Relativistic Quantum Mechanics
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Lecture - 15
Time reversal symmetry, The PCT invariance

Today, I am going to discuss the properties of time reversal symmetry and other consequences related to it. Last time, I mentioned that there were general classifications of a possible symmetries in quantum mechanics by Wigner, and which he showed by a theorem that either the symmetry is unitary and linear. And if not it has to be antiunitary and antilinear there is no other possibilities. Now, unitary and linear symmetries are quite familiar as a matter of fact most of the symmetries, which we deal within classical mechanics or quantum mechanics they are unitary and linear.

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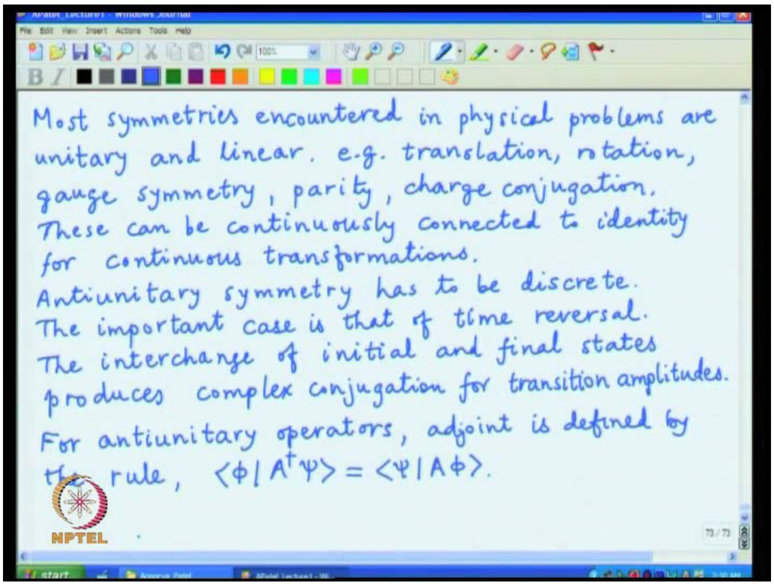

For example, the symmetries of say translation, rotation, gauge symmetry which are all continuous symmetries or even discrete ones like parity and charge conjugation, which I have dealt with earlier in the course. And a generic feature of this unitary symmetry is that in most cases we can define operators which generate these symmetries. And they define transformations which are continuously connected to identity; an identity can be just treated as a transformation which does not do anything at all.

So, these can be continuously connected to identity for symmetries which are continuous on the other hand the anti unitary symmetry is peculiar. Because it converts all the states to their complex conjugates transition amplitudes also to their complex conjugate. And there is no way that can be treated as a continues operation it is a big jump going from a complex number to its conjugate. So, the anti unitary symmetry it has to be discrete the continues symmetries are connected to identity by definition. And taking the deformation parameter to 0 for discrete symmetry there is no such analogue; continuous symmetry can also be discrete. For example, the symmetry which we encountered earlier the symmetry of parity transformation, but they do not always have to be discrete there can be continues versions as well.

On the other hand anti unitary symmetry cannot have a continues version at all. And this has only one important example and that symmetry is which I want to discuss and that is the symmetry of time reversal. The important case is that of as a matter of fact we do not know any other anti unitary symmetry which does not involve time reversal in some form or the other. And the anti unitaries nature is kind of built in time reversal, because the interchange of initial and final states; which has to be part of transformation that reverses the arrow of time, and produces complex conjugation for the transition amplitudes. And so it is indeed the result which we expect for any anti unitary symmetry.

Now, let us define this process explicitly in terms of various quantities involved; one can also keep a little thing in mind that the ad joint process for defining operator is little bit different. So, if we have anti unitary operators the joint is defined by the rule which can be said as phi acting on a dagger psi is the same as psi acting on A phi; in the normal unitary operators the ad joint is define as the same object on the left side equals A phi acting on psi. And what is happening here is extra operation of complex conjugation which flips the order of the initial and final states; as for as various coordinates are concerned one can easily now write down that various transformations for time reversal. The space and time coordinate change in a rather straight forward manner. Then the sense of the time reverses nothing happened to space.

