Relativistic Quantum Mechanics
Prof. Apoorva D Patel
Department of Physics
Indian Institute of Science, Bangalore

Lecture - 18
Groups and symmetries, the Lorentz and Poincare Groups

Now I am going to discuss a new topic that is the study of Lorentz symmetry. It plays a very important role in the subject of relativistic quantum theories; in particular it is an aspect of relativity which when combined with quantum mechanics produces very useful and important constraints, and all the theories various quantum field theories which we have constructed have this Lorentz symmetry built into them from the beginning. So, what is this Lorentz symmetry and what are its consequences? To understand it in a great detail we first need certain basics, and I will start with definitions of what is meant by a group.

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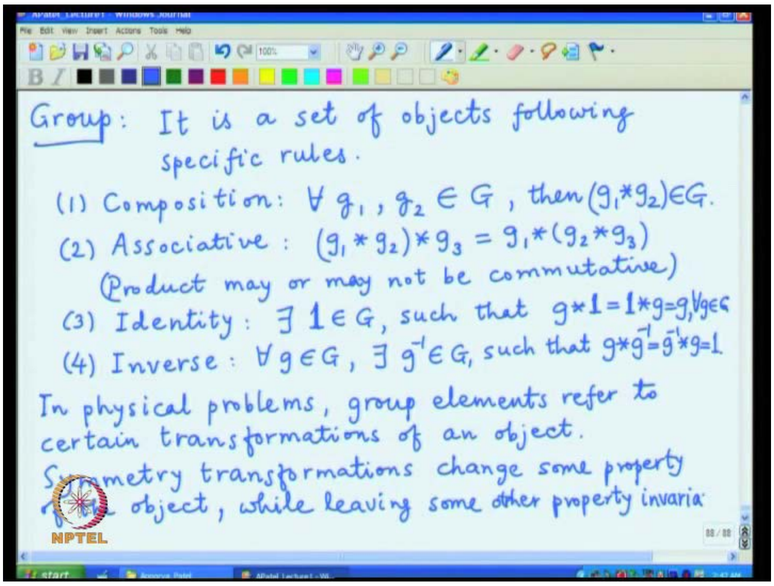

A group is a mathematical structure essentially defined by its algebraic properties, and it follows a certain set of rules which characterize the group of various types. So, it is a set of objects following specific rules, and I will just list them one by one. The first rule is a law for composition which states that if there are two elements g 1 and g 2 in the group, then their product element which is denoted by g 1 times g 2 also belongs to the same set. This product I have denoted here by the symbol star, but many times that symbol is omitted for the sake of gravity as in the case of usual convention for multiplication. So, this rule basically defines a property of closure that you can take any two elements, and from that two you can obtain a third one and that also must belong to the set.

There is a second property of this composition rule which is that the composition is associative; that means that g 1 star g 2 star g 3 can be also constructed by following different steps and this associativity ensures that when there are multiple elements present in which order we take the products by picking two out of them at a time does not matter. On the other hand the group elements do not have to be commutative which means g 1 star g 2 does not have to be the same as g 2 star g 1, and both kind of groups exist, and some of them are called commutative groups, and some of them are called non-commutative groups just depending on the property of the product.

The third property is existence of an identity element which means that there exists a particular element inside g typically denoted by 1 such that when it is combined with any other element you get the same element back. So, for every element in g it is kind of invariance statement that identity provides, and the last property in the list is an inverse that for all elements of the group there exists an inverse element in the group as well such that g combined with g inverse gives one in either order. And this sort of properties is a complete specification of what is mathematically defined as a group. It is a sort of simple list, and that is precisely the reason that it turns out to be very powerful in its predictions in many different branches of a science.

So, let us now look at in which context this theory of groups is used in physical situations. So, in physical problems the group elements refer to certain transformations of the object under considerations, and so you take the object, apply a particular transformation, and it changes to a new form, and if all such transformations obey the mathematical properties described above which refer to composition of one transformation with another, then those set of transformations will be called a group. Now this property of transformations is most useful when the transformations change some property of the object, but also at the same time leave some other property of the object unchanged.

And so one can say that the object has changed in some manner, but also remained invariant in some manner, and in this sense one can talk about a specific object according to the part which has remained invariant and talk about the transformation which has changed some other part; such situations in a physical problems are often labeled by the word symmetry. So, symmetry transformations change some property of the object while leaving some other property invariant, and when we deal with such transformations then they define so called symmetry groups which are most useful in study of physical problems.

