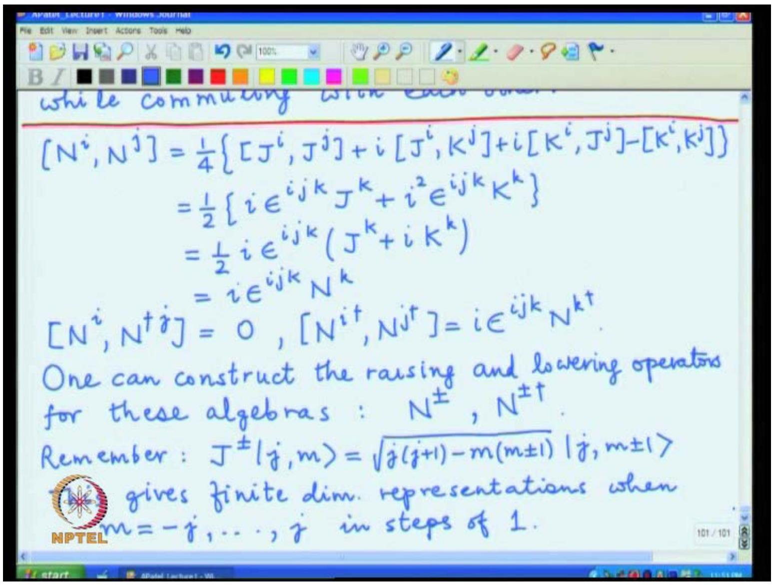

Now, this statement can be very easily proved. For instance, let me evaluate the commutator N i and N j substituting these factors explicitly. We have four terms and we know the commutation rules between J and K explicitly. In particular, J i times J j is nothing but i times epsilon i j k J k. And K i, K j gives the same term with a minus sign, but there is a minus in front of it. So, the two terms are actually identical. And the result can be then written as i epsilon i j k J k. And the factor now reduces from quarter to half because there are two terms.

Similarly, J i, K j gives i times epsilon i j k and together with the operator K k. And that is equal to the third term as well, because epsilon i j k is antisymmetric i and j; and this produces the same result. So, these two terms are identical; again, the quarter becomes half. I can write this thing as plus i now is occurring twice epsilon i j k times K k. And this now can be rewritten as factoring out common coefficients i times epsilon i j k times J k plus i K k, which is nothing but i epsilon i j k N k. So, this shows that, the components of N – there are three of them; they obey the same angular momentum algebra.

Similarly, one can reduce the N i dagger, N j dagger to the same kind of structure. What happens when one is N and the other is N dagger is that, one gets the same four terms all right, but there are differences in signs, because N dagger will have minus i instead of plus i. So, the last two terms, which I have written here, can be written with opposite signs; and the first two terms of the same sign. But, we saw that, in this particular case, the two terms added with the sign flips included that, two terms are exactly going to cancel. So, N i, N j dagger actually produces 0. And I can just write for completeness that, we have the angular momentum algebra again for N dagger operators. So, this is a complete factorization or separation into 2 parts of the lee algebra of the homogeneous Lorentz group.

And then one can identify all the representations by solving these two separated algebras individually. We know the answer for all the representations of the angular momentum algebra; and we will only concentrate on the finite dimensional representations. They are given by all the integer values for the number of states belonging to that particular representation.

And, in particular, if you look at the matrix dimensions, the matrix dimension starts with 1, which corresponds to the 0 angular momentum state and then it takes all the integers values, which are basically 2 J plus 1 components. The way we can arrive at the same algebra here in spite of the fact that these are not Hermitian operators anymore, is because of the simple fact that, N can still construct from this algebra the operators, which raise and lower the values of the third component of the angular momentum. So, one can construct the raising and lowering operators for these algebras; and one can just call them N plus or minus and N plus or minus dagger.

And, what these operators do when acting on specific states is the same kind of rules, which are well-known in angular momentum, states that the raising and lowering operators acting on specific angular momentum states produces a state with different value of m. But, there is a normalization constant in front. And because of that, one can obtain the finite dimensional representations when the allowed values of m go from minus j to plus j in steps of 1. And that is possible when j is either an integer or a half integer.

