Plasma Physics and Applications

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Week – 11

Lecture 55: Diffusion Equation

Hello dear students. We are currently discussing diffusion and collisions in plasma. We have seen what is the an bipolar diffusion and how it attributes to constant velocities or uniform velocities for particles, plasma particles with unequal masses. In today's class, we will try to understand few more aspects about diffusion processes in plasma physics. We will try to get a diffusion equation and we will try to see how plasma will diffuse between within a boundary, confined between two plates, how plasma will diffuse and what are the parameters which will play a role in the rate at which plasma diffuses into these plates. And we will also try to see how plasmas diffusion is affected in the presence of a magnetic field whether it is a weakly ionized plasma or fully ionized plasma.

So, we can start from the ambipolar diffusion itself wherein we have learnt the formation of electric fields which make the diffusion coefficient to be different for both the electrons and ions thereby resulting in equal speeds of these particles. Now, we have learnt that the ambipolar diffusion the particle flux which is given by gamma is minus dA del N. So, what is this? This is let us say we call this equation as 1. Let us try to recall this equation.

Gamma represents the particle flux and dA is the ambipolar diffusion coefficient which seem to depend on Te plus something like kB Te plus Ti by m mu. Here kB is the Boltzmann's constant, Te and Ti are the temperatures of electron and ions m mass and mu is the collisional frequency and delta N is the gradient of concentration in plasma. So, let us try to bring the equations of continuity for the plasma. So, what is this? This is the equation of continuity or dou N by dou T plus del dot N u is equals to 0. So, this N u is actually the flux.

So, we can rewrite this equation as dou N by dou T plus del dot gamma is equals to 0.

Let us say we call this as equation number 2 and this as equation number 3. Now, the first equation is just the flux which is defined in terms of the diffusion coefficient and the concentration gradient. The second equation is the equation of continuity and third is another form of continuity which is written in the units of or in the form in using the flux. So, we can use using equation 1 in 3.

$$\vec{\Gamma} = -D_a \vec{\nabla} n \quad \text{— } 1$$

$$\frac{k_B(T_e + T_i)}{m \nu}$$

Equation of Continuity

$$\frac{\partial n}{\partial t} + \vec{\nabla}\cdot(n u) = 0 \quad \text{— } 2$$

$$\frac{\partial n}{\partial t} + \vec{\nabla}\cdot(\vec{\Gamma}) = 0 \quad \text{— } 3$$

Using 1 in 3

$$\frac{\partial n}{\partial t} + \vec{\nabla}\cdot(-D_a \vec{\nabla} n) = 0$$

$$\boxed{\frac{\partial n}{\partial t} - D_a \nabla^2 n = 0} \quad \text{— } 4$$

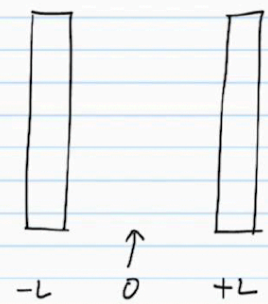
Then we can write dou N by dou T plus del dot minus dA del N is equals to 0. Doing some simplification we can write the rate at which the number density changes with respect to time minus dA del square N is equals to 0. So, dou N by dou T minus dA del square N is equals to 0. What is this? This is the continuity equation written by using the diffusion coefficient into it. Let us say we call this equation as 4.

So, this equation can be used to study the temporal and spatial evolution of plasma under the scenario of diffusion. So, this equation is the continuity equation. What are we trying to understand? So, this equation tells you how the mass changes with respect to time. So, the rate of change of mass should be equal to the difference of amount of flux that is getting in and the amount of flux that is going out. So, that is the simple notion of the continuity equation.

$$\frac{\partial n}{\partial t} - D_a \nabla^2 n = 0 \quad \text{—} \quad (5)$$

$$\frac{dn}{dt} - D_a \frac{d^2 n}{dx^2} = 0$$

$$\frac{\partial n}{\partial t} - D_a \frac{\partial^2 n}{\partial x^2} = 0 \quad \text{—} \quad (5)$$



separating variables

$$n(x,t) = X(x)\,T(t) \quad \text{—} \quad (6)$$

Using (6) into (5) and dividing by XT

$$\frac{1}{XT}\frac{\partial}{\partial t} X(x)T(t) = \frac{D_a}{XT}\frac{\partial^2}{\partial x^2} X(x)T(t)$$

But here under the context of diffusion, what is diffusion? Diffusion is movement of certain particles in order to negate the pressure gradient. And in this process they may encounter collisions. Collisions will have a frequency because the diffusion coefficient is something which comprises all this information. So, we have been able to write the continuity equation in the units of the amipolar diffusion coefficient. But this tells you how the spatial and temporal variations can be understood in the context of diffusion.