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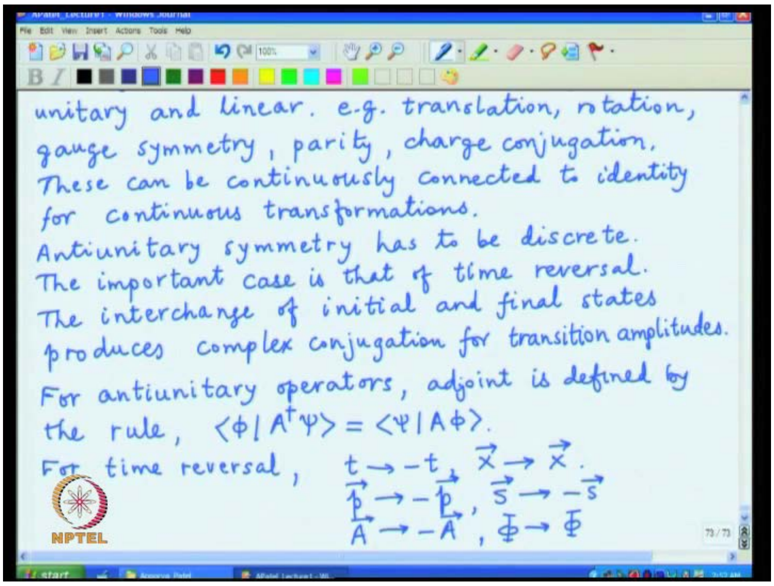

On the other hand the analogous operators for the fields or energy and moment can be now derived from what happens to space and time. For example, the momentum will reverse its sign; because the direction of velocity will flip it is given by time derivative; we also expect the spin to reverse. Because the direction of rotation will flip once you change the sense of time and similarly if you have a say electromagnetic field the vector field will flip its sign.

Because it is produced by current and we expect current to change its sense; when the arrow of time is reversed, but the scalar field remains unchanged. So, these are the basic things which we expect based on our sense of time and how the various operators are defined. But now we want to construct the transformation which satisfies all these properties; and then also satisfies the dynamical equations of motion for quantum mechanics. So, the wave function will have to be transformed consistent with all this sense of transformations of various operators.

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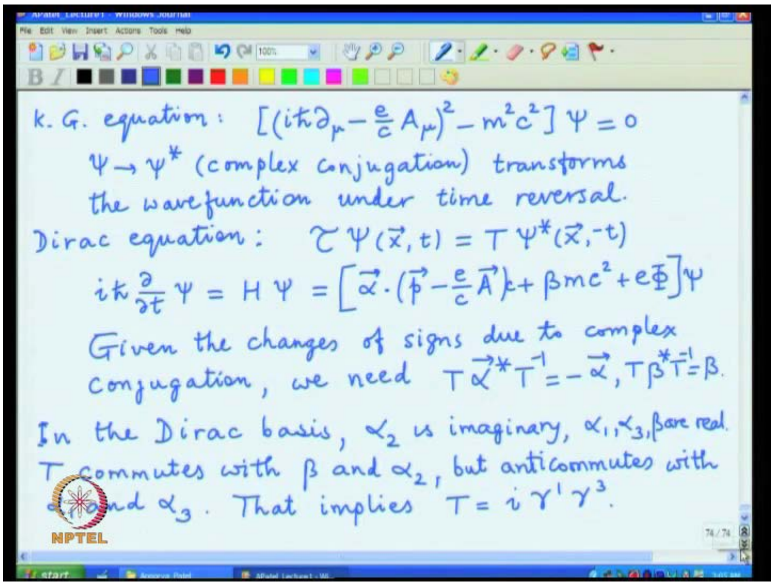

So, first let us look at the Klein Gordon equation it had the structure that; the derivative, covariant derivative. If I want to include the electromagnetic field as well square minus m square c square acting on some field psi is equal to 0. Now, we want to obtain a similar equation with all the coordinates transformed, nothing will happen to m square c square. It does not depend on either space or time; it just a constant. But one can now see how the various coordinate transformation will change the covariant derivative. The time component nothing will happened to a 0, on the other hand the Del by d t we expect to flip sign. So, the sign between this Del and a gets reversed for the time component. In case of space the vector component of a will flip sign, but Del by Del x does not change sign.