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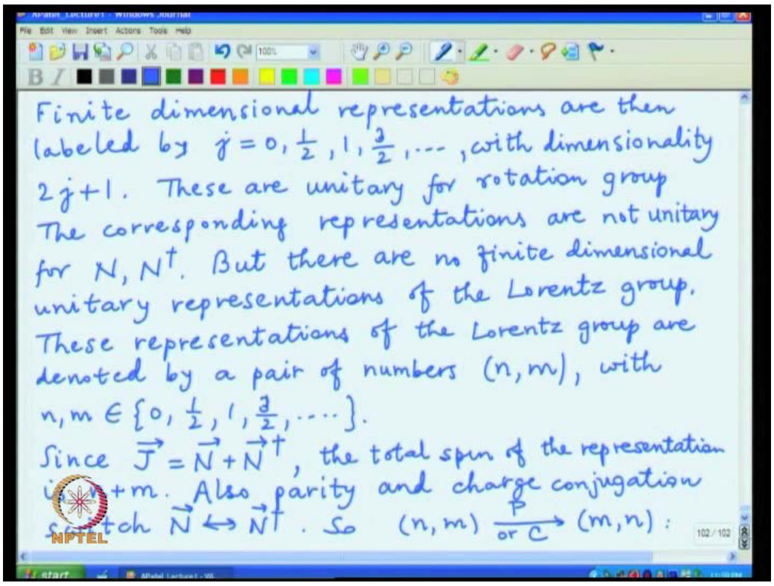

So, we can state that, finite dimensional representations are then labelled by j equal to 0, half, 1, 3-half, etcetera with dimensionality 2 j plus 1. So, there will be matrices of 2 j plus 1 times 2 j plus 1 size. And the allowed values are all these various familiar numbers. The same procedure can be repeated for this N, because we can construct this raising and lowering operators that does not rely on the fact that, the generators are Hermitian or not. In the case of angular momentum, the generators were Hermitian; in the case of N and N dagger, the generators are not Hermitian. But, one can still get all the finite dimensional representations.

These are unitary for rotation group. And the corresponding representations are not unitary for N and N dagger, which are the components of the Lorentz group. But, there are no finite dimensional unitary representations of the Lorentz group. This is a peculiarity, but it is a consequence of the Minkowski metric, which we have to follow;

and that lead to all the signs; and in particular, the factors of i in the definitions of N and N dagger.

We are happy to deal with these non-unitary finite dimension representations in many situations. And I will quickly summarize them. But, before that, we can just write down – these representations of the Lorentz group are denoted by a pair of numbers: n and m I will call them; and the values obviously belong to the same set. So, that gives a complete classification of all the finite dimensional representations of the Lorentz group, because we already know what the finite dimensional representations of angular momentum algebra were. It is a different group, but the algebra happens to be the same. So, one can just take over the results.

Now, one can ask, what these non-unitary representations are useful for. And for that, one can look at some of the properties of what happens to these operators – N and N dagger under various operations, which we are familiar with. One reason is very easy to see since the rotation operator is nothing but N plus N dagger. The total spin of the representation is the sum of these two numbers. So, that is one useful criterion, which we are familiar within non-relativistic physics.

What about the other property, which essentially can be interpreted as a difference between N and N dagger? And that can be easily seen by looking at a transformation, which can interchange N and N dagger. These transformations are again very easy to construct. We know the properties of N and N dagger under the discrete symmetries. Under parity, J does not change, but K changes its sign; it has only one space component and one time component.

So, parity as well as charge conjugation, because charge conjugation interchanges N and N dagger as well, because there is a complex conjugate operation, which is part of the definition of what happens under charge conjugation. So, then one has the simple result that, parity and charge conjugation – any one of them interchange N with N dagger. So, one can take a particular representation and apply either P or C; and one gets a representation, which is the same with the two labels interchange. So, this is the other part, which can be of use to extract the meaning of this representation of the sum, which is invariant under this interchange gives the spin. But, one can flip these indices if they are two, are not equal using the parity or charge conjugation.

