Plasma Physics and Applications

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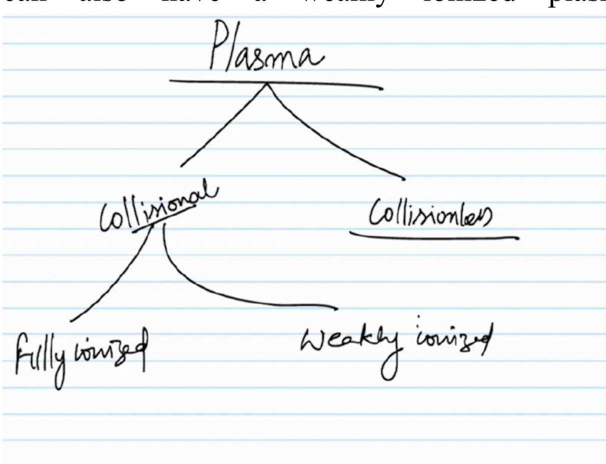
Week – 11

Lecture 56: Diffusion in Presence of B

Hello dear students. In today's lecture we will try to understand the role of magnetic field in plasma diffusion. So, in continuation to our discussion so far we have learnt diffusion will depend on many things it will depend on the collisional frequency, it will depend on the mass, it will depend on the temperature and the primary role of diffusion can be to negate the concentration gradients and in the process the collisions will happen and there are situations in which collisions may not actually be relevant. So, we have to understand if you can if you take plasma and if you are trying to understand diffusion processes we can see plasma in a broad characteristic as one with collisions the plasma in which there are collisions or we can call it as collisional plasma and another type of plasma can be collisionless. So, if you recall our earlier discussions there were some situations when in the absence of collisions there will be no energy transfer that happens from one point to another point and the wave motion can have any velocity it can even be greater than the speed of light. So, things like that and there can be some plasmas in which the plasma is if it is collisional then you can have a fully ionized plasma or you can also have a weakly ionized plasma or a weakly ionized plasma.



So, in fully ionized plasma you will have columbic interactions and in weakly ionized plasma you will have collisions with the neutral species. So, at the beginning of all these discussions we know very well that the particles or the plasma as a whole is highly subjected to a magnetic field. I mean the magnetic field can have a profound influence on the way plasma particles move. Now, the movement is what is represented by the diffusion or diffusion actually talks about movement or what happens in this movement as it collides as the particle collides.

So, now depending on the primary classification that we have just talked about. So, plasmas collisions can happen in all these different types and diffusion is a very much a very essential part of that. So, since the particles movement is highly influenced by the magnetic field it is natural for us to expect that diffusion will change by the application of magnetic field. So, why do we even talk about this because it is a central problem of thermonuclear fusion. If you are trying to solve problems or devise applications or in research related to thermonuclear fusion this is the central problem how to handle diffusion or how diffusion changes with the external magnetic field.

So, to study the effect of magnetic field on diffusion we consider a weakly ionized plasma in which electrons and ions are non uniformly distributed across the length of the plasma confinement and in the constant background of neutral species. So, what do we consider weakly ionized plasma weakly ionized plasma so that means there will be a lot of neutral species in the background and non uniform distribution. Why do we call or we bring this non-informed distribution into the picture because if there is no irregularity in the distribution of plasma then there is no provision to have this pressure gradient or the diffusion to happen. So, since the magnetic field does not affect particles movement in its own direction if the particles are moving in the direction of magnetic field the magnetic field cannot have any influence on those particles. So, we will consider the magnetic field to be in one particular direction and we will try to see what happens in the directions perpendicular to this magnetic fields direction.

— Weakly ionized plasma

— Neutral species in the background

— Non uniform distribution

$$D_a = \frac{k_B T}{m\nu} \; ; \; \underline{\mu}$$

$$\uparrow B$$

— Due to collision, particles will diffuse across the $\vec{B}$ into the walls.

