Relativistic Quantum Mechanics
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Lecture - 19
Group representations, generators and algebra, Translations, rotations and boosts

Yes, now we are ready to discuss various algebraic properties of the Lorentz group. Before starting on the explicit structure of the Lorentz group, I still want to give some general description of various features that arise in a group theory. When group theory is applied to quantum mechanics it requires only a specific type of representations to be physical.

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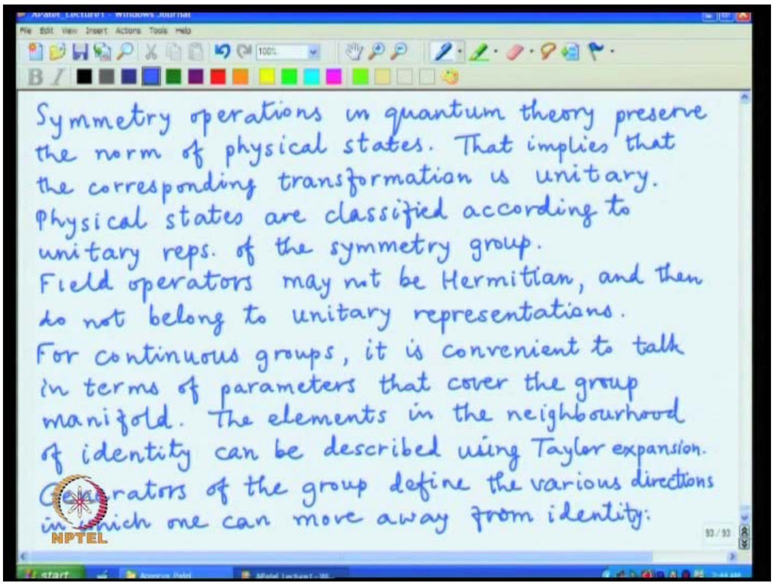

The reason is that symmetry operations in quantum theory preserve the norm of physical states. This is just a statement of conservation of probability which cannot be destroyed by just applying some transformation or changing the basis. Now, this restriction implies that the corresponding transformation is unitary, and so when acting on physical quantum states we have to restrict ourselves to unitary representations of the group. And in other words the physical states of a quantum theory are classified in terms of unitary representations of whatever symmetry group that we are looking at, so physical. On the other hand there may be other objects in the quantum theory which are not physical states; in particular there can be several types of operators which transform states, and they do not necessarily preserve the norm.

Example are the creation and annihilation operators which appear in field theory of many different types, and even in harmonic oscillators they are not Hermitian operators, and the transformations generated by them do not have to be unitary transformations. So, in such cases one may classify these operators according to representations which are more general; they do not have to be necessarily unitary. So, field operators may not be Hermitian and then do not belong to unitary representations, and in discussing Lorentz group, we will have an occasion to see both these properties. There will be field operators which do not belong to unitary representations, and there will be physical states which belong to unitary representations, and this is a generic feature which appears in quantum field theories in general.

Another concept useful is the structure of the group when the parameters describing it form a continuum manifold, and this is a case for many continuous groups including translations and rotations and Lorentz group as well as various gauge theory groups, and so it is a subject of study all by itself, and in this particular situations one talks not about just the group elements but rather the parameters which describe a group manifold. So, for continuous groups it is convenient to talk in terms of parameters that cover the, and so this now generates a description of group elements as belonging to another space. The space will have some dimensions; the parameters may be linear may be non-linear all those kind of features appear, but the most convenient thing to do is to start the discussion of continuous groups in the neighborhood of the identity element.

So, the elements in the neighborhood of identity can be described using Taylor expansion. The 0 s term is of course the identity itself, but the linear term then gives many directions in which the element can differ from identity, and the number of such directions becomes the dimension of the group manifold and the various vectors indicating the various directions on this manifold are called the generators of the group. So, generators of the group define the various directions in which one can move away from identity, and clearly the number of generators equals the dimension of the manifold and in defining the generators one just takes the linear term in the Taylor series.