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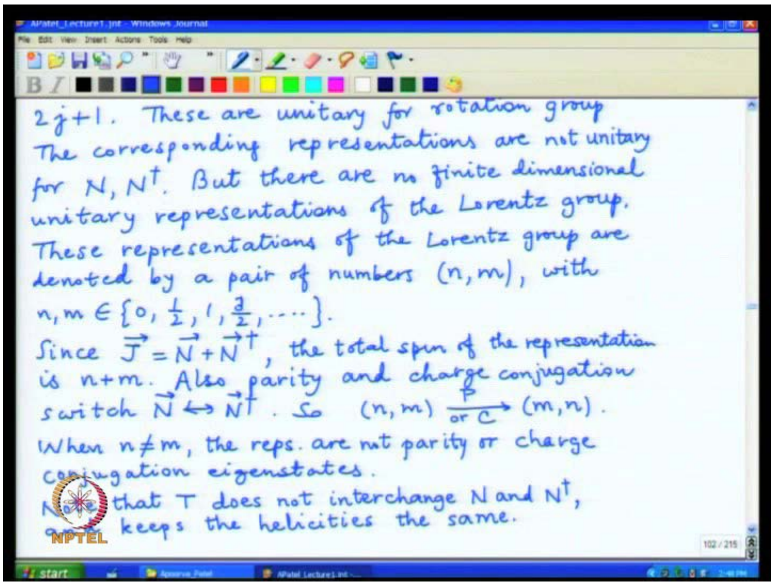

So, when n is not equal to m, the representations are not parity or charge conjugation eigenstates. So, this one I had to keep in mind, because there will be various theories, where one uses different kind of representations depending on whether the parity is a good symmetry or charge conjugation a good symmetry or not. And then one has to restrict oneself to appropriate states of fields, which belong to specific representations. Note that, the time reversal operation does not interchange N and N dagger, and keeps the helicities the same. It is actually based to illustrate all these properties with simple examples.

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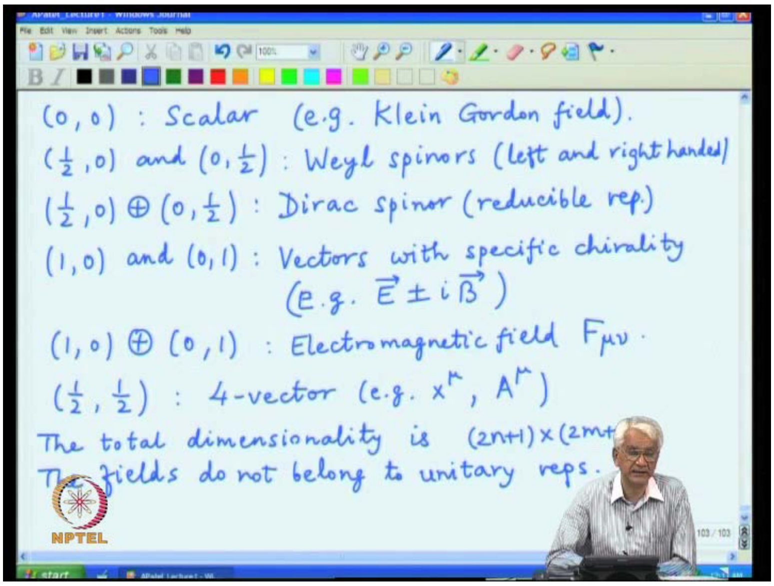

And, I will just list them. The smallest representation is 0, 0. It is a scalar with no spin and symmetric under parity and a charge conjugation. And this will define a field, which can be called say the Klein-Gordon field. Remember that, the field operators in particular – they will require a and a dagger of creation and annihilation operators are not Hermitian. And so they do not have to belong to unitary representations of the Lorentz group; and so one can easily use these non-unitary representations given by these two indices to describe fields with finite number of degrees of freedom. That is a sense in which these representations are getting labelled.

The next one is half, 0 and 0, half. Both are of the same dimensionality. They have two degrees of freedom. And these are related to each other by parity or charge conjugation. We have encountered these objects before. And they represent Weyl spinors; one of the two can be called the left and another one is called right-handed; one can pick a convention depending on the signs of j plus or minus i k. Say the first one is left-handed; the other one will become right-handed.

So, these have two components and they belong to the Weyl representations of spin half particles. The spin is half with the sum of these two. What about the Dirac spinors? We constructed Weyl spinors by decomposing the Dirac spinor into two parts. In the special case, mass equal to 0 and one can put those things back together. So, one has a combination half, 0 plus 0, half, which will become a Dirac spinor. And this is actually a sum of these two representations. So, it is a reducible representation of the Lorentz group. It is not irreducible representation like the Weyl spinors are. So, this is the next step in the hierarchy.