Primarily we will be looking at how the diffusion coefficient dA what is diffusion coefficient this k B T by m nu how this will change as a function of magnetic field or in the presence of magnetic field or how the mobility will change as a function or due to the presence of magnetic field. So, it is natural for us to expect that these two parameters will not change in the direction of magnetic field rather they will change in the perpendicular direction. So, now at this point in the presence of magnetic field due to collisions what will generally happen is that the particles due to collisions particles will diffuse across the magnetic field into the walls. So, to see how this is this can be understood how this comes about we can basically start with the basic fluid equations. So, the basic fluid equation contains the rate of change of velocity on the left hand side and on the right hand side all those parameters all those force terms which will balance the rate of change of velocity on the left hand side.

$$mn\frac{d\vec{u}}{dt} = \underbrace{\pm en\left(\vec{E}+\vec{u}\times\vec{B}\right)}_{L.F} \underbrace{-\vec{\nabla}p}_{P.T} \underbrace{- mn\nu\vec{u}}_{\text{Collisional term}} \quad \text{—} \; (1)$$

$$\frac{d\vec{u}}{dt} = 0 \qquad P = nk_BT$$

$$0 = \pm en\left(\vec{E}+\vec{u}\times\vec{B}\right) - k_BT\vec{\nabla}n - mn\nu\vec{u}$$

$$mn\nu\vec{u} = \pm en\left(\vec{E}+\vec{u}\times\vec{B}\right) - k_BT\nabla n \quad \text{—} \; (2)$$

$$\underbrace{\vec{B} = B\hat{z}}_{} \qquad E = (E_x, E_y, E_z)$$

So, the equation is very familiar to us we have written this equation so many number of times du by dt is plus minus m is the mass n is the number of particles per unit volume. So, this becomes a density actually plus minus En times E plus u cross B minus del p minus m n nu u. So, this term is the Lorentz force this term is the pressure term and this is the collisional term. Now all of these things can of course have an influence on the movement of the particle. Now this after this the mathematical treatment is basically rigorous algebra, but otherwise it is simple.

Now this can be done by neglecting the electric field altogether we can consider with the magnetic field we will still be able to arrive at the conclusions, but let us see how we proceed by including the electric field also. So, now assuming that the deviation from the equilibrium from the stability due to the inhomogeneity in the number of particles per unit volume or non uniform distribution is to be very small. So, assuming that the deviation from the equilibrium due to the concentration gradient to be very small we can assume a steady state we can say that the steady state exists and the rate of change of velocity in this case will be 0. So, pressure p can be written as N k B T using both of these things you can write 0 is equals to plus minus E n E plus u cross p minus k B T del n minus m n mu u or we can conveniently bringing the frictional term or the collisional term here m n mu u is equals to plus minus E n E plus u cross p minus k B T del n. Let us say we call this equation as 1 this as 2.

$$m n \nu u_x = \pm e n E_x \pm e n u_y B_z - k_B T \frac{\partial n}{\partial x} \quad \text{—— (3)}$$

$$m n \nu u_y = \pm e n E_y \mp e n u_x B_z - k_B T \frac{\partial n}{\partial y} \quad \text{—— (4)}$$

$$m n \nu u_z = \pm e n E_z - k_B T \frac{\partial n}{\partial z} \quad \text{—— (5)}$$

$$u_z = \pm \frac{e \cancel{n}}{m \cancel{n} \nu} E_z - \frac{k_B T}{m n \nu} \frac{\partial n}{\partial z}$$

$$u_z = \pm \left(\frac{e}{m\nu}\right) E_z - \frac{D}{n} \frac{\partial n}{\partial z}$$

$$\boxed{u_z = \pm \mu_{\parallel} E_z - \frac{D_{\parallel}}{n} \frac{\partial n}{\partial z}}$$

So, we will assume the magnetic field to be let us say in the z direction. Now this is familiar to us and you will assume the electric field to be E x, E y, E z. So, given these two things we can break this curl and write three component equations from equation number 2 itself. This is a very familiar exercise that we have done earlier multiple number of times. So, with that I will directly write the equations and you can derive it at your convenience.