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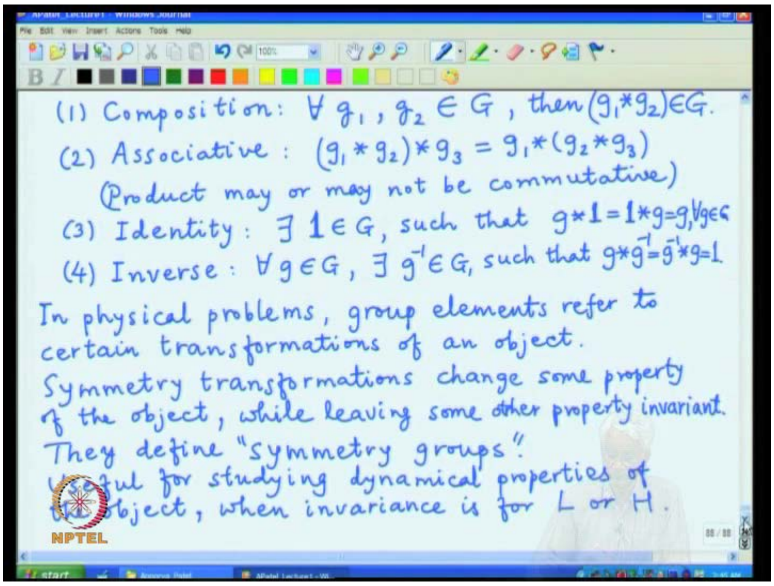

Particularly in situations where the transformation refer to some dynamical properties of the object when the invariance is for the dynamical equations of motion or equivalently for the Lagrangian or the Hamiltonian, and to problems of physics which we will encounter have the symmetry groups acting as invariances in this particular sense. And it is in a sense similar to a person changing clothes, the transformation which occurs when one looks different by wearing a different set of clothes at the same time something is unchanged. So, you can say that it is the same person wearing different clothes, and that helps identify the person as well as describe what are the changes that have occurred, and in physical problems such properties lead to powerful consequences, and that is the reason that group theory is important in the study of physical problems.

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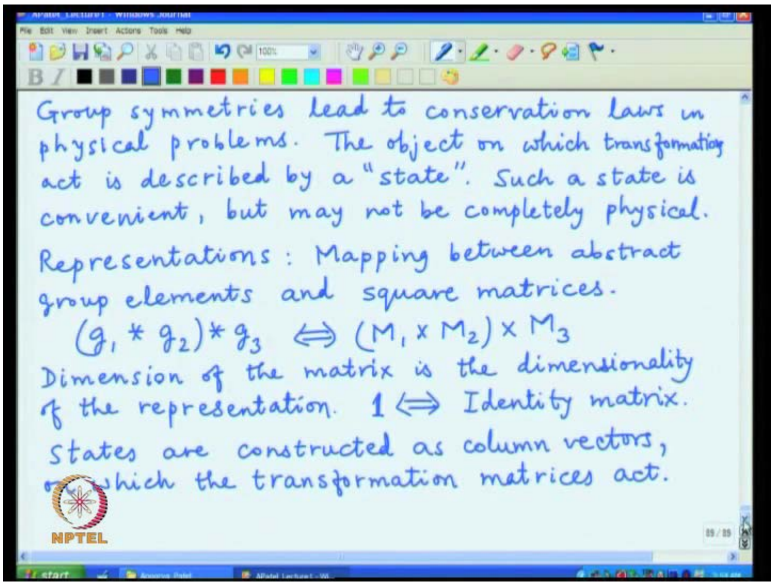

So, these group symmetries lead to conservation laws in physical problems. This is a powerful consequence, and we will see many examples of it. I cannot immediately describe how the symmetries and conservation laws are connected, but it is essentially a mathematical deduction, and this implies that there are quantities which are conserved, and one can connect them even backwards to mathematical structure obeyed by certain groups, and this will describe that what kind of transformations can occur while leaving some particular objects constant. Mathematically the set of transformations are enough to construct a group and describe its property, but in physical situations one more ingredient is typically inserted, and that is a definition for a description of the object on which the transformations act.

So, the object on which transformation act is described by a physical state, and the transformation is an operator that acts on the state. Such a state is convenient for understanding what is going on, but it may not be completely physical in the sense that one can observe this state in all its generality; such a thing is often true in quantum mechanics, whereas, state would be described by a wave function but not all aspects of the wave function, may be physical such as its global phase. One can only talk about transformations described by groups in full generality, a state turns out to be a convenient auxiliary concept which can be added to help mathematically formulate the problem, but at the end it may be filtered out when one is directly dealing with the consequences of group symmetries. So, this is much about the general philosophy of the groups.

