Plasma Physics and Applications

Prof. MV Sunil Krishna

Department of Physics

Indian Institute of Technology Roorkee

Week – 12

Lecture 57: Diffusion in Presence of B-II

Hello dear students. In the last lecture we have seen what is the role of diffusion and how a magnetic field can change the rate at which diffusion happens or recombination or loss of plasma can happen. In today's class we will try to understand some important aspects of the discussion that we had about a weak a weakly ionized plasma in the presence of collisions and in the presence of magnetic field. How diffusion will occur due to the collisions as well as due to the magnetic fields effect. So, in that discussion we have derived this diffusion coefficient which we called as d perpendicular which is k B T nu m by q square B square. Now if you remember the discussion started on the context of how the magnetic field can affect the particles movement or how collisions can be affected in the presence of magnetic field.

As long as the particles motion is parallel to the direction of the magnetic field there will not be any effect. So, that is why we considered the magnetic field to be in such a way that the velocity components are perpendicular. And the resulting diffusion the coefficient of diffusion was thus named d perpendicular. Now from this d perpendicular can be directly proportional to the frequency.

1) $D_\perp = \dfrac{k_B T \nu m}{q^2 B^2} \Rightarrow D_\perp \propto \nu$, $D_\perp \propto \dfrac{1}{B^2}$, $D_\perp \propto m$, $D_\perp \propto T$

2) $D = \dfrac{k_B T}{m\nu} \Longrightarrow D \propto T$, $D \propto \dfrac{1}{\nu}$

(a) Collisions are needed for diffusion in the presence of $\vec{B}$

What is this frequency? This is the collisional frequency and d perpendicular is also proportional to m and d perpendicular is proportional to 1 by B square and d perpendicular is proportional to the temperature. Now if you put it to contrast in comparison to the parallel diffusion coefficient or diffusion coefficient in the absence of any magnetic field you can write it as k B T by m nu. Now from this expression you can observe that the diffusion coefficient or the parallel diffusion coefficient is directly proportional to the temperature it is proportional to 1 by frequency. So, in the absence of magnetic field the collisional frequency more the collisional frequency lesser will be the diffusion that is what it this expression means. Let us try to highlight the key differences between these two.

So, one important difference is that the diffusion coefficient depends or is proportional to the inverse of collisional frequency when there is no magnetic field or when the magnetic field is parallel whereas the diffusion coefficient is directly proportional to the collisional frequency in the presence of magnetic field or in a perpendicular diffusion coefficient. So, collisional frequency what is the physical idea of collisional frequency? The particle in the presence of magnetic field is gyrating in an orbit something like that and whenever it collides it will start a new orbit something like that. So, the rate at which collisions happen it is obvious for us to think that if there are more collisions the diffusion is hindered the diffusion is not very fast. So, that is what it is. So, in a regular situation if there are more number of collisions if there are if the collisional frequency is larger the diffusion will be slower, but you see the exact opposite in the case of perpendicular diffusion de-perpendicular is directly proportional to the collisional frequency.

$D_\perp \propto m$

$\nu \propto \frac{1}{\sqrt{m}}$

$D_\perp = \frac{k_B T \nu m}{q^2 B^2}$

$\boxed{D_\perp = C\sqrt{m}}$

$e^- \downarrow \quad D_\perp \downarrow \quad r_L \downarrow$

e^- diffuse slower

$r_L = \frac{m v_\perp}{qB}$

$D \propto \frac{1}{m}$

$\boxed{D \propto m^{-1}}$

$e \downarrow \quad D \uparrow$

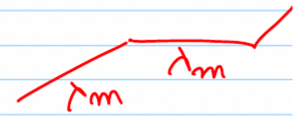
$D = \frac{k_B T}{m\nu} = \frac{v_{th}^2}{\nu \leftarrow} \approx \frac{\lambda_m^2}{\tau}$