Now using this equation we can examine how the plasma that is created between two parallel planes will decay by diffusion. So, we will consider two parallel planes something like this. Let us say at the centre we refer it to as 0 and this is plus L and this is minus L. The scenario is we can examine how the plasma that is created between these two parallel planes decays by diffusion. Decays means plasma can recombine all the electrons and ions can recombine resulting in a net neutral state of the gas.

$$\frac{1}{\cancel{X}T}\cancel{X(x)}\frac{\partial}{\partial t}T(t) = \frac{D_a}{X\cancel{T}}\cancel{T(t)}\frac{\partial^2}{\partial x^2}X(x)$$

$$\underbrace{\frac{1}{T}\frac{\partial}{\partial t}T(t)}_{f(t)} = \underbrace{\frac{D_a}{X}\frac{\partial^2}{\partial x^2}X(x)}_{f(x)}$$

$$\frac{\partial n}{\partial t} = D_a \nabla^2 n.$$

$n(x,t)$ $\quad n = XT$

$n(x,t)$

constant

Let us assume $= -\frac{1}{\tau}$

$$\frac{dT}{dt} = -\frac{1}{\tau}T \quad \text{---} \quad (7)$$

So, you will use this equation and find out how the plasma will diffuse when it travels towards these walls. And more importantly we will try to understand what are the parameters which will decide the rate at which the plasma diffusion will happen and the rate at which plasma will decay. So, let us say we can write the equation which this dou N by dou t is equals to dA del square N. So, which is we can write dou N by dou t minus dA del square N is equals to 0. So, this term is variation with respect to time the number density variation with respect to time that is related to the variation of maybe the second order variation of number density with respect to space.

So, this has all the dimensions in it. So, we call this as 5 and we can write this equation in one dimension as let us say dN by dt minus dA d square N by dx square is equals to 0. So, we are not including the y and z directions since there are only two directions I mean time is one dimension and there is the space dimension. So, we can also write it as dou N by dou t. So, we are evaluating the variation of N partial variation of N while keeping x constant with respect to t minus dA dou square N by dou x square is equals to 0.

$$\frac{D_a}{X}\frac{d^2X}{dx^2} = -\frac{1}{\tau}$$

$$\Rightarrow \frac{d^2X}{dx^2} = -\frac{1}{D_a\tau}X \quad \text{—} (8)$$

$$(7) \Rightarrow \frac{dT}{T} = -\frac{1}{\tau}dt$$

$$T(t) = T_0 \exp(-t/\tau)$$

$$\ln T = -\frac{1}{\tau}t + \text{constant}$$

$$@\, t = 0 \Rightarrow k = \ln T_0$$

$$n(x,t) = X(x)\underline{\underline{T(t)}}$$

$$\ln\frac{T}{T_0} = -\frac{t}{\tau}$$

$$\Rightarrow \frac{T}{T_0} = e^{-t/\tau}$$

$$\boxed{T = T_0 e^{-t/\tau}}$$

Now, this equation how do we solve this equation? This equation can be solved by using the method of separation of variables because it is the form of the equation is convenient enough for us to use this method of separating variables. So, we can solve it by variable separable. So, by separating variables. So, in this method what do we do? So, if N is the solution of this equation which is dependent on x as well as t, we can write the N as something a function which is purely dependent on x and another function which is purely dependent on time. So, this method is familiar to us.