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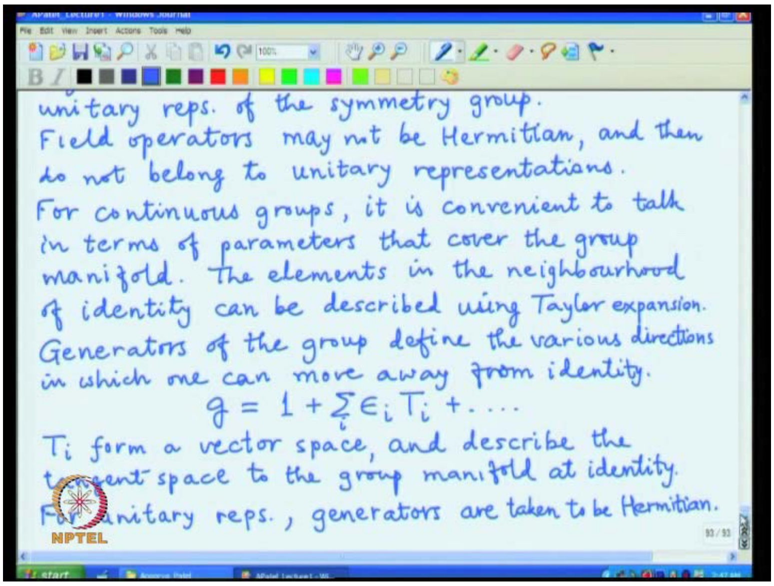

For example, a general group element will be written as 1 plus sum over i some; in fact that is our parameters epsilon i times T i which represent the various generator directions plus the higher order terms, and these generators now form a vector space. So, T i form a and this is an important property satisfied by the generators one can make linear combinations of them as in any vector space, and these vector space is nothing but the tangent space to the group manifold at the location of the identity element. So, these generators actually are very useful concept, and it is quite convenient to take the generators according to useful conventions.

So, in particular case of unitary group elements or unitary representations the generators are chosen to be Hermitian which is a standard mathematical convention that when one expands the unitary matrix about the identity one can write the expansion as 1 plus i times a Hermitian matrix, and it is convenient to take the generators to be Hermitian in this particular case. So, that is a nomenclature, but we will see that the matrices do not have to be always unitary, and in that particular case the generators will not be Hermitian.

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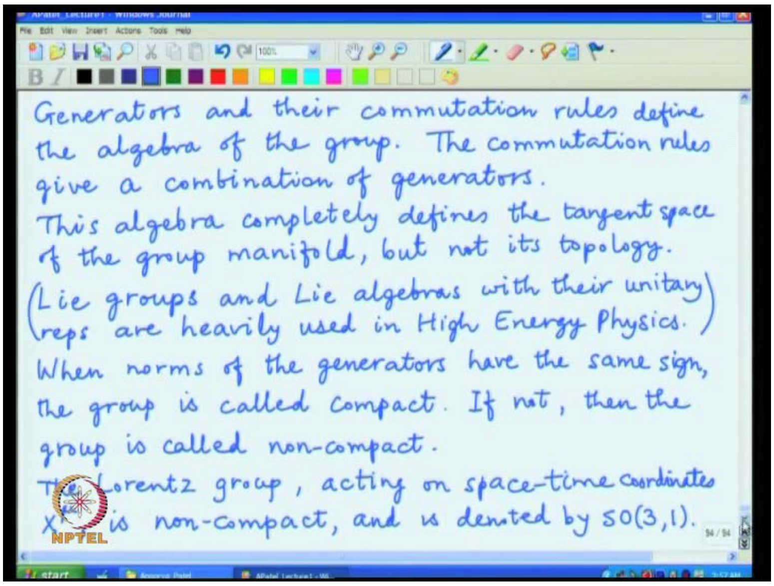

So, now the generators and the linear space in which they act define a whole structure that is called the algebra of the group. The generators are linear operators by constructions, and they may or may not commute. What is required is that the commutation rules always give a closer property of the group which means that the commutator of two generators will be some linear combination of other generators, and that rule which will be different for different groups defines the so called algebra of the group, and this algebra in a sense contains most of the features of the group but not all. So, this algebra it completely defines the tangent space of the group manifold and what is left out is a global property which is often called the topology of the manifold.

And in particular there can be instances where a single algebra can define a structure which can be combined with different topologies to give rise to different group manifolds. So, this much is a good enough definition. I should also mention that this particular type of groups which are completely defined by the unitary representations are called Lie groups, and the corresponding unitary representations are heavily used in the subject of high energy physics in particular quantum field theory of many different types, but those are only restricted to discussing unitary groups. The Lorentz group actually turns out to be not a unitary manifold, and we have to go beyond the machinery of the Lie groups to discuss Lorentz group.

