Plasma Physics and Applications

Prof. MV Sunil Krishna

Department of Physics

Indian Institute of Technology Roorkee

Week – 02

Lecture 09: Numerical Problems on Debye Shielding

Hello dear students. So far in our discussions we have understood what is Debye shielding, what is plasma frequency and how we can derive expressions for both these parameters. And in terms of these two parameters we have been able to define the plasma criteria which are essential conditions for calling an ionized gas as plasma. In today's class we will take up some numerical examples based on all these topics. So, the first question is based on the Debye shielding which is based on the thought experiment that we did to establish the nature of Debye's potential. So, which is a test charge plus Q is immersed in a plasma chamber, a plasma consisting of electrons and ions.

Show that the effective charge inside the Debye sphere because of all the shielding enables the test charge. This is the concept itself. So, where we have learnt that if you put a positive charge all this electron cloud will be sufficient to nullify this positive charge. So, to do this to prove that the total charge in the electron cloud will be minus Q.

So, we can start from the expression for the density which is E times Ni minus Ne. We know that the number of electrons is N0 e to the power of minus E phi by kT i, Ne is N0 e to the power of minus plus E phi by kT e. Substituting both of this, the charge density rho of R can be written as En0 e to the power of minus E phi by kT minus e to the power of plus E phi by kT. Expanding the exponential, we can write rho of R is equals to En0. This is similar to what we have done in the last lecture, En0 into 1 minus E phi by k T i minus 1 minus E phi by kT e which is equals to we have a minus e square N0 by k times 1 by T i plus 1 by T e.

Using what we know for 1 by lambda d square, you can remember from the last class, we can rewrite this expression as rho of R is minus N0 e square by k times 1 by T i plus 1 by T e. So, we know that 1 by lambda d square is equal to N0 e square by epsilon0 kB times 1 by T i plus 1 by T e. Using this, what do we use it for? All of this N0 kB for this part, all of this actually can be written as N0 e square. So, epsilon0 is missing, these two things are one and the same. So, if we make a simple substitution, we will be able to write this equation as rho of R is equals to minus, minus is still there epsilon0 phi, there is a phi here, epsilon0 phi divided by lambda d square.

1) A test charge $+Q$ is immersed in a plasma consisting of e^- & ions. Show that the effective charge inside the Debye sphere cancels the test charge.

$$\rho(r) = e(n_i - n_e)$$

$$n_i = n_0\, e^{-e\phi/kT_i}, \quad n_e = n_0\, e^{+e\phi/kT_e}$$

$$\rho(r) = e n_0 \left[e^{-e\phi/kT} - e^{+e\phi/kT} \right]$$

$$\rho(r) = e n_0 \left[\cancel{1} - \frac{e\phi}{kT_i} - \cancel{1} - \frac{e\phi}{kT_e} \right]$$

$$= -\frac{e^2 n_0}{k} \left[\frac{1}{T_i} + \frac{1}{T_e} \right]$$

So, everything now you are able to follow. Rho of R is epsilon0 phi by lambda d square. Now we know the charge density seems to be function of the device potential and the device length and the constant. Now, we know that phi the potential, the device potential is 1 by 4 pi epsilon0 into Q by R e to the power of minus R by lambda d. We can use this and write the combined expression for the charge density is minus epsilon0 by lambda d square times 1 by 4 pi epsilon0 times Q by R times e to the power of minus R by lambda d.

When we know the charge density, charge density is charge per unit volume. When we know the charge density, we can find out the charge by integrating the charge density over the entire volume. So, total charge, the charge density, where is the charge density available? This is available within the device sphere. We can calculate the total charge by 0 to infinity rho of R 4 pi R square d R. Substituting what we know already is minus Q by lambda d square integral 0 to infinity R exponential minus R by lambda d.

We have the epsilon0 will get cancelled here itself and 4 pi, this factor 4 pi will get cancelled with the 4 pi that appears here and effectively what will remain is this. This is the total charge that is available within the device sphere. So, this is similar to what we know. In mathematical physics, you must have studied. If I make a substitution by R by lambda, calling it as x, so d R by lambda d becomes d x.

$$\rho(r) = -\boxed{\frac{n_0 e^2}{k_B}\left[\frac{1}{T_i} + \frac{1}{T_e}\right]}\phi$$

$$\text{Using} \quad \frac{1}{\lambda_D^2} = \frac{n_0 e^2}{\epsilon_0 k_B}\left[\frac{1}{T_i} + \frac{1}{T_e}\right]\phi \implies \frac{\epsilon_0}{\lambda_D^2} = \frac{n_0 e^2}{k_B}\left[\frac{1}{T_e} + \frac{1}{T_i}\right]$$

$$\rho(r) = \frac{-\epsilon_0 \phi}{\lambda_D^2} \qquad \rho = \frac{Q}{V}$$

$$\phi(r) = \frac{1}{4\pi\epsilon_0} \cdot \frac{Q}{r} e^{-r/\lambda_D}$$

$$\rho(r) = -\frac{\epsilon_0}{\lambda_D^2} \frac{1}{4\pi\epsilon_0} \frac{Q}{r} \cdot e^{-r/\lambda_D}$$

So, total charge within the device sphere is minus Q integral x e to the power of minus x d x. d R is lambda d d x here and if you substitute R is equals to x lambda, lambda d square will appear in the numerator that will cancel this lambda d square. Since, you have already used R in terms of x, it will be x e to the power of minus x. So, this integral is a pretty standard integral which is called as the gamma function. So, total charge inside the device sphere is minus Q times gamma of 1.

