Plasma Physics and Applications

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Week – 02

Lecture 06: Debye Potential - II

Hello dear students. In today's lecture, we will try to understand what is Debye shielding and we will derive an expression for the form of the potential which exists within the electron cloud. So far what we have understood is the basic definition of Debye shielding and how we can construct a mathematical picture for understanding the shielding of positive charge or test charge by the electron cloud. In the last class, we have derived this second order differential equation which is d square x by dr square minus x by lambda d square is equal to 0. So, in the last class itself, I told x is not the position coordinate that we are familiar with rather x is defined to be r phi where phi is the potential that we are trying to understand, r is the radial coordinate and x is the variable which stands for this. So, lambda d is a substitution that we have made, lambda d is defined to be square root of epsilon naught k B T e divided by n e square.

Epsilon naught refers to the permittivity of free space k B the Boltzmann's constant, T e is the electron temperature, n is the charge density or number of particles per unit volume within the plasma, e is the magnitude of the electron charge. Now, we made the substitution to get this equation in a simple terms. Now, we have to solve this differential equation so as to derive what is phi of r. In order to get here, we have assumed the potential to be symmetric in theta and phi and it has a variation only with respect to r.

Now, once we have a differential equation what we do is we take solutions which are acceptable for this differential equation and upon substitution of these solutions the differential equation will be 0 or it will be valid. So, let us say I take one such solution where x is e to the power of r by lambda d plus B e to the power of minus r by lambda d. This is an exponential solution x is equals to because x is the dependent variable here, x is A e to the power of r by lambda d plus B e to the power of minus r by lambda d. Now, if you substitute you take d x by d r then you take d square x by d r square and substitute in let us say we call this equation as 1. If you put this in equation 1, we will see that this solution satisfies the differential equation.

Debye's Shielding

$$\frac{d^2 x}{dr^2} - \frac{x}{\lambda_D^2} = 0 \quad \text{--- (1)} \qquad \underline{x = r\phi}$$

$$\lambda_D = \sqrt{\frac{\epsilon_0 k_B T_e}{n e^2}}$$

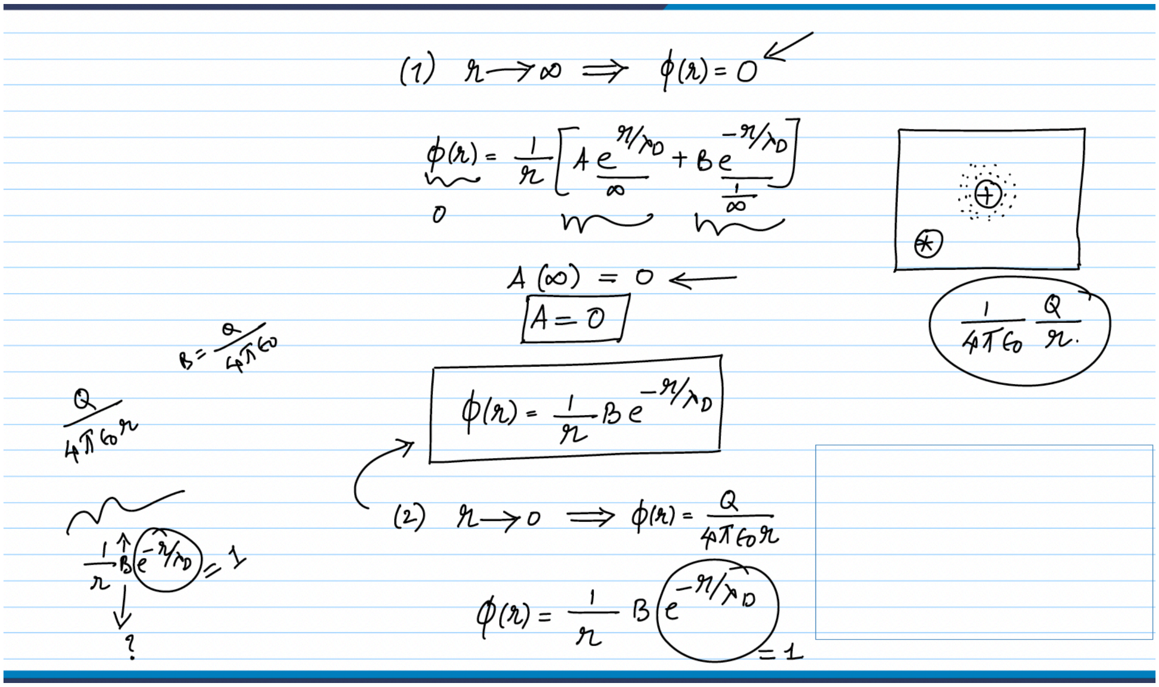
$$x = A e^{r/\lambda_D} + B e^{-r/\lambda_D}$$