Now, one can go to representations of dimension 3. One can again construct two different combinations, which can be called 1, 0 and 0, 1. The two are again related by chirality and they represent objects with spin 1, but with definite chirality. So, these are actually combinations of vector fields with specific helicity and they can be constructed from the well-known electromagnetic fields. For example, these are vectors with specific chirality. And those components we know from electromagnetic waves that, one can have circular polarizations of the wave with specific chirality.

And, they can be described by the two pairs E plus or minus i B. This make up six components. And by separating in these two particular parts, if one picks only one sign, they will belong to say one part, which is 1, 0 – the left-handed or left circularly polarized photon. And E minus i B will then become a right circularly polarized photon. In nature, we have actually both of them together and they make up again a direct sum of these two photons. And this has now 6 degrees of freedom. So, this is electromagnetic field. These components can be put together in the structure F mu nu. It again is a reducible representation; you can break it up into these two chiralities above. So, this now is an object, which had 3 degrees of freedom and we put them together.

The next one available is the object with 4 degrees of freedom, which can be generated by the two components: half and half. Now, this one is already has a symmetry under parity and charge conjugation, because both these numbers are equal. In addition, it has spin 1. So, this is also a vector field. But, now, it has four components. So, this is a 4-vector and they can occur in various combinations. For example, the coordinates; or, one can even have the electromagnetic potential. They belong to this representation half and half. So, these are actually the most frequently encountered representations of the group. The total dimensionality is product of the two factors: 2n plus 1 times 2m plus 1. And the fields, which we have used, do not have to correspond to unitary representation.

So, we are completely comfortable labelling them in this particular fashion. This set is actually the most useful part in discussing all the components of the standard model. That is the major investigation in a quantum field theory. The one object, which is left out, which is not inside is gravity; and gravity corresponds to fields of spin 2. Literally speaking, it is not the Lorentz group, which is useful anymore, but one can look at the Lorentz group as a tangent space, which is useful for describing weak gravitational fields. And in particular, they can describe the quanta of gravity labelled as gravitons.

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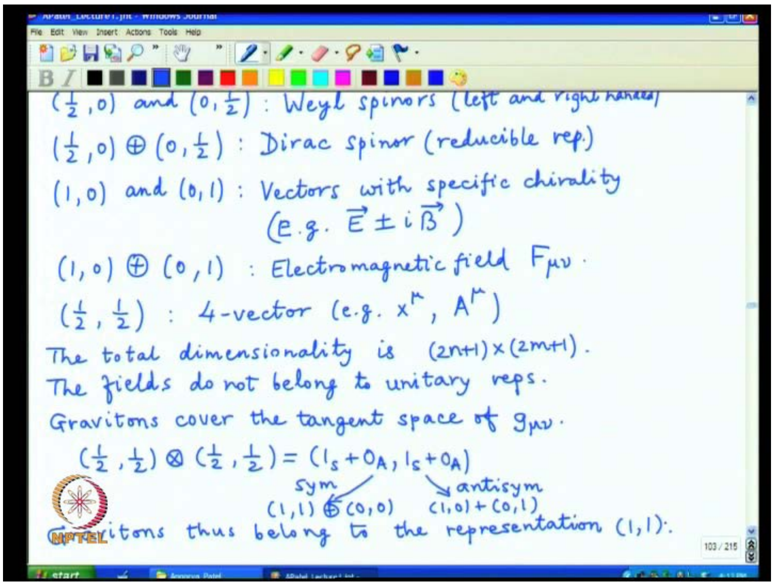

So, the gravitons can be still incorporated in this Lorentz group language. They cover the tangent space of the symmetric metric tensor G mu nu with 10 degrees of freedom. With a little group algebra, we can construct the two index objects as tensor products of one index objects. The tensor product of two half representations gives the symmetric 1 representation and antisymmetric 0 representation. The fully symmetric part with 10 components is 1, 1 plus 0, 0, which describes gravitons and a scalar respectively. The fully antisymmetric part is 1, 0 plus 0, 1, which can describe the electromagnetic field F mu nu as mentioned above. The gravitons with spin 2 thus belong to the representation 1, 1 and are symmetric under parity and charge conjugation. We really do not need gravitons to discuss the usual standard model, but Lorentz group is capable enough to describe graviton as a useful degrees of freedom. So, this is all one can say about working out the completely algebra and solving it to get all representations of the Lorentz group.