So, gamma of 1 is 1. So, the total charge is minus Q which means that for a positive test charge of magnitude plus Q that you kept out inside the plasma, the total effective charge that is trying to nullify this inside the device sphere is exactly opposite minus Q. So, this is the proof for device shielding in a way. Now, let us say we take another example, example number 2. So, we can write the formula for device length by substituting all the known values of constants.

$$\text{Total charge} = \int_0^\infty \rho(r)\, \underline{4\pi} r^2 dr$$

$$\text{Total charge} = \frac{-q}{\lambda_D^2} \int_0^\infty r \exp\left(-r/\lambda_D\right) \underline{dr}$$

$$r/\lambda_D = x \Rightarrow \frac{dr}{\lambda_D} = dx$$

$$\text{Total charge} = -q \int_0^\infty x\, e^{-x}\, dx$$

$$\text{Total charge} = -q\, \Gamma(1)$$

$$= -q$$

We can write it as 69 times square root of T by N when the temperature is in Kelvin. So, we know that 1 electron volt is equal to 11600 Kelvin. So, we use the unit of electron volt to describe temperatures in plasma or lambda d can be written as 7430 times K T by N raise to the power of half when you are using energy as K T in electron volt or the plasma parameter N d can be written as 1.38 10 to the power of 6 T to the power of 3 by 2 divided by N to the power of half when T is in Kelvin. Do not think how I got this formula.

We have already derived the formulas. You just have to substitute the known values of constants, you will get this. Using this because it is very essential that we always remember plasma criteria. Using this we can consider a variety of plasma that is available in the universe and calculate these parameters. For example, if you consider, I will list the various types of plasmas, let us say interstellar plasma.

So, the number density of interstellar plasma is of the order of 10 to the power of 6. This number density is written in the units of per meter cube that is number of particles per meter cube and the temperature of plasma in the units of electron volts is of the order of 10 to the power of minus 1. If you calculate the plasma frequency for this using the plasma frequency in hertz which will be like 9 square root of N hertz. What is this is the frequency. It will be of the order of 10 to the power of 4 hertz and the lambda d, device length would be approximately equal to 1 and the plasma parameter N d would be equal to 4 10 to the power of 6.

Lambda d has the units of meters and N d has the units of per meter cube. Then another type of plasma you can find in solar corona which has a typical number density of 10 to the power of 12 and the temperature is 100 electron volts. The plasma frequency will be of the order of 10 to the power of 7 hertz and the device length is 10 to the power of minus 1 meters and plasma parameter is 4 10 to the power of 9 per meter cube. And then you have chromosphere, the density is very high 10 to the power of 18 that electron temperature is 1 10 to the power of 10 will be the plasma frequency in hertz and 7 10 to the power of minus 6 meters is the device length and it will be 1.4 10 to the power of 3 per meter cube the plasma parameter.

$$\lambda_D = 69\sqrt{\frac{T}{n}} \qquad T \text{ is in } K$$

$$\lambda_D = 7430\left(\frac{kT}{n}\right)^{1/2} \qquad kT \text{ in eV} \qquad 1\,eV = 11{,}600\,K$$

$$N_D = 1.38\times10^{6}\,\frac{T^{3/2}}{n^{1/2}} \qquad T \text{ is in } \underline{K}$$

Ionosphere is another region where plasma can be naturally found. The typical densities are of the order of 10 to the power of 12 per meter cube. The temperature is 10 to the power of minus 1 and plasma frequency becomes 10 to the power of 7 hertz. The device length is 10 to the power of minus 3 and device parameter the plasma parameter is 3 to the power of 4 per meter cube. And you have another manmade plasma tokamak which will have densities very high densities and at the same time very high temperatures.

The plasma frequency will be of the order of 11 and device length would be 7 10 to the power of minus 5 meters and 1.4 10 to the power of 8 per meter cube. So, these are all you can calculate using the formula yourself and get the same values. But I am trying to make a point here. The point is you see the densest plasma is 10 to the power of 20 per meter cube which is very dense.