So, we can write m n mu u x is equals to plus minus E n E x plus minus E n u y B z minus k B T dou n by dou x or m n mu u y is equals to plus minus E y minus plus m u x B z minus k B T dou n by dou y m n mu u z is equals to plus minus E n E z minus k B T dou n by dou z. So, we call these as 3, 4 and 5. I hope there is no confusion in this. So, we have just expanded this E plus u cross B and written three component equations and along each direction I have taken the equivalent term which appears in the gradient of the concentration. Now just take the equation number 5 you can write u z is equals to plus minus E n by m n mu E z minus k B T by m n mu dou n by dou z.

$$u_x = \pm \frac{enE_x}{mn\nu} \pm \frac{enu_yB_z}{mn\nu} - k_BT\frac{\partial n}{\partial x}\cdot\frac{1}{mn\nu}$$

$$\frac{e}{m\nu} = \frac{e}{m\times\frac{1}{T}}$$

$$u_x = \pm \frac{eE_x}{m\nu} \pm \frac{eB_z}{m\nu}u_y - \frac{k_BT}{mn\nu}\frac{\partial n}{\partial x}$$

$$D = \frac{k_BT}{m\nu}$$

$$\boxed{u_x = \pm \mu E_x \pm \frac{\omega_c}{\nu}u_y - \frac{D}{n}\frac{\partial n}{\partial x}} \quad \text{— (6)}$$

$$\omega_c = \frac{eB}{m}$$

Cyclotron frequency.

$$\boxed{u_y = \pm \mu E_y \mp \frac{\omega_c}{\nu}u_x - \frac{D}{n}\frac{\partial n}{\partial y}} \quad \text{— (7)}$$

So, we can cancel n here and you can write u z as plus minus E by m nu times E z minus d by n k T by m nu is the diffusion coefficient times dou n by dou z or u z is plus minus mu E z minus T by n dou n by dou z. So, this equation is familiar if you go like two or three lectures before we have derived a very similar equation in which the mobility and the diffusion coefficient both of them appear. Now the point that I wanted to make here is that the velocity component along the direction of the magnetic field will depend on of course these parameters, but it is unaffected by the magnetic field. So, they will remain a constant. So, we can call these mobility and diffusion coefficients as the parallel components because they are parallel to the magnetic field.

Now let us talk about the other two equations which are equation number 3 and 4. We can simplify these equations by sending all the multipliers of u x and u y to the other side and we can write u x is equals to plus minus E n E x divided by m n mu plus minus E n u y B z divided by m n mu minus k B T dou n by dou x and dou 1 by m n mu. You see this the velocity component u x has this u y and the other terms. Similarly, u y component has this u x. That means that we can use this equation number 4 and we can substitute into equation number 3 and get a combined solution for all that.

Similarly, we can write or you can simplify this as u x is plus minus E E x by m nu plus minus E B by m nu times u y minus k B T by m n nu dou n by dou x or u x is equals to because E by m nu is the mobility we can write u x is equals to mu times E x E B by m. What is E B by m? Something very familiar to us which is omega c the gyration frequency divided by nu u y k B T by m nu is the diffusion coefficient B by n dou n by dou x. This is u x. Let us say we call this equation as equation number 6 and this is an important equation. We will highlight this.

$$u_x(1+\omega_c^2\tau^2) = \pm \mu E_x - \frac{D}{n}\frac{\partial n}{\partial x} + \frac{\omega_c^2\tau^2 E_y}{B} \mp \omega_c^2\tau^2 \frac{k_B T}{eB}\frac{1}{n}\frac{\partial n}{\partial y} \qquad (8)$$

$$u_y(1+\omega_c^2\tau^2) = \pm \mu E_y - \frac{D}{n}\frac{\partial n}{\partial y} - \frac{\omega_c^2\tau^2 E_x}{B} \pm \omega_c^2\tau^2 \frac{k_B T}{eB}\frac{1}{n}\frac{\partial n}{\partial x} \qquad (9)$$

$$\tau = \frac{1}{\nu}$$

$$u_x = \frac{\pm \mu E_x}{(1+\omega_c^2\tau^2)} - \frac{D}{n}\frac{\partial n}{\partial x}\frac{1}{(1+\omega_c^2\tau^2)}$$

$$u_y = \frac{\pm \mu E_y}{(1+\omega_c^2\tau^2)} - \frac{D}{n}\frac{\partial n}{\partial y}\frac{1}{(1+\omega_c^2\tau^2)}$$

So, this is u x. In a very similar way, you can also write an equation for u y as u y as plus minus mu E y minus plus omega c by nu. I hope you remember why we are using this some symbols of plus minus and minus plus T by n. Even if you do not remember, you can go back and watch the earlier videos. This will be clear. So, equation number 6 and 7 are nothing but 3 and 4 written in a clear form.