So, what happens in case of a Lorentz group is that Minkowski metric has both plus sign and minus sign, and that is not a positive definite metric, and because of the consolations between the signs you can have objects which have positive norm like d x mu d x mu as well as negative norm, and that is a departure from the topic of Lie groups where if we concepts norms of the generators in Lie groups all of them will have the same sign, and those groups are referred to as compact groups. So, when norms of the generators have the same sign; in particular when the convention is taken that the generators are Hermitians the sign is positive. The group is called compact; loosely speaking compact means that one will have the parameterization of the group manifold where the parameters go over a finite range.

On the other hand if both plus and minus sign appear in calculation of the norms then the group is called non-compact, and in this particular case the parameters which describe the manifold will go over an infinite range. So, this is again some terminology, but it is useful to know, and in particular the Lorentz group acting on space time coordinates. Let us call them x mu is non-compact and is denoted by SO 3, 1 where this three and one refer to a metric with three signs of one type and one sign of the opposite type in the convention which I have used the time direction is the positive sign in the metric and the space is negative, and with this particular definition of the norm the group is orthogonal. In a strict sense orthogonal group will be where all the signs will be of the same type, but here the two different combinations of signs are listed as three and one, and the s in front of the notation denotes a spatial group which means that the transformation generated by this group will have determinant equal to one.

It will preserve the norm of the state explicitly, and this is a nomenclature, but the various features which have gone into it are important to understand when we write down the algebra. The group is non-compact; the parameters will cover an infinite range. There will be plus and minus sign in defining the norms, and that will be reflected in the properties of the generators in particular some matrices will be unitary while some may not be or generators may be Hermitian in some cases and anti-Hermitian in some other cases. So, now let us go and discuss this various algebraic properties one by one.

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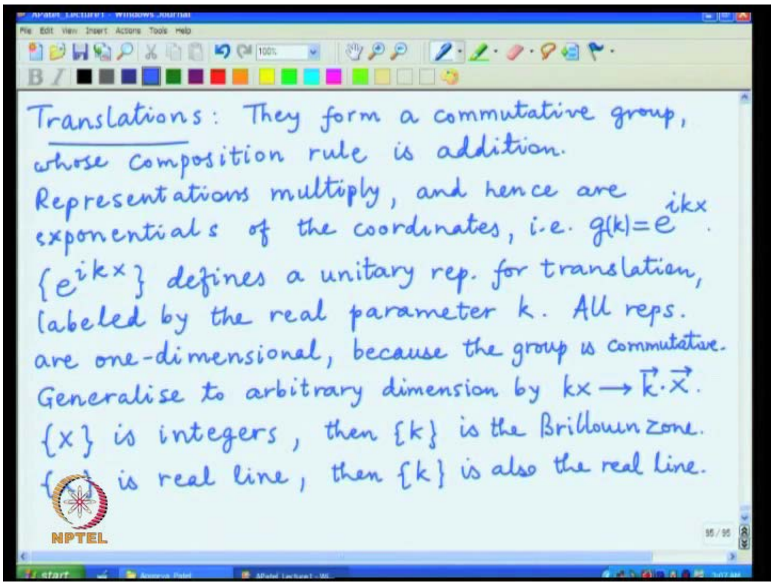

So, the easiest part is to discuss the so called inhomogeneous part of the group which defines translations. So, this form a commutative group whose composition rule is addition; that is what we mean by translating an object from one location to another, take its position at some constant which characterizes the translation, and that will describe its new position. Now our group machinery is defined in terms of a composition rule which was multiplication. So, how do we convert this intuitive picture of translations being additions in the coordinates to representation which will be described by a multiplication rule; I will just put these coordinates in the exponents. So, when we multiply two exponentials the exponents add.

So, in the exponents therefore, we can have the rule for translation, but the representations will be defined by the complete object, and they will be multiplied together. This is in a sense describing the famous transformation defined in terms of Fourier components for any description in a position space. So, the representations they multiply and hence are exponentials of the coordinates that is g which will be a function of k can be written as e raise to i k x. And so now these set of elements e raise to i k x where x covers the whole manifold of translations which we are dealing with defines unitary representation for translation labeled by the real parameter k and so for every value of k you have a new representations clearly it is a unitary.