So, again the relative sign of these things needs to be flipped. And the operation which straight forwardly does that is psi going to psi star which is a complex conjugation, transforms the wave function under time reversal. If we take a complex conjugation it flip the relative sign between Del and A. Because of that star involved here. And then the change of their space and time coordinate flips the relative sign again. So, both this things together leads this whole operator acting invariant and the complex conjugation would have change psi to psi star. So, that is the transformation for the Klein Gordon equation. And now we can proceed to the little more intricate situation of Dirac equation; where we will define a operator which in addition to making a complex conjugate of the wave function, will mix its internal components as well.

And, that is because the matrices which we are using the alpha and beta have also a particular property under complex conjugation. And we have to take into account that transformation to live the total equation invariant. So, the complex conjugation is a necessary component here. As well, we will define a generic time reversal process by some operator, tau acting on psi which gives time a reversed field. And that is equal to T psi star and change of coordinates. So, this is essentially the definition of the time reversal operator. Something will happen as far as, the complex conjugation and the coordinate dependent is concern and T is a matrix which may now have to construct. So, that the Dirac equation retains its structure. So, now, the Dirac equation can be written as i h cross Del by Del t psi, which is equivalent to Hamilton acting on psi; which is now equal to alpha dot p minus e by c A plus.

There is extra factor of c here, beta m c square plus a scalar field the whole thing acting on psi. So, again we can look at what all things are going to happen, when we apply a complex conjugation process? And on the left hand side, the I flip its sign as well as Del by Del t flips its sign. And so the complex conjugation does not change the operator. As such and now, on the right hand side we have to produce a same effect and here we saw that the relative sign between p and A will be flipped by the time reversal process, just as in the case of Klein Gordon equation. And we have to do something to the operator alpha to take care of it. And similarly; we have to get psi star and there is also a complex conjugate involved in the process. So, we can look at it again term by term, e phi this a scalar term nothing happens to it which is fine.

Beta m c square something can happen to it. Because the beta can undergo a complex conjugation process and we have to make sure that; then the representation which we are using it produces the necessary sign. A is going to flip its sign and the special coordinates which can be written as; i h cross Del by Del x nothing happens to Del by del x. But i h cross will flip again it is sign under complex conjugation. So, once you take this complex conjugation out of this whole operation, we will get a relation between what the matrix T will do to alpha and beta. And then what we need is given the changes of signs due to complex conjugation, which are no different. Then what we saw for the case of Klein Gordon equation; we need the additional sign to compensate from this matrix T. And that amounts to the transformation that alpha star T inverse is equal to minus alpha; that is transformation acting on the first term. Because this T minus a term is going to flip sign and that sign has to be compensated.

On the other hand, the beta m c square nothing is happening. So, it has to obey the rule that T beta star T inverse has to be equal to beta. So, this is a demand that the equation retains its form under time reversal. And now for a particular representation of this matrix alpha and beta, one can try to solve this relation and figure out? What the operator structure is in the Dirac basis. We have to find the solution and in which case this alpha 2 is imaginary, while alpha 1, alpha 3 and beta are real. So, it is a straight forward way to figure out, what happens beta star is actually equal to beta. So, whatever this matrix T it has to commute with beta. On the other hand, the matrix must anti commute with alpha 1 and alpha 3.

Because they are real and the commutation has to compensate for the minus sign. And for alpha 2 again minus sign is produce by complex conjugation. So, T has to commute with alpha 2. So, T commutes with beta and alpha 2, but anti commutes with alpha 1 and alpha 3. And this condition fixes, what the operator must be up to a phase. So, the answer is that T is a phase which is, commonly chosen to be i times gamma 1 gamma 3 that satisfies all the necessary conditions, because beta is going to commute with a gamma 1 and gamma 3. There are 2 commutations; both giving minus 1 and so minus 1 square is equal to 1. Alpha 2 is a product of beta and gamma 2 and both beta and gamma 2 commute with this product of 2, other gamma matrices gamma 1 and gamma 3.

On the other hand, for alpha 1 and alpha 3 they will commute with 1 factor and anti commute with another factor, and then the necessary sign comes back as required. So, this is a solution the phase kind of is a convention and it is chosen here. So, that the T times T dagger is essentially 1. If you square this object gamma 1 square will give minus 1, gamma 3 square will give minus 1.