That is it. Where E by m nu this is charge by mass into 1 by time collisional frequency. And diffusion coefficient is k B T energy by m nu and for reference omega c is E B by m or q B by m. This is the cyclotron frequency not the collision frequency. This is the cyclotron or gyration frequency. Now, we have to solve equation number 6 and 7.

That is the main task. This is simple algebra. You can solve it by yourself. Just have to substitute one term into other and then make some changes you will get the form anyway. So, task is to have in the u x term in the u x equation you should not have any u y and in the u y equation you should not have any u x. So, you have to decouple these two equations and then you will get a solution of this form which is u x times 1 plus omega c square tau square.

What is tau? Tau is the inverse of collisional frequency. So, u x times 1 plus omega c square tau square is plus minus mu e x minus d by n dou n by dou x plus omega c square tau square e y divided by B minus plus omega c square tau square k B T by e B 1 by n dou n by dou y. Similarly, u y times 1 plus omega c square tau square is plus minus mu e y minus d by n dou n by dou y e square tau square e x divided by B plus minus omega c square tau square k B T by e B 1 by n dou n by dou x. It is a very good exercise that you can do this algebra and get these two equations.

$$\left.\begin{aligned}\mu_\perp &= \frac{\mu}{1+\omega_c^2\tau^2} \\ D_\perp &= \frac{D}{1+\omega_c^2\tau^2}\end{aligned}\right\}$$

$$\omega_c = \frac{qB}{m}$$

they have decreased by $\frac{1}{1+\omega_c^2\tau^2}$

is only because of B_z

Strong $\vec{B} \Rightarrow \omega_c^2\tau^2 >> 1$

$$D_\perp = \frac{D}{\omega_c^2\tau^2} = \frac{k_BT}{m\nu}\cdot\frac{\nu^2}{q^2B^2} = \frac{k_BT\nu m}{q^2B^2}$$

This is 8 and this is 9. Now, what is tau? Tau is 1 by frequency. So, if frequency tells you how many number of collisions will happen per unit time, tau is the relaxation time. It is time between the collisions or between two successive collisions. Now, you have four terms on the right hand side for each velocity.

$$D_\perp = \frac{k_BT\nu m}{q^2B^2} \qquad D = \frac{k_BT}{m\nu}$$

$$\boxed{D_\perp \propto m} \qquad \boxed{D \propto \frac{1}{m}}$$

$$\boxed{D_\perp \propto \nu} \qquad \boxed{D \propto \frac{1}{\nu}}$$

$$\nu \propto \frac{1}{\sqrt{m}}$$

We have 1, 2, 3, 4. So, the first two terms of the equation represent the mobility and diffusion in a direction parallel to the gradients in potential and density. e x represents the gradient in the potential, dou n by dou x represents the gradient in the number density or the concentration. But they are perpendicular to the magnetic field, the mobility and the diffusion are perpendicular to the magnetic field because they are characterizing the movement in x direction which is perpendicular to the z direction. The last two term if you see carefully represent the e cross B drift and the diamagnetic drift. Now, if you wonder what are these e cross B drift and the diamagnetic drift, then probably you will have to go back and see the earlier lectures.

Q1) In a weakly ionized plasma (2eV) mobility is 120 $m^2/V\text{-}s$

find 'D'

$$D = \frac{k_B T}{m\nu}$$

$$\mu = \frac{e}{m\nu}$$

$$\frac{D}{\mu} = \frac{k_B T}{e}$$

$$D = \frac{\mu_e k_B T_e}{e}$$

$$D = \frac{120 \times 2 \times 1.6 \times 10^{-19}}{1.6 \times 10^{-19}}$$

$$\boxed{D = 240\ m^2/s}$$

When you talk about e cross B drift, you are referring to the particle description of plasma when plasma is subjected to simultaneous E and B fields which are perpendicular. It will experience a drift which is called as e cross B drift and the diamagnetic drift is when you call this plasma as a fluid and then you can get that diamagnetic drift. Now, these two things e cross B drift and the diamagnetic drift by definition will be perpendicular to the gradient of potential and gradient in the number density perpendicular not parallel. This is parallel to the gradient this is perpendicular to the gradient. Now, with these equations what we will do is, we will not bother ourselves with the e cross B drift and the diamagnetic drift because they do not tell you anything about the diffusion or the mobility.