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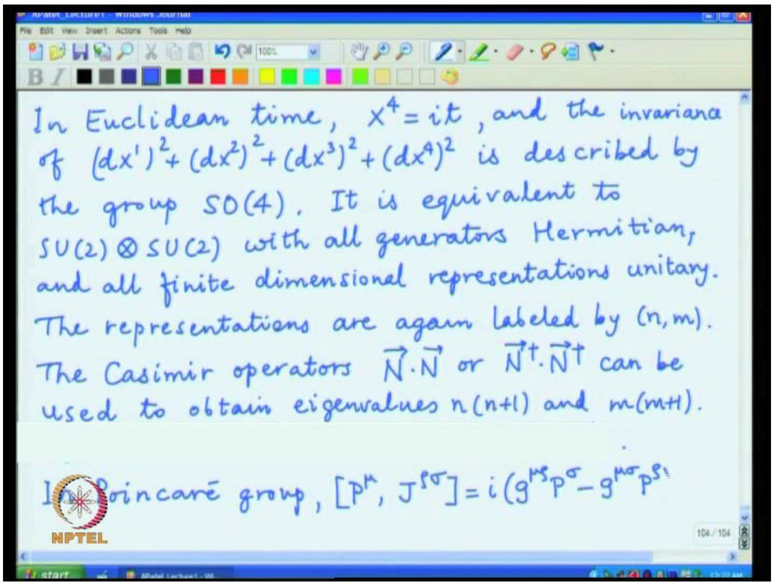

Now, I will mention some of the stuff related with the Lorentz group, which can be obtained by slight extensions of the analyses, which we have already gone through. One common technique used frequently is going to calculations in so-called Euclidean time, where the coordinate x four is used as i times t. And in that particular case, the Minkowski metric can be replaced by an identity matrix, which is much simpler to handle; and the invariance of d x 1 square plus d x 2 square plus d x 3 square plus d x 4 square now does not have any negative sign. And that is described by the group SO 4. And this is an orthogonal group acting on four components, and the norm of the vector is preserved. This is a much familiar or simpler version than the SO 3 comma 1, which we analyse as the Lorentz group.

The Euclidean group can be easily separated the same way as the Lorentz group. The N and N dagger, which we used, had J plus or minus i K. Now, the corresponding Euclidean version will have the i missing. So, it will be just J plus or minus K. The structure or the algebra completely reduces to two independent angular momentum algebra with all the generators Hermitian, because i has disappeared.

So, this is equivalent to the group SU 2 cross SU 2 with all generators Hermitian and all finite dimensional representations unitary. So, this is a so-called Euclidean space-time, where things actually turn out to be simpler, because the group actually becomes a standard lee group with finite dimensional unitary representation and it is much easier to handle. That is also useful many time in studying quantum field theory, because for various mathematical convenience, the time coordinate is extended to a complex plane and one does calculations by rotating from real time to imaginary time. And in that particular case, the symmetry algebra becomes that of the Euclidean group in 4 dimension. One can do the same analysis for this thing completely. There will be again the same two indices: n and m; labelling them. And one can even describe the eigenvalues in this particular case, which become Hermitian operators. They were not Hermitian operators in case of Lorentz group.

The representations are again labelled by this pair of indices. There are various ways to calculate the appropriate values of n and m; one of them is to count the degrees of freedom as I did for various types of fields just listed on the last page. And the other way is to algebraically calculate them. The relevant objects are eigenvalues for the operators, which are the square of N and N daggers. These are the so-called Casimir operators N dotted with N or N dagger dotted with N dagger. And one can evaluate the eigenvalues of these operators as well to find out what are the values of n and m in a specific representation. And we know what these values are. It is n times n plus 1 and m times m plus 1. This statement about Casimir operators is true both for the Lorentz group as well as the Euclidean version. The only thing is whether these operators are Hermitian or not, that will depend on which group you are using. But, the fact that they will have this particular eigenvalues is not a doubt. So, this is all one can say in case of Lorentz group.