$$\frac{dx}{dr}, \frac{d^2 x}{dr^2} \Rightarrow (1)$$

$$r\phi = A e^{r/\lambda_D} + B e^{-r/\lambda_D}$$

$$\phi(r) = \frac{1}{r}\left[\underline{A} e^{r/\lambda_D} + \underline{B} e^{-r/\lambda_D}\right]$$

So, once you define x is equals to A e to the power of r by lambda d B e to the power of minus r by lambda d, we know that x is actually r phi, r phi becomes A e to the power of r by lambda d plus B e to the power of minus r by lambda d. So, phi of r is going to be 1 by r times A e to the power of r by lambda d plus B e to the power of minus r by lambda d. Now, we have to find out the values of these constants A and B. So, generally we know how to do it, the process is very simple. We have two constants, so we need to apply boundary conditions which will tell us what is the value or of this function at the beginning then we can get rid of few constants.



So, let us say we use the boundary condition number 1 here which is when r tends to infinity what happens to phi of r? So, this is not a fact actually, this is something that we impose within the given scenario in which we have taken an enclosure in which plasma is present and we have kept it positive charge. The idea is that this positive charge will be surrounded by many thermal electrons whose objective is to nullify the charge but they are not able to do so because of the thermal energy or the inertia that they have. So, as a result they keep moving around this positive charge and thus constitute a shield of negative charge which will neutralize the positive charge for any part of the plasma which is beyond the cloud. This is the basic objective. What it means is that plasma by the means of its collective behavior has an ability to shield this positive charge or to make this positive charge invisible for most of it.

$$\phi(r) = \frac{1}{r}\left[A e^{r/\lambda_D} + B e^{-r/\lambda_D}\right]$$

$$A = 0 \; ; \quad B = \frac{Q}{4\pi\epsilon_0}$$

$$\frac{1}{r}, \; e^{-r}$$

$$\phi(r) = \frac{1}{r}\left[\frac{Q}{4\pi\epsilon_0} e^{-r/\lambda_D}\right] \leftarrow$$

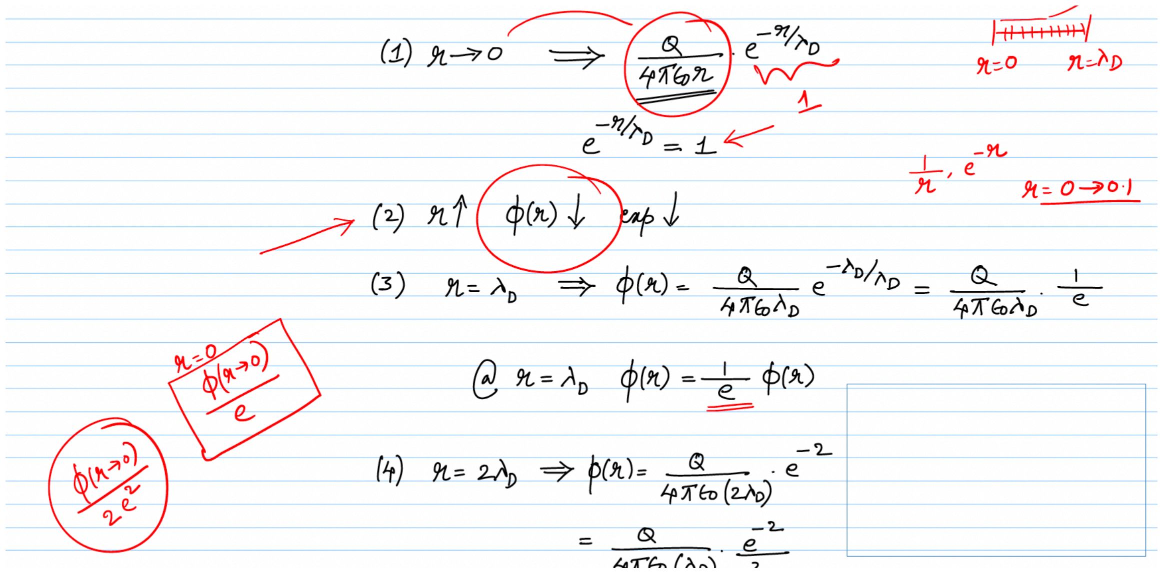
$$\phi(r) = \frac{Q}{4\pi\epsilon_0 r} \cdot e^{-r/\lambda_D}$$