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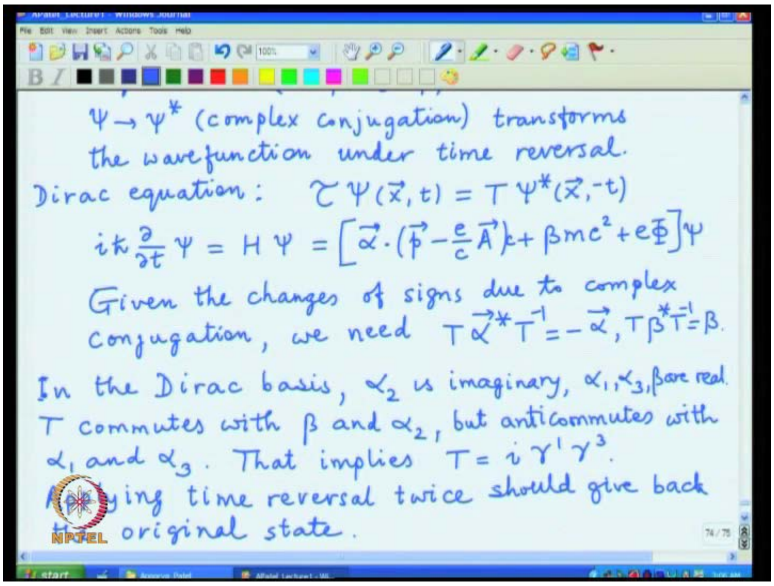

And then i times minus i will give plus 1 and that is operation which is required if you want to apply the symmetry twice and applying time reversal 2 times should give back the original starting state. And you need T and T star. Because first time we apply there is a complex conjugation and then the second one has to be acted on by the complex conjugated operator and that is the reason for choosing this convention. But as in quantum mechanics the phases are overall undetermined factors and one can choose any other convention if required. So, this is in a sense the transformation for the case a of Dirac equation. And various objects which are the operators, momentum, spin etcetera, which we saw how they come out under this particular process.

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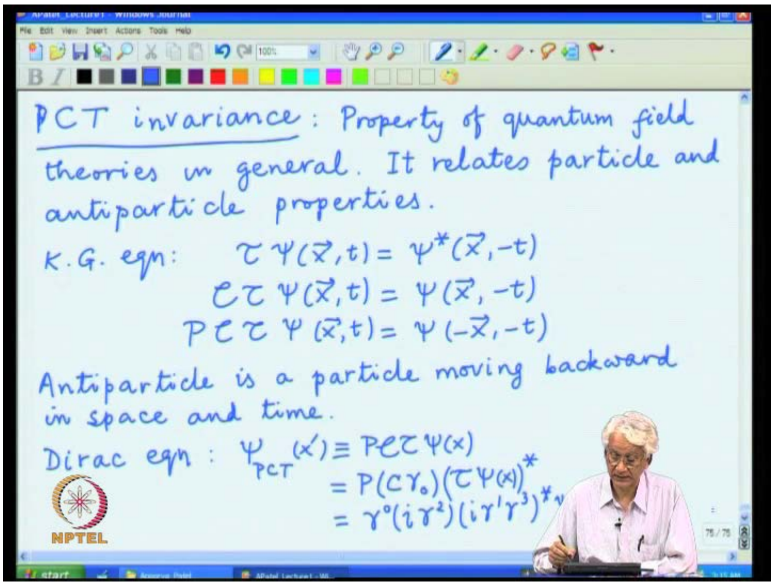

So, now let us combine the various discrete symmetries together and that plays a important role in the particular combination of P for parity C for charge conjugation and T for time reversal. And it turns out that this is the good symmetry is a property of quantum field theories in general. And it has consequences which relate particle and antiparticle properties. We have seen this interpretation of particle and antiparticle before. But now it will come out in a complete mathematical structure. So, let us just work out, what this combination in cases. So, we have this again the simpler case of Klein Gordon equation. Where the time reversal, we already saw in the previous example that it produces psi going to psi star.