And all these representations are one dimensional; that is a necessity, because the group composition rule is commutative. So, if you have a general matrix matrices do not necessarily commute, and since we have to follow the composition rule it brings us down to representations which are all one dimensional, or other words the matrices are simple complex numbers. And now one can generalize this thing to any arbitrary dimension by a simple transformation where k x is just converted to k dot x. It is a linear vector space, and one can just add various components; they are all going to be commuting with each other. So, there is no problem in generalizing it in such a simple fashion, and the range of k now depends on the values of x which one is dealing with if the set of x is integers which is what happens when the points are all on a simple lattice.

Then the set of allowed value k is the corresponding Brillouin zone. On the other hand if x is the whole real line then the set of k is also the real line. So, in very simple terms one has a complete description of all possible representations of the translation group just the Fourier coefficients e raise to i k dot x, and depending on the values available for x one can easily figure out what are the values available to k, and this algebra can also be written in terms of the momentum operator which is quite commonly used in a quantum mechanics, and that defines the role of the generator rather explicitly.

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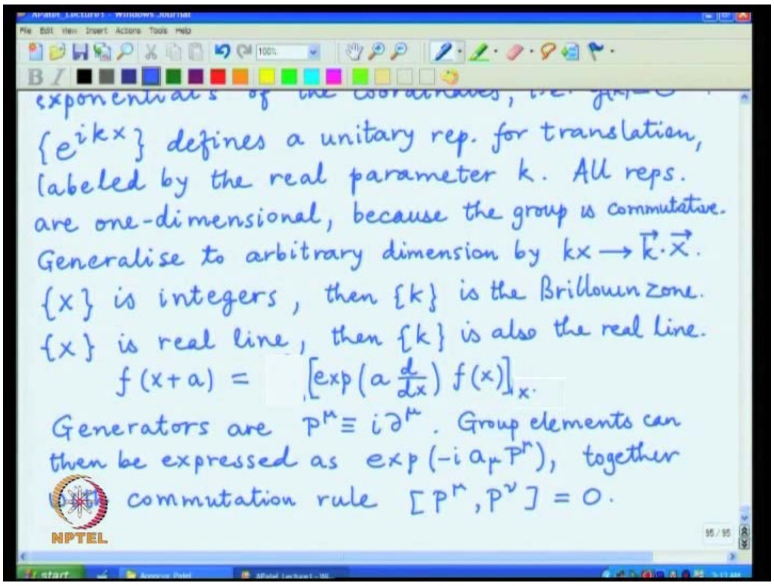

So, one can have a function say f of x and one wants to study it in the neighborhood of the point x, and that can be easily written as a Taylor series, but now I will write a Taylor series in a simple form as a exponential of the derivative operator. And that now acts on f of x, and one must evaluate these derivatives at the point x. and this definition can now be looked upon as an expansion of the function in the neighborhood of point x; what we need for translation is expansion about the identity element which corresponds to x equal to 0. And one can now easily look at the various terms as the various generators.

There will be one generator for every direction of the translations, and these generators are nothing but these operators d by d x in an abstract language. So, the generators are the so called momentum operators which we define in quantum mechanics as the derivative operator, and then the arbitrary group elements can then be expressed as exponential of minus i a mu p mu, and this can now give an abstract definitions of the group. There is a generator; there is a prescription to give any unitary representation of the group in terms of exponential of the generator, and we have the simple commutation rule which is all these generators commute with each other.

And that allows us to construct any group element in a general sense without referring to a specific choice of coordinates, but in practice we will always be picking up a coordinate system and all these things can be turned around to construct specific operations required for changing the coordinate system. So, this was the translation part of the group which is rather easy to deal with it. It is a commutative subgroup by subgroup I mean a subset of the Lorentz group which is a group by itself. The non-trivial part of the Lorentz group is actually the other set of transformations corresponding to rotations and boosts.