So, we will confine ourselves to the discussion of only the mobility and the diffusion. So, then we can write the equation u x as plus minus mu E x divided by 1 plus omega c square tau square minus d by N dou N by dou x times 1 by 1 plus omega c square tau square and u y is plus minus mu E y times 1 plus omega c square tau square minus d by

N dou N by dou y times 1 by 1 plus omega c square tau square. So, this is a modified scenario we just have everything else pushed to the right side. Now, the mobilities and the diffusion coefficients that we have we see here are basically perpendicular to the magnetic field. So, we can now define the perpendicular mobility and perpendicular diffusion as mu perpendicular is whatever the mu is there E by m mu divided by 1 plus omega c square tau square.

And similarly the diffusion coefficient the perpendicular diffusion coefficient is the normal d divided by 1 plus omega c square tau square. So, what it means is that the diffusion and the mobilities in the perpendicular direction to the magnetic field are decreased by a factor of they have been scaled down they have decreased by a factor of 1 by 1 plus omega c square tau square. Now, you see if you just compare this two equations with this equation yeah with this equation you see this equation has no hint of magnetic field. So, this you get the motion of particles will only be dependent on the diffusion parallel diffusion or the mobility, but there is no hint of magnetic field because the magnetic field is parallel to this direction. But if you put this equation in contrast with these two equations what you see? You see the same form the basic form is still the same, but everything appears to be scaled down by this additional factor 1 by 1 plus omega c square t square.

Now, this scaling or the decrease is only because of only because of B z. Now, you do not see a B z here, but omega c is q B by m there it is the magnetic field is implicitly appearing in this scaling factor in terms of omega c. So, this is the basic conclusion of this topic which is the magnetic fields effect is to decrease the mobility and the diffusion coefficient by this particular factor. Now, when you consider a very strong magnetic field very strong magnetic field what will happen is omega c square tau square will become much greater than 1. Omega c is q B by m the cyclotron frequency.

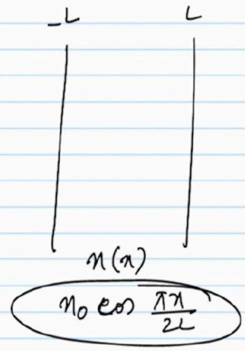
The cyclotron frequency tells you how many gyrations the particle can make per unit time and tau gives the collisional relaxation time. How much duration is required between two successive collisions? So, these two things are exactly kind of opposite to each other. One is telling you how much time it will require between two successive collisions and another is telling you before one collision what is the frequency of this motion of the particle. But when you consider the magnetic field to be very very strong so this numerator becomes large q B by m becomes very very large. So, omega c square t square will become much larger than 1.

Q2) Weakly ionized slab

$$n(x) = n_0 \cos\frac{\pi x}{2L} \qquad -L<x<L$$

$$Re\,(\text{rate}) = -\alpha n^2$$

Find 'n' $\qquad e^- + X^+ = X + h\nu$



$$\text{Loss due to diffusion} = D\frac{d^2 n}{dx^2}$$

$$= D\, n_0 \left(\frac{\pi}{2L}\right)^2 \cos\frac{\pi x}{2L}$$

$$\frac{dn}{dt} = D n \left(\frac{\pi}{2L}\right)^2 = -\alpha n^2$$

$$\boxed{n = \frac{D}{\alpha}\left(\frac{\pi}{2L}\right)^2}$$

So, then the D perpendicular diffusion coefficient perpendicular diffusion coefficient can be written as d by because see earlier it is just a number all of this that appears in the denominator is just a number because you are adding it to 1. So, it is just a number is a scaling factor. Now, so, D perpendicular becomes d by omega c square tau square. So, we can write it as D we know as k B T by m nu times tau square is nothing but nu square and omega c square is q square B square divided by m square which will give you k B T nu m divided by q square B square. Let us see the D perpendicular is k B T nu m divided by q square B square.

The D that we know very well is k B T by m nu. So, you see the difference the perpendicular diffusion coefficient is directly proportional to mass and the diffusion coefficient otherwise is universally proportional to the mass. So, this drift is of course perpendicular to the magnetic field that is what we have been telling from the beginning. So, the perpendicular diffusion coefficient is proportional to the collisional frequency. So, this is also one more D perpendicular is proportional to the collisional frequency.