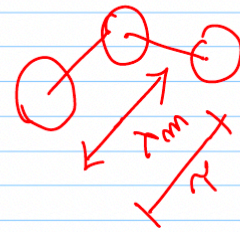
$\frac{1}{2} m v_{th}^2 = k_B T$

And temperature of course both of the situations it is the same and mass in this situation mass is slightly complicated not direct as such. But the perpendicular diffusion coefficient is proportional to 1 by b square it is natural in the presence of magnetic field the diffusion coefficient will of course depend on the magnetic field strength. So, in the presence of magnetic field and in the direction of which is perpendicular to the magnetic field the diffusion or movement of charged particles parallel to the magnetic field will remain a constant that means will remain unaffected. So that is why you have this let us say we call this as 1 and we have 2. So, both of these are nothing but diffusion coefficients, but one in the presence of magnetic field and another in the absence of magnetic field.

So, as we see that de-perpendicular is proportional to 1 by collisional frequency the diffusion coefficient in this case in the case of perpendicular magnetic field we can see that the collisions are very much needed for the diffusion to happen in the presence of magnetic field. So, one very important conclusion is we need collisions. Collisions are needed for diffusion to happen for diffusion in the presence of the magnetic field whereas if you look at diffusion parallel to the magnetic field or in the absence of magnetic field the collisional frequency or collisions will retard the rate at which the diffusion can happen. So, this is the basic role of collisional frequency in both the situations. If you talk about mass you know that the de-perpendicular is proportional to m.

$$D = \frac{\lambda_m^2}{\tau} \quad \lambda_m \longrightarrow \text{Mean free path}, \quad \tau \longrightarrow \text{Relaxation time.}$$





$$D_\perp = \frac{k_B T \nu_m}{q^2 B^2}$$

$$\sim \frac{m v^2 \nu_m}{q^2 B^2}$$

$$\sim \frac{m^2 v^2 \nu}{q^2 B^2}$$

$$\sim r_L^2 \nu$$

$$D_\perp \sim \frac{r_L^2}{\tau}$$

$$\frac{m v_\perp}{qB} = r_L$$

$$\nu = \frac{1}{\tau}$$

So, whereas d is proportional to 1 by m, but we know that the collisional frequency is proportional to 1 by square root of m. So, if you put this into this expression de-perpendicular is equals to k B T mu m divided by q square B square if you put into this what you will realize is de-perpendicular is or some constant times square root of m. So, it is not directly proportional to m rather it is proportional to directly proportional to square root of m that is the basic difference, but whereas in this case it is still the same d is proportional to m power minus 1. So, this is the key difference that you see in the diffusion coefficient in terms of mass or the dependence of diffusion coefficient on the mass. What we have seen in the last slide is that the dependence of diffusion coefficient two different types of diffusion coefficient on the collisional frequency.

Random Walk

Step length $\longrightarrow$ Mean free path $\quad B = 0$, $B \parallel v$

Radius of gyration $\quad B \neq 0$, $D_\perp$

$$\boxed{\begin{gathered} D \approx \frac{\lambda_m^2}{\tau} \\ D_\perp \approx \frac{r_L^2}{\tau} \end{gathered}}$$

What we have understood is that collisions are very much required for the diffusion of plasma to happen in the presence of magnetic field whereas, the same collisions will retard the diffusion in the absence of magnetic field. So, here when we combine these two or when we put these two dependences in contrast to each other we can say that in this picture in this perpendicular picture we can say that electron mass being smaller we can say that the diffusion coefficient will also be smaller. Whereas, in this case electrons being lighter the diffusion the rate at which these particles will diffuse in the absence of magnetic field will be larger. So, this means the electrons are smaller they have larger velocity they will diffuse faster, but here the electrons are lighter, but because of the presence of magnetic field they would not diffuse faster. So, what will the magnetic field do? The magnetic field will make this electrons gyrate and they gyrate between two successive collisions they would not travel directly because of the magnetic field they will every time they suffer the movement of electron between these collisions will be in a gyration or gyrating orbit.

So, the electrons because the electrons mass is much smaller. So, this the radius of gyration of electron will also be smaller. So, R L will also be smaller because if you remember R L is m v perpendicular by q B. So, m is small the radius of gyration would also be small. So, as a result so electron cannot cover larger distances in the presence of collisions because electron will not be able to free freely move from one point to another point because it has to gyrate while it moves and before it moves a certain distance it encounters a lot of collisions and these collisions will make the electrons movement look like a random walk.