But now, writing it explicitly one can put the coordinate's etcetera, in we had this relation. On top of that we now, apply the charge conjugation operation which I am going to deal. On by this curly c what it does is, it produces another complex conjugation as we have seen before. So, that cancels this first conjugation, but the arguments now remain unchanged.

Now, you apply the parity transformation on top of this thing already. And that parity transformation, we saw is going to reverse the special coordinate as well. So, the action of the 3 operators together produces a wave function which now is completely reversed in space and time coordinates. And this becomes an interpretation of what an antiparticle is? That an antiparticle is a object created by this P C T transformation acting on a particle state. And it is corresponds to a particle moving backward in space as well as time. And that backward is the reflection of this negative signs which are appearing in the arguments of the wave function.

So, antiparticle is define by this operation is, a particle moving backward in space and time. So, this P C T operation actually can be taken as what you mean by antiparticle in a mathematically explicit form. And this interpretation of a particle moving backward in space and time is heavily used in doing calculations of field theory. It is a diagrammatically represented as a arrow which instead of moving forward, it goes in the opposite direction. And that is a picture which is common place in various Feynman diagrams used to describe a process in quantum field theory.

So, this is a simple enough case because there were no matrices involved in the Klein Gordon equation. Let us do the same thing for Dirac equation. Let me just call this thing psi just the different notation to illustrate the various arguments; we define this thing as P C tau acting on psi of x. And now we just do it 1 by 1. The operation for tau we have already seen the operation for charge conjugation is multiplying by this matrix, which was called C gamma 0 and that has to act on tau psi the whole thing star.

So, now with sticking the effect of all the changes of various types; P is going to reverse the space coordinate tau is going to reverse the time coordinate. And then there are these various matrices in the internal space. So, flipping all this signs and putting a phase, phase again I am not going to worry too much about, but the P produces an operator which is gamma 0. This C gamma 0 produces an operator which is i times gamma 2. Then tau produces a change of complex conjugation which is canceled by the complex conjugation provided by this charge conjugation parts. So, that cancels out, but the matrix T still undergoes this complex conjugation process.

So, there is an i gamma 1 gamma 3 complex conjugate, and all the changes of coordinates takes x to minus x both in space as well as in time. And now all this stuff can be multiplied together, there is a complex conjugation gamma 1 and gamma 3 happened to be real. So, that will give minus I, there is a plus i here and they will cancel out and then you have a product of gamma 0 gamma 2 gamma 1 and gamma 3. It can be written as gamma 0 gamma 1 gamma2 gamma 3 with a minus sign coming from anti commutation. And that is related to the definition of the fifth matrix which we have been calling gamma 5. And so this result is equal to i times gamma 5. Gamma 5 is actually i times product of gamma 0 gamma 1 gamma 2 gamma 3. So, that is the i coming in and then psi of minus x.

So, this is how that the Dirac wave function will transform the x going to minus x is a same as in Klein Gordon equation. But now we have an extra factor of gamma 5 and that produces certain effects which we also expect to see in the structure of the various degrees of freedom, and their physical interpretation. So, 1 can now see what happens to those internal labels, because of this extra gamma 5.

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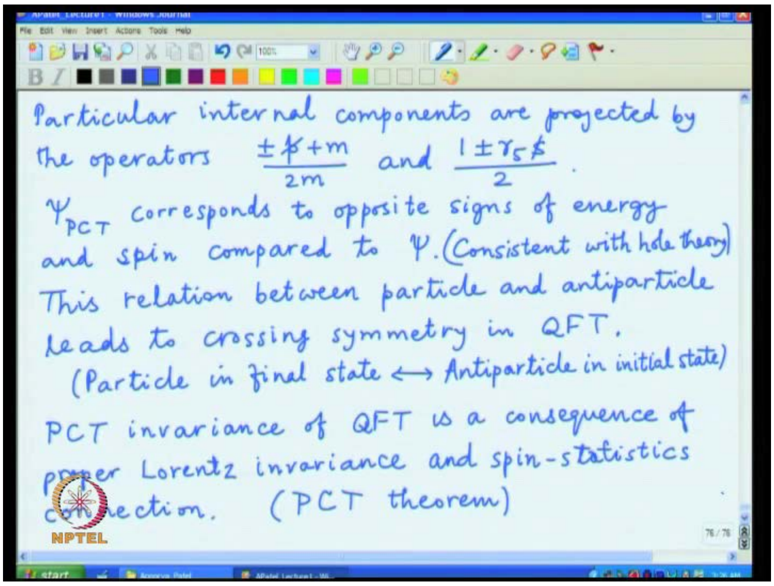