Coulombic part: $\frac{1}{r}$ — r is very small

exponential part: e^{-r} — $r \to \infty$

Some part is of course influenced but this influence is only to nullify or to make the charge positive charge invisible for most of it. So, ideally if you want a potential to exist because of this positive charge, so we can write what will be the fundamental nature of potential due to a charge 1 by 4 pi epsilon0 Q by R. If you want this potential to exist everywhere then there is no point of shielding or there is no idea of shielding at all. But if you want for example when R tends to infinity, when R is going to very large distances we can simply say that our phi of R should be 0. So, at very large distances of R, phi of R should become simply 0.

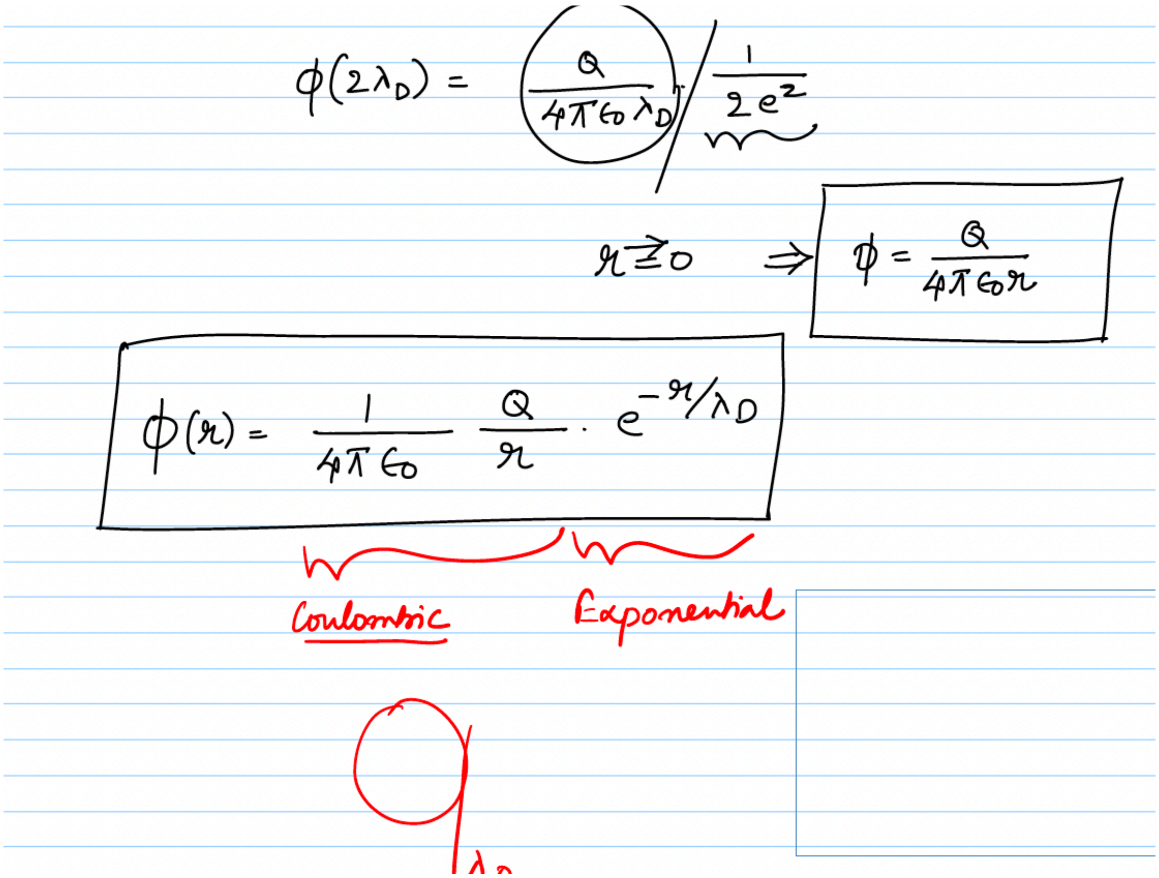
Now let us say you have what the potential phi of R the solution is 1 by R A e to the power of R by lambda d plus B e to the power of minus R by lambda d. Let us substitute this boundary condition 1 into this equation. What happens when R tends to infinity? When R becomes very large, what will happen? This will become infinity of course and this will become infinity but e to the power of minus. So, it will be 1 by because of that. So, this part will of course become 0 directly.