Diffusion of Plasma (fully ionized) across a magnetic field

$$\frac{d\vec{v}}{dt} = 0 \quad \text{(Steady state)}$$

$$\vec{J} \times \vec{B} = \vec{\nabla} p \quad \text{— (1)}$$

$$\vec{J} = \sigma (\vec{E} + \vec{u} \times \vec{B}) \quad \text{— (2)} \qquad \text{Generalized Ohm's law.}$$

σ — is the conductivity.

Eliminate $\vec{J}$ from (1) & (2)

$$\sigma (\vec{E} + \vec{u} \times \vec{B}) \times \vec{B} = \vec{\nabla} p$$

$$\sigma \left\{ (\vec{E} \times \vec{B}) + \underbrace{(\vec{u} \times \vec{B}) \times \vec{B}}_{} \right\} = \vec{\nabla} p$$

So, every step you have electron gyrating in a circular orbit. So, that is why the electrons even though they have lighter mass they move slower or they can we can say that electrons diffuse slower. It makes sense because collisions at the same time you see collisions are helping diffusion to happen faster, but if you bring mass into perspective you will decide that you will realize that the lighter mass particle which you obviously expect to diffuse faster or not diffusing faster because of the collisions and the radius of gyration. So, in the absence of magnetic field we have seen the diffusion coefficient is proportional to the inverse of mass. So, we can write this diffusion coefficient as k B T by m nu.

So, we can say that half mv square mv thermal square is equal to k B T or half k B T for easy representation. We can rewrite this expression as v thermal square divided by the frequency or you can also write it in terms of distance that the particle travels because you have involved the collision frequency here. So, if there are new number of collisions per unit time. So, it will obviously take an inverse of that the number that you get will be the average duration between two collisions. So, that is called as the relaxation time which is generally represented by tau.

$$\sigma\left(\vec{E}\times\vec{B} - U_\perp B^2\right) = \vec{\nabla}p$$

$$U_\perp = \underbrace{\frac{\vec{E}\times\vec{B}}{B^2}}_{\vec{E}\times\vec{B}\text{ drift}} - \underbrace{\frac{\vec{\nabla}p}{B^2}}_{\text{diffusion velocity in the direction of } -\vec{\nabla}p}$$

$$\boxed{\vec{\Gamma} = n U_\perp = \frac{-n\vec{\nabla}p}{\sigma B^2}}$$

So, this is lambda m square divided by tau. So, distance by time is distance is lambda m is actually the distance measure of distance. So, lambda v if you want to replace this v th square you have to write lambda m square by tau square and one tau will get cancelled and you will get the diffusion coefficient for the case of no magnetic field lambda m square divided by tau. So, what is lambda m? Lambda m is the mean free path. So, the particle is moving like this it goes here it suffers a collision and then it goes here it suffers a collision.

So, this process is something that resembles what is called as the random walk process. So, lambda m is the mean free path between two successive collisions and lambda m is the mean free path and tau is the relaxation time. So, this is the distance that so, if you look in the space dimension this is the distance and the time that it takes for this distance to be travelled is tau the relaxation time. Now, if you convert the d perpendicular also to look something similar to this d perpendicular is k B T mu m divided by q square B square which can approximately be written as m v square. So, k B T the thermal energy is equated to the kinetic energy that is why you got m v square mu m divided by q square B square or we can write it as m square v square the frequency of course, q square B square.

This looks in a form that is familiar to us because m v perpendicular by q B is nothing but the radius of gyration of the particle. So, this is like r L square nu or nu is 1 by tau r L square by tau. So, this is the perpendicular diffusion or diffusion in the presence of magnetic field can be considered as a random walk process with a step length of r L. Every time the particle suffers a collision or before that the particle is moving as if it is exhibiting random walk process. So, we must remember that the parallel diffusion the step length of the random walk was the mean free path.