And, that can be written as particular internal components are projected by the matrices which we have seen earlier, which are plus or minus, P slash, plus m by 2 m. I am leaving out the factors of C and the other operator was 1 plus or minus gamma 5 S slash by 2. The first 1 is for projection for energy sign, and the second one is a projection for the spin sign. So, if we take a particular state of a given sign of energy and a given sign of spin then when the P C T operation is applied, to the Eigen state of some combinations over there.

The new operator will take the wave function to an opposite signs of both these quantities. Because when we apply gamma 5 to a state which is an Eigen space of P slash plus m by 2 m. The gamma 5 can be anti commuted through this P slash. And the new wave function gamma 5 times psi will be an Eigen state of the same thing. But with minus P slash instead of plus P slash. Same thing happens in this spin case also because of the gamma 5 anti commuting with S slash the sign flips.

So, the operator psi p C T corresponds to opposite signs of energy and spin compared to the original wave function psi. That is the effect of the gamma 5 explicitly appearing, but this is also a property which we saw in the discussion of whole theory; that a antiparticle is interpreted as a absence of a particles. And that absents is responsible for flipping the sign of both the energy and the spin and the basis which we have defined for the upper and lower component of Dirac spinners. Already took care of this change in the various signs of the particle and antiparticle can be easily interpreted. So, this is consistent with the whole theory.

So, this is what happens in general under this P C T transformation. And the interpretation of an antiparticle moving backward in space time gives rise to a particular feature commonly referred to as crossing symmetry. So, this relation between particle and antiparticle leads to crossing symmetry in quantum field theory. And the crossing symmetry takes that a particle in some final state is equivalent to antiparticle in initial state.

And, it can be demonstrated by changing the arrows of time on the diagrams are used in denoting quantum field theory processes. Or equivalently, just changing from 1 side of the equation some particular object to the other side of equation and making it to antiparticle, which is what is describe by whole theory and the fact that this final state and initial states are interchanges a crucial part of it. And that is where the role of P C T comes in that. We have to flip all the signs required by the operations which I just discussed.

And, the crossing symmetry has very important relations in quantum field theory. And describing various amplitudes in scattering or, decays or, production processes and relating one of them to another it is sort of automatic rule. And it is a useful constraint, because you do not have to calculate the same thing over and over again once you notice. The relations between processes due to crossing, there is also a very general feature of this symmetry P C T invariance of a quantum field theory; is a consequence of proper Lorentz invariance, which means the theory obeys special relativity. And what is known as a spin statistics connection? Which means, that boson operators will commute and they will have integer’s spins? And the Fermi operators will anti commute and they will have half integer spins. And once you have both this property P C T invariants follows and wise a versa. P C T invariance will guaranty a spin statistic connection and this is a very general result.

And, the proof of it is slight complicated detail of quantum field theory will not go into it. But many times this is also refer to as a theorem. But the fact is that the theories which we built based on various conditions already include special relativity and the spin statistics property. So, whatever equations we construct are guaranteed to follow P C T invariance. And in that sense it is part of the formalism and not really testable in experiment. Because calculations which we do already assume the particular property coming from P C T invariance. So, if you want to check P C T invariance experimentally, you have to actually work a little harder develop a theory which does not assume spin statistic connections; or, a Lorentz invariants construct is mathematical framework, work out its prediction and then see whether those consequences match with a P C T invariance prediction.

So, experimental check of P C T need a Q F T formulation, which goes beyond the usual framework dictated by special relativity and spin statistic connection. And I am not going to go into any more detail, but that make us the checks of P C T experimentally hard. I will only say that experimental checks have so far given no indication that P C T is violated anywhere. So, everything is fine and we are perfectly happy dealing with theories which obey this P C T theorem. So, that much is a generic feature of this particle antiparticle relations and the combined symmetry of P C T which follows from them.