The physical meaning of this is because the collisions are needed for plasma to diffuse across the magnetic field. Whereas, diffusion in the diffusion coefficient in the absence of magnetic field or in direction per parallel to the magnetic field is proportional to 1 by nu. So, it is because how do we explain this? Because the collisions generally retard the movement they would not facilitate an easier movement from point to point. So, this is the basic difference the diffusion coefficient proportional to the frequency and the diffusion coefficient proportional to 1 by frequency. So, now if you remember the fact that the collisional frequency is generally proportional to 1 by square root of m that means electrons diffuse more rapidly than the ions because of their lower mass and higher mobility or higher thermal velocities.

So, in a perpendicular diffusion electrons generally move slower than the ions. So, this is some discussion about how particles will diffuse in the presence of magnetic field. Now what we have seen so far is that it is a weakly anise plasma in which particles are the diffusion the coefficient of diffusion and the mobility are strongly influenced by the presence of magnetic field. So, based on these concepts it is essential that we solve a couple of numerical problems so that we appreciate the physics even better. So, we will consider a simple numerical question 1.

So, let us say in a weakly ionized plasma characterized by 2 electron volt mobility is 120 meter square per volt second. Find the diffusion coefficient. So, we know that d by mu is k B T by E because d is k B T by m mu and mu is E by m mu. So, d by mu why we use this relation why cannot we directly calculate the diffusion coefficient because we know the formula we do not know the mass we do not know the frequency, but mobility is given.

So, d by mu is equals to k B T by E. So, diffusion coefficient d is equals to mu E k B T divided by E. The temperature in the units of electron volt is already given to us. So, d is equals to 120 times 2 electron volts into 1.6 10 to the power of minus 19 divided by the charge again 1.

6 10 to the power of minus 19. So, the diffusion coefficient d is 240 meter square per second. So, this is one problem that we can quickly solve based on the understanding between the mobility and the diffusion coefficient. We have one more numerical which can be solved by using the diffusion equation and involving the fact that the density gradient comes into this diffusion equation. So, the problem is you have a question number 2. We have weakly ionized plasma the plasma is weakly ionized.

So, we are still doing problems which are relevant to topics that we have understood so far we still have not gone into fully ionized plasma. So, weakly ionized plasma the slab geometry is given like this. Slab geometry permits a concentration gradient which is like N of x is equals to N naught cos pi x by 2 L between minus L 2 L like this. So, we have the slab arrangement in which the density the concentration of charged particles seems to be arranged or following this particular relation. Now suppose that I am not writing this text here just saying it so that you can write down or you can grab the context.

Suppose the plasma decays both by diffusion and recombination and assume the rate of loss due to recombination the recombination rate is minus alpha N square is minus alpha N square where alpha is just a constant. Find N find the value of N at which the rates of recombination and the diffusion are equal. So, what have you been given? You have been given an arrangement in which plasma is not uniformly distributed rather following a distribution which is N of x which is N naught cos pi x by 2 L. So, you can put various values of x between minus L to L and get what is the concentration at each and every point of this particular slab geometry. Now the plasma is diffusing from this higher concentration to lower concentration while it happens it also undergoes recombination.

Recombination is simply electron plus an ion giving you a neutral species. And at the same time it is also going through diffusion. Now we know the rate at which diffusion can take care of plasma now we have to find the concentration gradient such that both these two rates are same. So, loss due to diffusion is dN by dt this is equals to d times d square N by dx square this is what just we did. So, since we know the value of N we can substitute N is equals to N naught cos pi x by L obtain two derivatives and then what we get is d times N naught pi by 2 L whole square cos pi x by 2 L.

Now this is of course this is this multiplied by this is still N naught is still N. So, this is d N times pi by 2 L whole square this is what this is d N by dt due to diffusion. Now we have been told that this rate is actually also equal to minus alpha N square this is by the recombination both of them are same. Now we have to find N such that this equality is valid. So, N obviously becomes d by alpha times pi by 2 L whole square.

So, N is equals to d by alpha pi by 2 L whole square what is what is so specific about this form of N if N follows this distribution then the loss due to the recombination will be equal to the loss by diffusion. So, both these processes will have the same speed. So, we will try to have one more lecture for diffusion to understand the diffusion process in the presence of a magnetic field in a completely or fully ionized plasma in the next lecture. Thank you.