Now this description does not give an explicit mathematical structure, and that mathematical structure can now be constructed from this set of properties, and that structure is referred to as representations of the group. Now these representations are nothing but a mapping between these abstract group elements, and what can be called as matrices in particular the matrices have to be square matrices, and if one can find for every element a particular matrix, so that the matrix multiplication and the group composition rule obey the same structure. So, that when one says g 1 star g 2 is equal to g 3 there will be a corresponding matrix multiplication rule as well such that this product can be written as ordinary matrix product.

And when one can find such a mapping the corresponding set of matrices are said to form a particular representation of the group; the dimension of the matrix is also called the dimensionality of the representation, and obviously, the element one or the identity is always mapped onto the identity matrix that leaves any other matrix invariant. So, this offers a very specific structure that now instead of writing abstract elements, one starts writing down matrices and multiplying them together, and they correspond to various transformations in linear algebra; the states are then constructed as column vectors on which the transformation matrices act.

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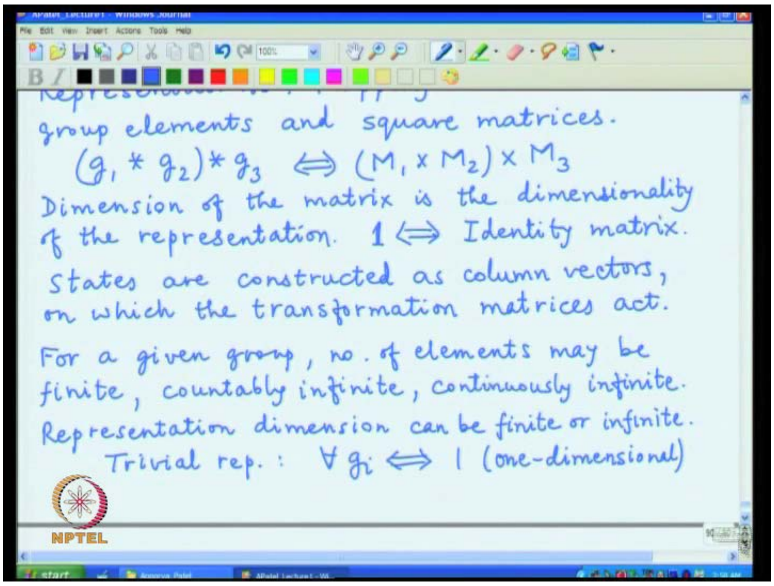

Now for a particular group the number of elements may be finite, they may be countably infinite as in case of integers, or they may be even continuously infinite as in case of real numbers, and similarly the representation dimensions it can be finite or infinite; in particular there is always a so called trivial representation where all the group elements g i the corresponding element is just one. It will trivially obey all the composition rules, and so the properties are all satisfied, but nothing much happens in terms of its physical constraint, and sometimes this is also referred to as a one dimensional trivial representation of the group.

Any other representation will be referred to as a nontrivial one, and for a particular group there will be some nontrivial representations if there has to be a correspondence to a nontrivial physical problem. One more label which is used typically in describing various representations; if the matrix has a particular set of basis where the matrices become block diagonal instead of being completely filled then such representations are called reducible.

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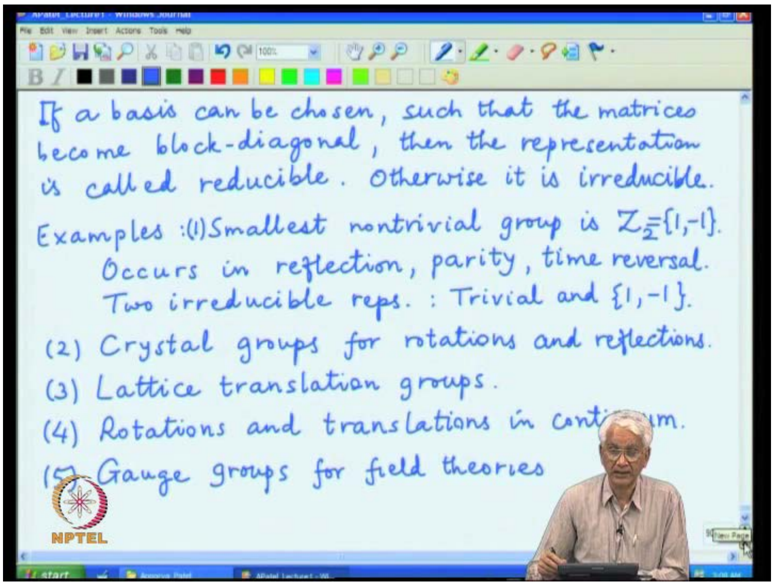