To do the full analyses of the inhomogeneous Lorentz group or the Poincare group, now, we have to act to this algebra – the commutation rules between translations and the rotations and boost. We have done the two parts separately. That can be easily added, because you already have all the necessary definitions. So, we need the extra commutation rules between P and J and one can easily work them out. These are nontrivial. The right-hand side is not 0. So, clearly, one cannot use all the generators simultaneously to label physical states.

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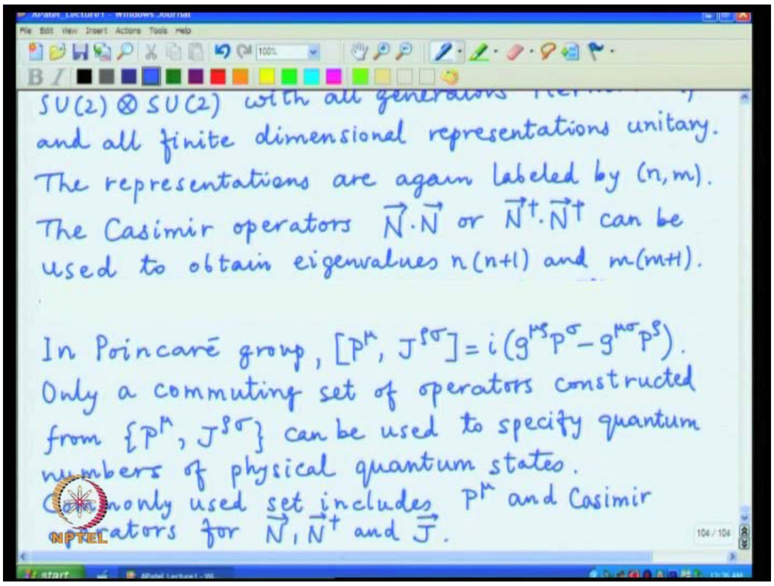

The only commuting set of operators constructed from P mu and J rho sigma can be used to specify quantum numbers of physical states. So, we have to answer the question that, which of these 10 things mutually commute and we can use them. The commonly used set includes the four momentum operators, which mutually commute; no trouble in simultaneously giving that eigenvalues. And in addition to that, these two Casimir operators, which we constructed; and the corresponding operators, which we know from non-relativistic quantum mechanics, which is j square. They are all so-called quadratic Casimir operators, which are used to specify momentum and angular momentum values for these various states.

And, the only thing, which is left for us to answer, is how far we can extend this sort of operators to completely specify the state with where the maximal set of quantum numbers? And we will have to answer that question by constructing the whole set of mutually commuting operators. That we will do shortly in our next lecture. But, right now, I want to digress to another interesting limit, which can be constructed from the same Lorentz algebra.

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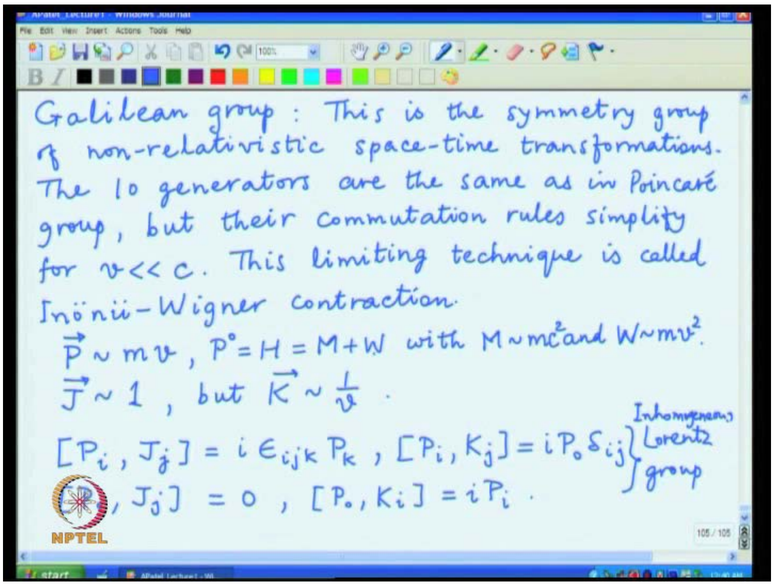