So, it is directly the collisions in the absence of this is lambda m and this is lambda m things like that, but when the particle is moving in the presence of magnetic field between those collisions the particle is gyrating and moving from one point to another point. So, that is why the step length of random walk in the presence of magnetic field let us write it. So, we have this process which is called as random walk. What is random walk? The particles movement in the presence of collisions or in the and in the presence of magnetic field appears to be a random walk process. This random walk process is characterized by what is called as the step length.

So, in the case of B is equals to 0 or B parallel to the velocity the step length is the mean free path and when B is not equals to 0 or we write B perpendicular the step length is the radius of gyration. This is the most important inference that we can draw at this point of time. So, I will just write those expressions D is approximated to lambda m square divided by tau whereas, D perpendicular is approximately equal to R L square divided by tau. This is the conclusion of the discussion that we had about diffusion of plasma in the presence of magnetic field, but most importantly this is valid when the plasma is not fully ionized. There are many number of neutral atoms or neutral species present in addition to the electrons and ions or simply put diffusion of weakly ionized plasma in the presence of magnetic field is dependent is a random walk process with a step length of radius of gyration.

$$P = p_e + p_i = n k_B (T_e + T_i)$$

$$\vec{\Gamma} = \frac{n k_B (T_e + T_i) \vec{\nabla} n}{\sigma B^2}$$

$$\boxed{\vec{\Gamma} = D_\perp \vec{\nabla} n}$$

$$\boxed{D_\perp = \frac{n k_B (T_e + T_i)}{\sigma B^2}}$$

Classical diffusion coeff for a fully ionized Plasma

$$D_\perp \propto \frac{1}{T} \qquad \sigma \propto T^{3/2} \qquad D_\perp \propto \frac{1}{B^2} \qquad \propto n$$

Diffusion of weakly ionized plasma in the absence of magnetic field is a random walk process with a step length of mean free path. Now, moving on we will try to understand how the diffusion process can be seen for a strongly ionized plasma or fully ionized plasma. So, this discussion is actually the next topic which is diffusion of plasma of course, plasma, but fully ionized across a magnetic field, fully ionized plasma across a magnetic field. So, we have to consider plasma as a fluid which is moving from one point on that point. Now, the moment you say it is fully ionized we have already seen there will be no role of collisions.

So, we can assume for simplicity we can assume the steady state to be existing, steady state wherein you make the derivative of velocity with respect to time to be 0 that means, the particles are moving in a steady state. So, they are moving with a constant velocity which is not changing with respect to time. That means, there are no collisions inside which can disturb the steady nature of the velocity. So, we can say that for simplicity we say dv by dt is equal to 0 which is also called as the steady state. Now, if you have considered this plasma we have to see what are the forces that can exist on plasma.

$$D = \frac{k_B T}{m\nu} \qquad \lambda_m$$

Weakly ionized plasma with collisions

$$D_\perp = \frac{k_B T \nu_m}{q^2 B^2}$$

$\longrightarrow$ Random walk

$\longrightarrow$ r_L

fully ionized plasma

$$D_\perp = \frac{n k_B (T_e + T_i)}{\sigma B^2}$$

So, the equation of motion in the presence of magnetic field of course, and if it is a fluid there has to be some pressure gradient force. Now, since you have equated the rate of change of velocity to be 0 all these forces added up will become 0. That means, they do not have any effect as such. So, we have to write the force that is because of the magnetic field is equal to del P. Let us say we call this as equation 1.

The first term is because of the magnetic field, the second term is the pressure gradient term. Now, we can write the generalized Ohm's law in the simplified form as we can write J the current density is sigma times E plus u cross P. We call this as 2. We do not know J, while we wrote equation number 1 we do not know what is J, I mean at least how J can be evaluated. What is sigma? Sigma is the conductivity.

What is sigma? Sigma is the generalized Ohm's law. Now, we can eliminate J from 1 and 2. By eliminating J we are trying to just get one equation which represents both these two equations. So, we can substitute all of this that appears on the right hand side of equation number 2 in the place of J in equation number 1. So, sigma times E plus u cross B cross B is del B.