Its temperature is also very high. But you see the device length is very small whereas interstellar plasma is very weak 10 to the power of 6 particles per meter cube is very weak actually. Temperatures is also less but device length is almost 1 meter and the number of particles within the device sphere will be of this order. Now the point is it reinforces the understanding of device shielding. What we have learnt in device shielding that if the temperature is very large the device length would increase and if the concentration of charged particles is very small or if the concentration is small the device length would increase if the concentration is high the device length would decrease.

That is why the plasma is so dense that it takes very small distance to screen out or to shield out the external electric fields or external charges. So, these things go hand in hand. We can take one more example to visualize how much density are we talking about or what is the relevance of density. Example number 3, calculate the density of ideal gas at 0 degree Celsius and 760 torr of pressure. Number 2, calculate the density of ideal gas near vacuum at 10 to the power of minus 3 torr of pressure at 20 degree Celsius.

This example can help you visualize how weak or how strong is this plasma in comparison to the average molecular densities that we have in the atmosphere or in the air that we breathe. We have been given the temperature at 0 degree Celsius and at 20 degree Celsius and we also been given the pressure 760 torr. We know that the standard temperature and pressure under STP a mole of an ideal gas contains nearly 6.022 10 to the power of 23 molecules which is called as the Avogadro number and these many molecules occupy 22.

	m^{-3}	eV	$f = 9\sqrt{n}$ Hz Hz	m	$/m^3$
	n	T	f_p	λ_D	N_D
Interstellar	10^6	10^{-1}	10^4	1	4×10^6
Solar Corona	10^{12}	10^2	10^7	10^{-1}	4×10^9
Chromosphere	10^{18}	1	10^{10}	7×10^{-6}	1.4×10^3
Ionosphere	10^{12}	10^{-1}	10^7	2×10^{-3}	3×10^4
Tokamak	10^{20}	10^4	10^{11}	7×10^{-5}	1.4×10^8

4 litres of volume. We know that how many molecules are there and how much volume they would take up. What we have to find? We have to find what will be the density per unit volume. So, that can be found that the density or number of atoms or molecules per unit volume, let us say per meter cube is equals to 6.022 10 to the power of 23 divided by 2.24 10 to the power of minus 2 which will be made a conversion of litres to meter cube 2.

(3) Calculate the density of ideal gas

(a) At $0^{\circ}C$ & 760 Torr of pressure

(b) Vacuum at 10^{-3} Torr at $20^{\circ}C$

(a) 1 mole $\underline{6.022 \times 10^{23}} \Rightarrow \underline{22.4\ lt}$

$$\text{Density per } m^3 = \frac{6.022 \times 10^{23}}{2.24 \times 10^{-2}} = 2.66 \times 10^{25}\ m^{-3}$$

(b) $PV = NRT$

$$n = \frac{N}{V} = \frac{P}{RT}$$

$$\frac{n_1}{n_0} = \frac{P_1 T_0}{P_0 T_1} \quad \Big| \quad n_0 = 2.66 \times 10^{25}/m$$

66 10 to the power of 25 per meter cube. What is this number? This number tells you how many number of atoms or molecules are there within a unit volume at the surface at 0 degree Celsius. If you put this to perspective with plasma, plasma is very weak at least if you put this at least there is 5 orders of magnitude weaker than the air that we breathe. Although plasma has all these characteristics, but it is a very weak distribution. Now, the second part this is the answer for the first part.

For the second part B, we know that p v is equal to nRT where n is n by v which is equals to p by RT. So, that means that if you remove the constant R, we can write n 1 by n 0, n 1 can be something that you want to find out and n 0 is something that you know already is p 1 T 0 by p 0 T 1. So, n 1 can be taken as this one 2.66 10 to the power of 25 per meter cube and you have been given 2 temperatures at 20 degree Celsius. Then you can find out what is n 0 has to be this n 0 has to be 2.

66 10 to the power of 25 per meter cube, you have to find the value of n 1. So, n 1 is n 0 p 1 T 0 by p 0 T 1 which will be 2.66 10 to the power of 25 multiplied by 10 to the power of minus 3 273 0 degree Celsius by 760 in torr into 273 plus 20 at 20 degree Celsius. So, the pressure is 2 pressures are given 10 to the power of minus 3 torr and 760.

$$n_1 = n_0 \left(\frac{P_1 T_0}{P_0 T_1} \right)$$

$$= 2.66 \times 10^{25} \left(\frac{10^{-3} \times 273}{760 \times (273 + 20)} \right)$$

$$n_1 = 3.30 \times 10^{19} \ m^{-3}$$

So, the value that you will get is 3.30 10 to the power of 19 per meter cube which will be n 1. What is this n 1? n 1 is the number of molecules that will be present at this pressure and at this temperature. In order to get this what you have used is you have used a known value of temperature and pressure and the number density we are calling it as n 0. So, which means that since it is the atmosphere it will have this many atoms even at such a low pressure near vacuum conditions also there are this many molecules per unit volume. So, based on all these aspects we will try to discuss some more numerical problems in the next lecture.