So, if a basis can be chosen such that the matrices become block diagonal, then the representation is called reducible; otherwise, it is called irreducible, and mathematically it is the list of all irreducible representations which give a complete characterization of the group. From this irreducible representation one can construct reducible representation by just using them as blocks, and so the theory of groups often boils down to characterizing all its irreducible representations. And each irreducible representation gives a particular transformation rules for a certain type of objects, and in that manner one can understand different properties of the objects and under which category the various objects fall into as well.

So, this is a set of abstract rules and definitions, and now let me give some simple examples of the various type of groups which we have seen already may be in a different language. So, the smallest nontrivial group is a set of two elements often referred to as z 2, and it can be represented by two numbers plus 1 and minus 1; the square of either of the elements is identity, and that closes the composition rule for the group completely, because one of the element is identity itself. And this group occurs in many physical situations where repeating the operation twice leads one back to the original state occurs in, say, reflection, two reflections take us back to where we started the examples which we have included earlier parity and time reversal; they are also operations if you perform them twice.

You go back to the original configuration and charge conjugation always has the same property again. So, this is a useful group. It has this simple property, and it has basically two irreducible representations. Both are one dimensional; one of them is a trivial representation, and the second one is these two elements which I denoted by these numbers themselves, because when you multiply them they will produce all the corresponding multiplication results. So, this is the simplest group one can talk about, and then one can talk about many other operations which may occur with a finite set of objects or an infinite set of objects. For a finite set of object there are many of the so called crystallography groups for rotations and reflections.

They are certain discrete operations which can be combined together, then they are groups for lattice translation which are infinite on an infinite lattice they will roughly correspond to a set of integers on a periodic lattice, then one can have continuous groups as well. So, rotations and translations in continuum where we have our usual Cartesian coordinates and one can talk about various angles or distances over which one performs a certain transformations, and there are many such instances which we will come across. And one particular instance which is useful in the context of gauge theories are the so called gauge groups for field theories, where the transformation rules will be what will be called a gauge transformations, and they will change the states in a particular way and lead to certain consequences when the theory has a gauge symmetry in particular various types of conservation rules that will follow. So, this is some background on group theory and how it shows up in physics.

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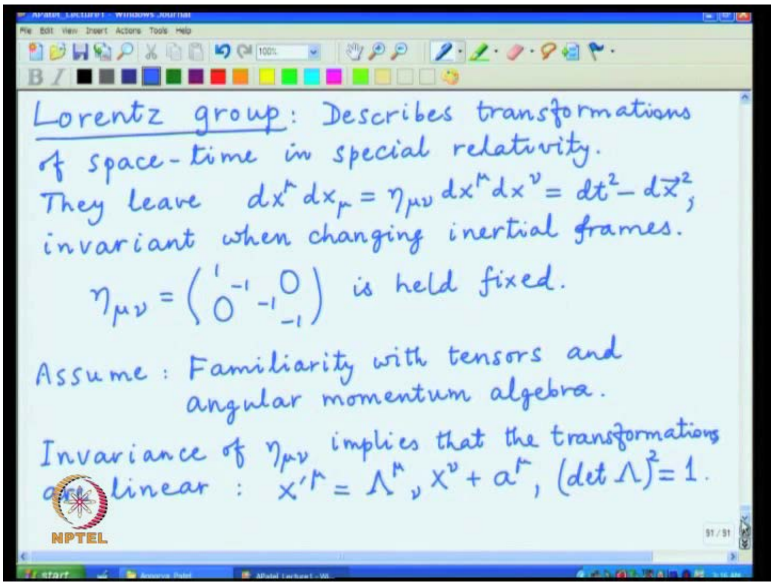

Now I can focus onto the particular group that we want to analyze in detail and that is the so called Lorentz group. Now this is a set of transformation which is symmetry operations of the theory of special relativity. So, they describe transformations of space and time in spatial relativity. Mathematically, it can be defined as transformations between inertial frames, and the characterization of this inertial frames is that they leave the combination d x nu contracted with itself, and in explicit time and space notation it can be written as d t square minus d x square invariant under change of frames not only this vector d x nu square remains unchanged; this metric so called Murkowski metric for which I have chosen the convention of plus 1 for the time and minus 1 for the space direction is held fixed under this changing of frames. So, in all the frames the metric has the same particular value.