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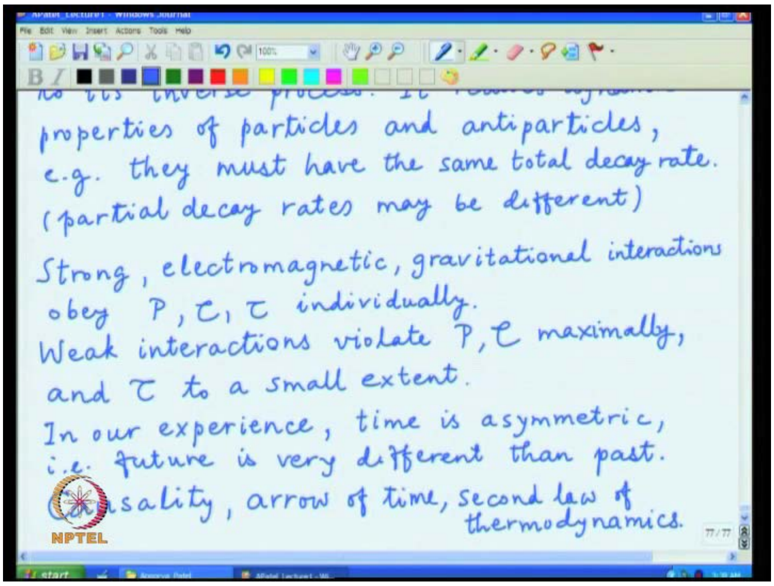

And, I can say that because time reversal is part of it this total operation. P C T is anti unitary and relates any process to its inverse process crossing symmetry is a consequence of this particular rule. But more than that it relates the properties of particles and antiparticles and that is something again which is checked experimentally as well. So, it relates dynamical properties of particles and antiparticles. For example, they must have the same total decay rate; this again follows from the considerations of how the states change and how the operators change?

We saw that is static properties flip going from particle to antiparticle those are all the charges of various type. Mass is the same, but now, the dynamic properties the particle is unstable. It will have some decay rate and the antiparticle will have the same total decay rate. Partial decay rates it should be noted, that they do not have to be the same under this P C T symmetry that if there are many different ways of a particle to decay they have individual decay rates in various channels. And the antiparticle will have the corresponding decay rates onto the channels of the antiparticles of the other one. The partial fractions among these various types of decay may be different.

But the total decay rate; when you add up all the pieces must be the same. And that is a property of P C T dictating how the particles and the antiparticles should behave. So, this is something which has been again checked in experiment to a high accuracy can take unstable particle. And its analogous unstable antiparticle measure how fast both of them are going to decay and then compare the numbers and it has indeed been observed that the decay rates always match. It has also been observed that there are some particular processes of decay where, the partial decay rates between particles and antiparticles do not match. And that give rise to violation of not P C T, but some partial parts of it in particular. If the time reversal processes is violated one can have the partial decay rates different for particle and antiparticle.

So, now let me just write down various examples of what happens in various interactions which we know very well. And these are the various cases; that we have strong electromagnetic and the gravitational interactions they obey P C and tau individually. But so called weak interactions violate P and C maximally. And an example of this maximal violation is existence of neutrino. Where, the transformation converting a left hander neutrino, to a right handed neutrino will be symmetry except that one of them just does not exist. And that is labeled as a maximal violation. If the proportion of the 2 had been unequal, it could have been a partial violation, but it became 100 percent on one type.

And, nothing on the other side it is a largest violation one can have and weak interaction end of violating these 2 maximally. And the time reversal to a rather small extent there is a violation of time reversal and weak interaction. But the magnitude of that thing is rather small. And this small violation now comes back to bug us because in our experience time is asymmetric, I e this future is very different than, past.

And, that kind of produces a puzzle that if the violation of time reversal was so tiny. How come it is such a profound effect in our experience, there are various ways of phrasing; that this is a very strong sense and of them is causality another phrase which is uses arrow of time. Yet another consequence which is part of our experience is a second law of thermodynamics. And this we cannot bypass as just some insignificant features coming from a tiny violation. And we need to understand in a complete physical theory; how this strong sense of time asymmetry arises and that is a question which is not fully settled in theory; we have certain ideas. But they require certain ingredients to produce these results and I will discuss that in the next lecture.