Now if you want the left hand side to be 0, you would expect that this part A into infinity cannot be 0. But on the left hand side you have a 0. So, in order to achieve this equality you must say that A is 0. I hope you have understood what I have done here. Phi of R is the potential which is assumed to be existing in this form.

This form is perfectly fine with us because it satisfies the differential equation which is a result of all that discussion that we had. So, this differential equation contains all information about the shielding process, about the potential, about the symmetry of the potential, everything. That means if you are able to solve this differential equation then you are actually accounting for the form of the potential which keeps all that shielding intact as simple as that. Now when you go about solving it, this is what you get. But what you have is you have two constants A and B.

You do not know what those A and B's are. Having A and B's nonetheless still satisfies the differential equation but we cannot take this constant as it is. We should know them in its full empirical form or analytical form. So, in order to do that generally the accepted method or the method that we follow is we see what will happen to the solution at the ends of the system. So, when R tends to very large values you would naturally want the potential to vanish that is why you take the phi of R to be 0.



If you substitute this, so you have this is 0. When R tends to infinity this is A into e to the power of this exponential of infinity of becomes infinity and since exponential of minus infinity becomes 1 by exponential of R by lambda d, 1 by infinity is 0. So, in order to achieve this A into infinity is equal to 0, we require the constant A to be 0. What about B? We do not know about B. So, B can be anything.

It does not matter but this term this entire term is 0 at infinity. So, let us say at that point we can write the form of the potential as phi of R is 1 by R B e to the power of minus R by lambda d. You understand this is what I have done. So, if you are wondering why I did not take A or just now I said it B can be anything. So, B is still unknown for us but we know for a fact that A has to be 0 when R tends to infinity.

When R becomes sufficiently large the constant A will tend to be 0. So, at that point we have B e to the power of minus R by lambda d. Let us take another extreme condition or another boundary condition. Let us say when R tends to 0. What happens when R tends to 0? When R tends to 0 you are at the positive charge itself.

So, in that case it is natural that we can only expect the potential to be phi of R to be columbic in nature which is 1 by 4 pi epsilon naught R. When R tends to 0 you would expect that the potential will be phi is equals to Q by 4 pi epsilon naught R. If you use this into this equation so we have the nature of potential. So, phi of R if you match these two things what you will realize is e to the power 1 by R is as it is there B e to the power of minus R by lambda d. This factor has to vanish if you have this then you can probably say that the constant B is Q by 4 pi epsilon naught but for this potential to exist everywhere you need this constant to be equal to 1 e to the power of minus R by lambda d should be equal to 1.

Now going back and using both these things together we can realize that phi of R is 1 by R A e to the power of R by lambda d plus B e to the power of minus R by lambda d. Using the fact that A is 0 and B is Q by 4 pi epsilon naught phi of R can be written as 1 by R Q by 4 pi epsilon e to the power of minus R by lambda d. Let us say so what are the parameters that we have here 1 by R times Q by 4 pi epsilon naught e to the power of minus R by lambda d I will write it or rewrite it for convenience like this. Now you see what I have done you have the coulombic potential, coulombic part as it is and then you have the exponential part. This is where things become interesting, coulombic part and exponential part.