So, such transformations are the set of elements that form the Lorentz group. Now certain consequences follow immediately from these properties, and I will assume a certain amount of familiarity in discussing them, and this is the familiarity with tensor language where all these vectors appear with various indices and corresponding rules for raising and lowering them and also the algebra of angular momentum operators. So, what does follow from these particular properties? In particularly the metric eta mu nu is a tensor with two indices. So, it has a transformation rule when one goes from one frame described by coordinate x to another frame described by coordinate x prime, and the metric will change by linear operators which will have the structure d x prime by d x, and if the metric is supposed to remain invariant then these factors of d x prime by d x have no other option but to be constants.

And so we have an immediate consequence that the transformation are linear explicitly x prime is equal to some particular operator lambda mu nu times x mu plus another vector a mu where both lambda and a do not depend on the space time coordinates. So, this will make d x prime by d x factors which appear in transformation of the metric constant, but because the metric has two indices there will be two such factors appearing when one write downs a transformation rule for eta. And the fact that eta remains the same means the product of two such lambda matrices has to be such that it produces a factor one, and that leads to a constraint that it is a linear transformation.

But in addition the square of it characterized by the determinant is equal to one, and this then becomes a definition of the Lorentz transformation which is equivalent to writing down what is held invariant many times, people start from describing this kind of linear transformations and constructing the machinery of Lorentz group starting from that. But this particular definition also makes it obvious that there will be some transformations which are peculiar, and they will be characterized by the sign of determinant lambda.

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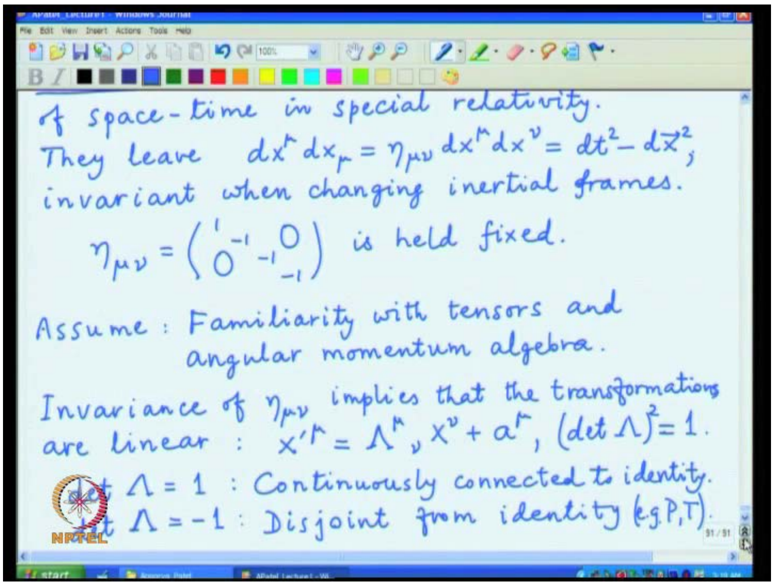

So, the transformations where determinant lambda is equal to one they include the element identity which is just a constant identity matrix for lambda. So, these are continuously connected, but there is a other part of this sort of transformation where determinant lambda is equal to minus 1, and they are disjoined set of operations from the element identity, and in particular the operation of parity and time reversal fall into this class where parity will change sign of three space directions, and time reversal will change the sign of one time direction, and clearly under that transformation the determinant is minus one. So, everything is kind of included over here, but for the purpose of study now one can break up this whole structure of Lorentz group into smaller components that can be studied more easily, and that classification goes as follows.

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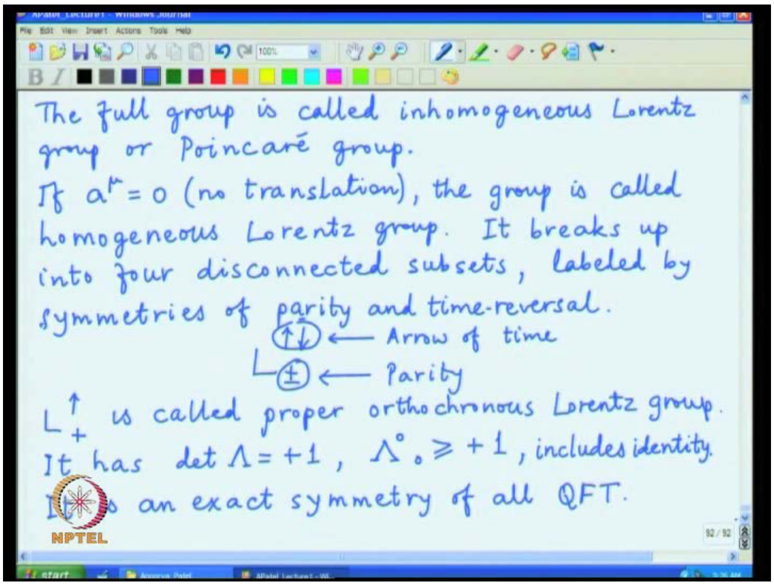