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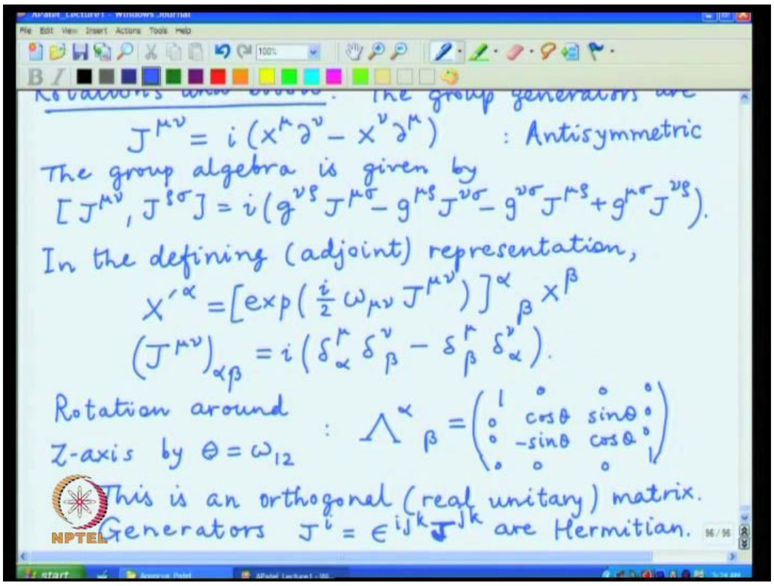

And that is what we have to turn next in terms of the algebra. This again are continuous transformations which can be described in terms of the generators, and now with the terminology already introduced for the translation group it is very easy to write down the group generators, and they are defined in an abstract senses J mu nu equal to i times x mu d nu minus x nu d mu, and this combination is nothing but a simple extension of the well known property of definition of the angular momentum as r cross p. In particular the antisymmetric structure buried inside here is nothing but that of a cross product and given this definition now it is very easy to work out the various commutation rules.

So, the group algebra is given by the commutations rule, and this can be easily evaluated by just working out the derivatives acting on the coordinates and clearly because the derivative is first order coordinates also appear in the first order. So, the result basically produces the structure of the same type, and one can work it out explicitly to be of the form where if you figure out just the first term the rest of the term can be written following the total antisymmetry of the indices involved. So, this gives the commutators of J also proportional to J, and that is the property of the algebra ensuring that the whole machinery closes; one does not get anything new, and one has a complete description of the tangent space to the group manifold by these rules.

Now keep in mind that the metric tensor appears all over the place in these calculations, and the signs will change depending on whether their index is up or down, and we have to keep track of it to be able to write down the transformations explicitly and not make any mistakes. So, given this structure now what we can do is look at the specific defining representations, how this generators become when they act on the coordinates explicitly in the defining representation which is also often referred to as the adjoint representation. The arbitrary change in the coordinate will be described as x prime alpha is exponential of i by 2 some parameters which indicate the amount of change multiply by the corresponding generator, and we have to keep track of the indices so that they are contracted properly.

So, this particular operator will have to be defined with one index up and the other index down, and that is where all the caveats of the Minkowski metric will come in. And now one can look at what happens to the coordinates in our usual basis when there is a rotation and when there is a boost and from that looking at the infinitesimal transformation deduce an explicit structure of what this abstract operator J mu nu will look like in this adjoint representation, and that relation turns out to be a simple combinations of chronicle deltas in this particular scheme of writing, but if one needs to a raise or lower a tensor index, the matrix signs will automatically appear as this is required in writing down the transformation between x prime and x.

And we know the well known situations, and I will give an illustration of the particular matrices described by the structure as two cases; one is a rotation around the z axis by an angle which in this notation will be called omega 12. And in this particular case the transformation matrix which is lambda in our notation with one index up and the other index down has the form of identity for the indices corresponding the time and the z coordinate while cosine theta and sin theta factors for the x and y coordinates which change in the rotation around the z axis. So, this is a familiar matrix from the rotation transformation and the normalizations have been chosen such that the angle is precisely this omega 12. The reason for one half appearing in the definition between x prime and x is this product omega mu nu J mu nu.

Each combination gets counted twice mu nu in some particular order, and then nu mu in the opposite order contribute the same result, and that cancels the factor of two just a consequence of a simple antisymmetry of mu nu in the definition of both omega mu nu and J mu nu, and this is an orthogonal matrix. Orthogonal in particular means that it is real and unitary at the same time, and this is a familiar result known from the study of rotations that we have this particular structure. Since the matrix is unitary if one finds the corresponding generators for rotations for the infinitesimal transformations the generators are Hermitians. Generators are denoted now by a single index in our common terminology compared to this two index notation, and they are conveniently defined using this antisymmetric tensor epsilon i J k.