And, that is the algebra of the non-relativistic quantum mechanics and which is also commonly called the Galilean group. So, this is the symmetry group of non-relativistic space-time transformations. It has the same time generators. So, one can start out by looking at what is happening in the Poincare group and take a particular limit, where the velocity v is much smaller than the speed of light in magnitude. And in that particular limit, what happens is some of the commutators simplify in the sense that, the right-hand side may become of the order of v by c; and then one can equate that to 0. And so one has a simplified algebra of the same 10 generators, which will describe the Galilean group. And this technique of starting with a one group and taking a limit to obtain a different algebra is a useful technique in the study of several continuous groups. This limiting technique is called Inonu-Wigner contraction.

What it does is essentially power counting. You take all the commutation rules; scale all the generators appropriately depending upon their magnitudes and basically throughout all the terms, which are of lower order; keep only the leading order terms. And that gives a new set of commutation rules. So, for that, I can write down again the various commutation rules for J, K and P. The parts, which are different involves power counting of various terms.

And, we can now give their various order of magnitudes in this particular case. The power counting rules for the various generators are the 3-vector for the momentum is of the order of m times v. The zeroth component, which is also the Hamiltonian, has two terms M and W; where, M is the rest mass and W is the remaining part of the energy, which will be of the order of m v square. This is m c square. And that is suppressed compared to the rest mass by 2 powers. This includes both the kinetic energy and the potential energy, but both of them are of comparable magnitude much smaller than the rest mass energy. So, this is how the momentum components are going to scale.

The angular momentum will just be of the same order 1. It is actually quantized in units of Planck's constant in quantum theories and has no dependence on the velocity of the particle at all. I should say that, I have omitted Planck's constant in all the discussion so far, because what we are doing is pure mathematics; and in that, the algebra of J or P can be written without referring to the Planck's constant if one wants to bring those back to go to physical units and all the labels for representations J, M, N – they all have to be measured in units of Planck's constant.

So far, I have written them as ordinary integers. But, these things are fairly easy to understand and it is not difficult to reinsert factors of Planck's constant or even factors of speed of light whenever needed. So, the angular momentum actually does not scale in this non-relativistic limit, but the boost operator does scale, because it involves a time component and the scaling is important. It actually goes as 1 over the velocity. And this is the place, which one has to pay attention to in doing the power counting and deciding what terms to drop and what terms to keep. We will clearly see where this K scaling as 1 over v comes from.

And so now, we go back to the commutation rules which we wrote down for the Lorentz group; and I can write it in this particular language of angular momenta and P. So, we had these rules of commutator between P and J. These are just the subset of the Lorentz group relations I wrote down earlier. And this produces P. The similar rule for P and K is the P 0 component with a chronicle delta. And then there are the rules, which now involve, commutates as P 0 with J and K. These are the only non-trivial ones. This is rather trivially 0. One is a time component; another – the space components. And they do not have any mixture, but the commutator with boost operators is still dependent on momentum. So, these are rewritten commutators between P mu and J rho sigma, which I wrote down earlier in the case of Poincare or inhomogeneous Lorentz group.

Now, we can see what happens if one does a power counting for this particular rules. And that allows us to understand why K has to scale as 1 over v. The commutation between P and J does not suffer any change as far as the power counting goes both sides scale the same way. But, for the commutators involving P and K, we have a different rule, because P is order of the velocity while P 0 is of the order of the rest mass. And to cancel the scaling of v, one must take K of the order of 1 over v to maintain this commutation rule, because P 0 is order 1, P is order v, and K has to be order 1 over v, so that this relation is satisfied. The same way, the second relation P 0 is order 1 and K is now order 1 over v.

And, one has to see what now is going to happen to this commutation rule to become consistent. And what happens is that, there is a leading term in P 0, which a rest mass term; that actually commutes with everything; it is just a constant. So, that leading term in P 0 does not contribute to the commutator; what contributes is the sub-leading term, which is order v square, the term written as W. And with that order v square and K behaving as order 1 over v, one has the correct power counting for P, which is of order v. So, this particular power counting is consistent, but it now simplifies the algebra in writing down all the various commutation rules; and one can quickly summarize them. I will list those things next time.