So, the full group which I defined by this transformation rules for x prime and x being linear that group is called inhomogeneous Lorentz group or often also by the name Poincare group. If the parameters a mu are set to 0 which means there is no translation in the coordinate system, then the group is called homogeneous Lorentz group or many times just Lorentz group and now if the sign of determinant is taken into account as well. So, then this homogeneous group breaks up into four disconnected subsets labeled by symmetries of parity and time reversal; in particular the various sectors are denoted by appropriate subscripts and superscripts. So, there is an upper index which gives the arrow of time.

It has two possibilities forward or backward and a lower subscript which denotes the operation of parity whether parity is included in the transformation or not, and all together that makes up four possibilities, and the particular sector on which we are going to focus on is denoted by L forward in time and plus in parity, and that has been given the name proper orthochronous Lorentz group. It consists of the transformations which are continuously connected to identity, and the other sectors can be trivially obtained from this by multiplying appropriate signs of parity or time reversal. And this group can be now studied using the algebra which we will soon construct, and the characterization of this sort of transformation is given by determinant lambda equal to plus 1 and the 0 0 element which defines the arrow of time being greater than one.

And what we have learnt by many experiments and theoretical investigation that this particular group is an exact symmetry of all quantum field theories that we have found to describe nature. And so it has the consequences which are obeyed by all the various particles and all the various interactions that we discover in studying various parts of high energy physics. So, that is the importance of this particular group.

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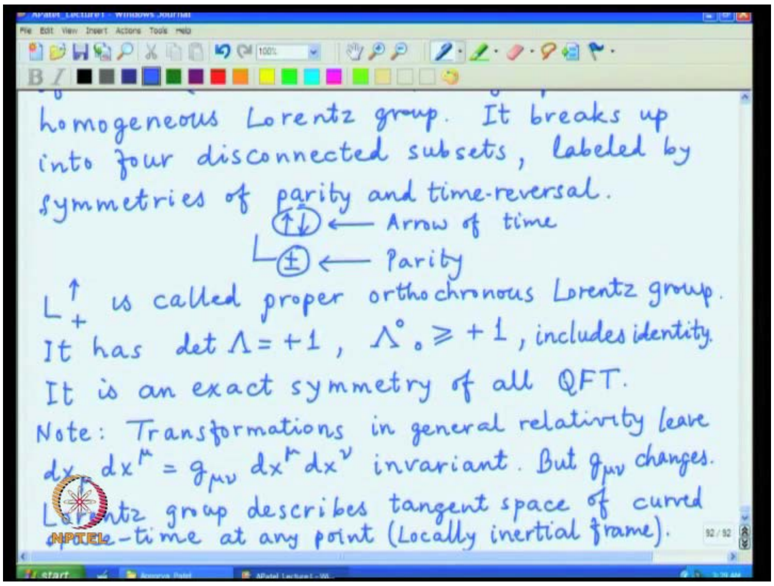

And as a side comment let me also mention a particular extension of quantum field theories or where we go in dealing with different kind of theories in physics. So, there is a obvious extensions of the linear space time transformations to arbitrary space times transformations and those form the description of general relativity, and this arbitrary transformations also have the property that d x mu contracted with itself remains unchanged. But the characteristics of general relativity is that the metric itself is longer constant; it can change when going from one coordinate system to another, and that gives rise to the nature of space time which is often said to possess curvature, and in that particular sense the so called Lorentz group and a linear transformation correspond to the tangent space that can be constructed at any location in general relativity.

So, Lorentz group describes the tangent space of the curved space time of general relativity at any point, and that is useful in studying general relativity as well, because this tangent space is referred to as a locally inertial frame in general relativity. And that frame allows simpler constructions of various quantities relevant to general relativity in an arbitrary setting of space time. So, this much is the background structure and definitions of Lorentz group. Next time we will study the consequences which follow from these mathematical definitions.