Just take this two index tensor J j k and multiply by epsilon i J k to get a single index tensors J i. And these are now restricted where both J and k as well as i are only the space directions, and when dealing with space directions I will not make any distinction between upper indices and lower indices. Both of them will mean the same thing which means that it is a conventional to choose the lower and the upper indices to have the same value, because there is no mixture in the matrix signs of the space directions themselves.

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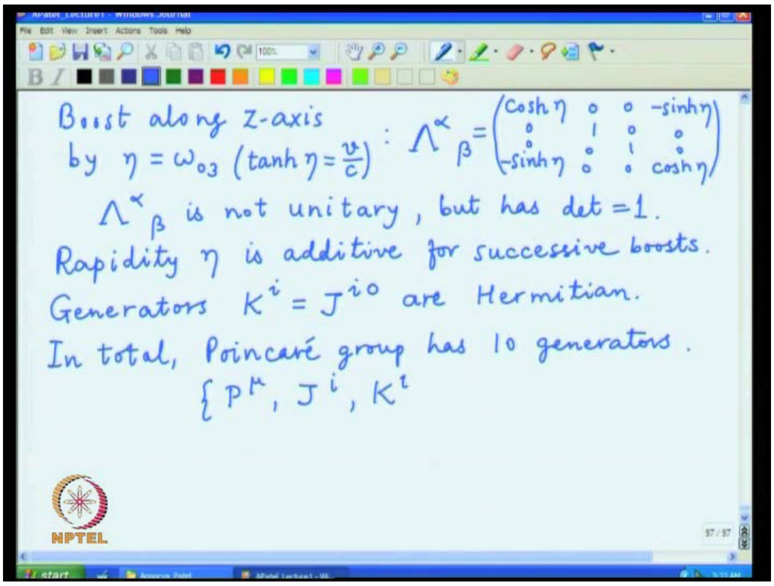

So, this is the rotation part; one can now also write down the boost part again familiar from the Lorentz transformation characterized by the parameter eta, and now I will label it with this index omega 0 3, and the definition of eta is that hyperbolic tangent of eta is v over c where v is the boost velocity, and with this particular convention the matrix has the same form as in case of rotation except that the trigonometric functions are now replaced by hyperbolic functions. And the action of that boost along z axis mixes the time and z components while leaving the x and y components untouched; it is a completely complimentary subset of coordinates compared to a rotation around z axis, and the difference in the structure now appears explicitly that this matrix is not unitary anymore.

They change from trigonometric to hyperbolic function essentially is responsible for the change in that behavior, and that change is essentially connected to the opposite sign in the Minkowski metric when time and space coordinates are mixed compared to when only two spatial coordinates are mixed. So, this lambda alpha beta is not unitary, but its determinant is still one, and if one attempts to parameterize it in terms of an infinitesimal transformation instead of getting Hermitian generators one gets anti-Hermitian generators coming out as an expansion with the same convention as in case of rotation. So, that is the standard form, and I should point out that it is quite convenient to describe boosts in terms of this quantity eta instead of the velocity, because this quantity eta which is also referred to as rapidity is an additive quantity.

So, one can apply successive boost where eta will just add very similar to apply successive rotations where the angles will just add, and that is the reason for choosing this hyperbolic transformations, and it also keeps track of what is really going on in the transformations. One can see the non-compact nature of the group in this matrix as well, because these elements hyperbolic cosine and sine are essentially unbounded. They will go all the way till infinity for large values of eta and the parameterization in terms of eta in that sense is non-compact. So, it is this boosts which make the Lorentz group’s non-compact nature quite explicit, and one can now write it is generators as well in a specific notations. The generators can be defined as two index notations with one of the index being the time direction, and that will give three different boosts and the conventional notation for that is a just one index tensor denoting the direction of boost.

And these particular objects are Hermitian by their own definitions; what makes these matrices non unitary is actually using these generators with one upper and one lower index provided one treats both the indices of the same type one can have a definition which respects the Hermiticity property in the abstract sense which was present in the definitions of this J mu nu. It is explicitly constructed so that this is an algebraically Hermitian operator in the same way as momentum is a Hermitian operator for any value of the index. When it is acting on a particular state one can do integration by parts to make this act on the complementary state and corresponding sign flip is absorbed by i going to minus I; that is what proves that this a Hermitian operator. So, one now has a complete description of Lorentz group and its time generators. So, in total Poincare group has time generators which we have seen, and we will now work out the consequences of the algebra of these ten generators in the next lecture.