Coulombic part has a dependence of 1 by R and the exponential part has a dependence of e to the power of minus R. Lambda d is the length R is also in the dimensions of length of course. Now let us try to understand the meaning of this. If you go back we look what we have done. We have taken the differential equation, we have assumed a solution for that differential equation and if the solution has to be valid we can keep phi of R to be in that form.

Then one boundary condition gave us that the constant A has to be 0 at R tends to infinity and using the other boundary condition we realized that when we are very close to the positive charge the form of the potential cannot be different than the coulombic potential. So, in that cases we have 1 by R times B e to the power of minus R by lambda d is there. We still do not know what is B. So, if all of this has to be equal to Q by 4 pi epsilon naught R. Then we said this factor has to be equal to 1 and this B has to be equal to Q by 4 pi epsilon.

We just have used our prior knowledge and re-designated the constants appropriately. So, using all that we have now got the form of phi of R the potential in this expression. So, we realize that it is coulombic in nature for let us say we can say it here. It is coulombic in nature. So, this part was relevant or this is because of the constant B.

So, the coulombic part is relevant when R is very small because when R is very small only then we have this e to the power of minus R by lambda d equal to 1 and all of this become non-zero or not negligible. And this is relevant when R tends to infinity. R is very large. So, what is the meaning of this renewed potential? So, the external discharge effect cannot be felt beyond lambda d that is what we have to prove. Now if you see 1 by R and e to the power of minus R we have two things here.

Something decays at the rate of 1 by R and something else decays at the rate of e to the power of minus R which is faster that is the basic question. Where is this valid? This is valid at very short distances and this is valid at very large distances. But both of them combinedly exist to determine the nature of potential that is generated because of a positive charge which is kept in plasma. In order to further appreciate the physical meaning of this potential let us say we take few situations.

Number 1, when R tends to 0. When R tends to 0 what will happen? You have Q by 4 pi epsilon0 R into e to the power of minus R by lambda d. So, obviously when R tends to 0 we can expect that e to the power of minus R by lambda d will simply become 1. When R tends to 0 you substitute R is equal to 0 in this you will realize that this exponential is becoming 1. The exponential part becomes 1 which means that this part becomes relevant or important or in a second situation we say when R increases what happens to phi of R? When R increases phi of R will decrease and the exponential part e to the power of minus R by lambda d also decreases. How do I say this? Q by 4 pi epsilon0 R when R is increasing the denominator is increasing the potential the left hand side will decrease.

e to the power of minus R by lambda d is 1 by e to the power of R by lambda d. When you have increasing R you are rising the power of this exponential the denominator will increase and as a result the numerator will decrease. With increase in R the denominator will only increase and whatever is being accounted for this will eventually decrease this is what is shown here. So, I hope you are able to follow this simple mathematics.

Then we have another situation. Let us say when R is equals to something called as lambda d because lambda d is there within your picture. When R is just equal to lambda d what happens to phi? Phi of R becomes Q by 4 pi epsilon0 lambda d e to the power of minus lambda d by lambda d right which is Q by 4 pi epsilon0 lambda d into 1 by e. What is it? So, what we can infer is at a distance R is equals to lambda d the potential phi of R becomes 1 by e th of the potential. Potential suddenly drops to very small values because 1 by e 1 by 2.

7 something of the original potential. So, at lambda d the potential is very small. Now we take another situation where R is equals to 2 lambda d you have gone beyond lambda d or beyond the device length. Then we can say that as a result phi of R becomes Q by 4 pi epsilon0 times 2 lambda d e to the power of minus 2 or Q by 4 pi epsilon0. We will take lambda d to be separate and e to the power of minus 2 by 2 or phi at 2 lambda d

becomes Q by 4 pi epsilon0 lambda d into 1 by 2 e to the power of 2. What is the conclusion? The conclusion is that if this is the potential at the edge of the Debye sphere at a distance of 2 lambda d the potential becomes even smaller.

The potential is getting scaled by nearly 1 by 2 exponential power to the power of 2. So, it is decreasing very fast. The conclusion is that at R is equals to 0 the potential is phi is Q by 4 pi epsilon0 R. We cannot write R is equals to 0 because if R is equals to 0 Q 1 by 0 which becomes phi becomes infinity.