So, just some vector operations E cross B plus u cross B cross B is del B. So, expanding this triple vector product all of you know the formula for that. So, sigma times E cross B the first term remains as it is minus u perpendicular B square. This term will survive only if you take u to be the u perpendicular. For those of you who are wondering how I got only one term here instead of the two terms that will appear out of this triple product, you just have to refer to simple vector identities you will get it.

Rather it is actually good if you can work out the some intermediate algebra steps that I intentionally skip. It will be you will see that your mathematical abilities will increase. So, simplifying this you will get u perpendicular is equals to E cross B by B square minus del P by B square. So, what is u perpendicular? U perpendicular is a normal component of velocity. Now these two terms are one is the familiar E cross B drift and this one this is velocity the dimensions are velocity.

So, this is this term is called as the or represents the diffusion velocity in the direction of minus del P. So, if you are wondering what are we actually doing let us just go back and see where we started with. We started to understand the diffusion of fully ionized plasma in the presence of a magnetic field across a magnetic field. For simplicity we assumed a steady state solution ideally we are supposed to start with the equation of motion which is just a force balance attributing the sum of forces to be equal to the rate of change of velocity. Now you assume the fluid that you call as plasma which is also fully ionized moving from one point on that point in the absence of any collisions.

You can assume that this flow is existing in a steady state that means there are no changes in the velocity. So, which makes that these two forces to be equal to each other and using the generalized Ohm's law we have replaced the current density with this and just some following some simple algebra you realize that the fluids velocity under such situations will be composed of two parts. One an e cross B drift term another which is giving you the diffusion velocity in the direction of the pressure gradient. Now if this is the velocity that we are talking about then using some law we can calculate the flux of plasma which is having both these terms together. So, the flux can be gamma which is gamma is N u perpendicular we know this relation already.

So, using this only the pressure gradient because the other term is e cross B term is sigma B square. What is N? N is the number of charged particles per unit volume. Now so far this is the flux. So, this will be the diffusion flux for a fully ionized plasma in the presence of a magnetic field. Now so far we have not differentiated between the electrons and ions so that means that we had actually considered a single fluid plasma single fluid fully ionized plasma.

But now if you make a change saying that let us consider a two fluid plasma then the pressure that you write P should be equal to the pressure of electrons plus pressure of ions which is equals to N k B T e plus T i. So, in that case the flux gamma can be written as N k B T e plus T i times del N the gradient on the pressure gets now transferred onto the number density sigma B square. So, the flux is if you compare this to this to this then you can write that all of this using the Fick's law you can write this in terms of this. That means whatever that appears to be multiplying this term delta del N should be the diffusion coefficient T perpendicular is N k B T e plus T i divided by sigma B square. So, this is also a diffusion coefficient is known as the classical diffusion coefficient for a fully ionized plasma.

What is this? This is the classical diffusion coefficient for a fully ionized plasma. Just for reference D perpendicular is proportional to 1 by B square and for Maxwell distribution of let us say velocities you will realize that the D perpendicular is proportional to the number density and we know that the sigma the conductivity is proportional to T to the power of 3 by 2 if it is an actual distribution which means that D perpendicular is proportional to 1 by T. Now let us write the three diffusion coefficients that we have derived throughout the discussion. One the diffusion coefficient that we know since the beginning is k B T by m nu where nu is the frequency, collisional frequency. And D perpendicular in the presence of magnetic field is written as k B T nu m divided by q square B square.

This is for weakly ionized plasma with collisions. The major significance or the major understanding that came out from this is about the random walk process. So, here the random walk step length is lambda m and in this situation the random walk step length is the radius of gyration. For a fully ionized plasma the D perpendicular is what we have just derived is n k B T plus T i divided by sigma B square. So, here D perpendicular also depends on what the conductivity. So, we have different things now right so this is fully ionized plasma in the absence of collisions.

So, this was some discussion about diffusion the importance of diffusion and how diffusion depends on critical physical parameters in the presence of magnetic field and how it is different from the situation that we have seen in the case of no magnetic field or parallel magnetic field right. So, this is a very important topic in plasma physics. So, try to work out the mathematics algebra steps that are in between all the derivations. I hope you will appreciate this topic and you can explore more about this in any of the standard reference books that are listed in the course page. Thank you.