We will only say that R tends to 0. Phi is Q by 4 pi epsilon0 R. What happened to the exponential term? Exponential term became just 1. So, there is no role of exponential term. Let us say this is the conclusion actually. Let us try to understand this in the fullest possible manner 4 by epsilon0 Q by R e to the power of minus R by lambda d.

You see this is what we have derived. I am doing it again actually. This is what we have derived. How did we get? This is the constant B and we got this by assuming that when R tends to 0 the potential will be columbic in nature. Columbic potential has 1 by R dependence. How did we get this? e to the power of minus R by lambda d we got this basically from the solution of the differential equation itself nothing else.

The solution of differential equation had this term e to the power of minus R by lambda d. Now we know how we got these terms. Let us try to understand the physical significance of these different terms. This is part from our understanding is columbic and this part is exponential. These two represent how the potential will decrease away from the positive charge.

What do we want? We want such a picture in which the potential becomes almost 0 once you cross the limit of lambda d because lambda d defines the length up to which this positive charges presence can be felt. Now in order to understand this we have constructed these four different cases. In the first case we took R to the extreme left saying that R is nearly 0. We know what happens when R is nearly 0. Because this is self-reinforcing because we constructed this picture and mathematically we found out those terms which are relevant to prove our assumption.

This is self-reinforcing each other. When R tends to 0 the potential is columbic because the potential is indeed columbic and the other term simply becomes 1. Both these terms coexist. For example, you are moving away from R is equal to 0. This is what the situation is now. At R is equal to 0 or R tends to 0 we only have this term valid.

This term becomes simply 1. Now let us move away from R is equal to 0. When R is increasing what happens? The potential phi of R which is the columbic part is of course decreasing and the exponential part is also decreasing which was just 1 which does not have any variation. Now it is decreasing as we go away from the R is equal to 0 limit. Both of them are decreasing. When how fast are they decreasing? You can realize by plotting something like 1 by R and e to the power of minus R.

You take R is equal to 0 to 0.1 and use any mathematical tool available online and plot 1 by R and e to the power of minus R. You will realize although both of these things combinedly decreasing the value of potential you will realize which is making this decrement very fast. But the message is very clear as you go away from R is equal to 0 the potential will decrease. At lambda d you will realize that the potential has already dropped to 1 by e of its value in the vicinity of R is equal to 0.

This is the most important thing. At R is equal to 0 there is nothing we do not we never discuss at R is equal to 0. In the vicinity of R is equal to 0 the potential is something let us say phi at R tends to 0. The potential at R is equal to lambda d is simply 1 by e of this potential is the message. Now the exponential decrease has actually started which means exponential decrease is a very fast decrease in comparison to 1 by R. Now the exponential decrease has started so it has reached to 1 by e of its original value.

You go further ahead you move away from R is equal to 0 even further take a distance of 2 lambda d twice the lambda d you will see that the device potential or the potential as a matter of fact with respect to phi at R tends to 0 becomes very small e to the power of minus 2 times by 2 or this entire potential scales. So, this is phi at R tends to 0 the potential at 2 lambda d becomes 2 this divided by 2 e to the power of 2. What am I trying to prove here? I am trying to emphasize that when you are moving away from the positive charge the potential is up to a particular distance it is falling with a 1 by R dependence because it is fast and after that the potential is falling at a very fast rate which is exponential in nature. Now let us say we can reinforce these four statements by including one more mathematical step. We can say that if R is equal to 0 and R is equal to lambda d are the limits we say that within short intervals spanning from here to here R is this somewhere the columbic part is very dominant and after some point the exponential part becomes dominant.

So, having said all that we can now say that the potential inside the device length decreases according to 1 by R and at the edges when it is about to reach lambda d it will decrease exponentially. So, we can put it in terms of a simple graph where we take R the distance away from the center of this positive charge and this is phi of R. The potential would be something like this which is columbic which is 1 by R and suddenly at lambda d it will be very fast. So, this is an exponential d. So, now we have understood how the

Debye potential works and how we can derive an expression for the Debye potential and the Debye length.

So, there are some more aspects about the Debye shielding that we will try to cover in the next lecture. Thank you.