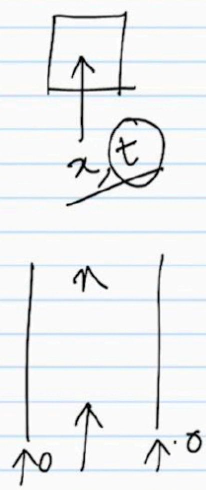
So, what we can do is so N of x can be used as x a function of small x capital T a function of small t. Let us say we call this equation as equation number 6. So, we can use this into using 6 into the this is also equation number 5 this is the same form into 5 and dividing by let us say capital X t. So, what will we get? We will get dou by dou t of x of x T of t into 1 by x t is equals to d A dou square by dou x square of x of x T of t and this is divided by x t. So, dou by dou t will only operate on t and dou square by dou x square will only operate on x of x.

So, the other term will remain a constant and will come out of the differential operator. So, we can write it as 1 by x t because we are dividing with x t times x dou by dou t of T of t is equals to d A divided by x t times T of t. It is implicit it just helps if it is written explicitly square of x of x. So, this x will get cancelled here and this t will get cancelled here and as a result you will get something like this 1 by t dou by dou t of T of t is equals to d A by x dou square by dou x square of x of x. Now what you have is now you have the left hand side only a function of time and the right hand side only a function of x.

What is the difference by the way? How is it different from the earlier form of the equation? Let us bring it. We have dou by dou t of n which is equals to d A del square del square n. You see this since n is a function of x and t on both sides of this equation we have terms which are function of x as well as t. Using this solution n is equals to capital X t what we have been able to achieve is we have been able to separate this equation on the left hand side we have only a function of time and on the right hand side only a function of x. But the interesting thing about this is that both of these variations are actually equal.

$$\frac{d^2X}{dx^2} = -\frac{X}{D_a\tau}$$

$$X(x) = A\cos\frac{1}{\sqrt{D_a\tau}}x + B\sin\frac{1}{\sqrt{D_a\tau}}x.$$



A & B are constants which can be evaluated based on the geometry of slab.

$x = \pm L$

$B = 0$

@ $x = \pm L \Rightarrow$ Plasma density is zero

@ $x = 0 \Rightarrow$ The density is max

$X(x)$ $\qquad n(x) = Max$

So, for having a consistent solution or a non trivial solution or for the equality to be valid we can only expect that both of these terms are equal to something and it is convenient to establish this something as a constant. So, both of these sides individually must be equal to a constant only then so it is quite trivial. Both of these should be equal to the same constant and now let us assume this particular constant as let us say let us assume this constant to be something minus 1 by tau. Now we can equate this minus 1 by tau individually to left hand side as well as the right hand side and then seek a solution for both capital T as well as x of x. So, it means that you have to construct a solution for capital T of t for which if you take a time derivative you will get a constant and you have to construct a solution for capital X of x which is purely a function of x for which if you take a double derivative with respect to x it will give you the same constant that you have seen on the left hand side.

So, mathematically the process is simple but what actually you are doing is what just I have explained. So, since you assumed this constant to be minus 1 by tau we can say that dT T capital T by dt. Now there is no need for any partial derivative because it is only a function of time capital T is only a function of time earlier n used to be a function of x as well as t. So, you have to take independent derivatives while keeping the others constant so that that derivative is nothing but the partial derivative. Variation in a physical quantity which is evaluated only with respect to one independent variable whereas the other independent variables are being held constant is the partial derivative.

$$\frac{L}{\sqrt{D_a \tau}} = \frac{\pi}{2}$$

$$\Rightarrow \tau = \left(\frac{2L}{\pi}\right)^2 \cdot \frac{1}{D_a}$$

$$n(x,t) = X(x)\,T(t) \longleftarrow$$

$$n(x,t) = n_0 e^{-t/\tau}$$

$$T(t) = T_0 e^{-t/\tau}$$

$$X(x) = A \cos \frac{\pi x}{2L}$$

$$n(x,t) = X_0 \cos \frac{\pi x}{2L} * T_0 e^{-t}$$

$$B = 0$$

$$A \cos \frac{1}{\sqrt{D_a \tau}} x = n_0$$

$$\frac{L}{\sqrt{D_a \tau}} = \frac{\pi}{2}$$

$$\frac{1}{\sqrt{D_a \tau}} = \frac{\pi}{2L}$$

Now that T is only a function of t we can shift it to the total derivative dT by dt is equals to minus 1 by tau capital T. So, let us say we call this equation as equation number 7. And similarly dA by x d square x by dx square should also be equal to minus 1 by tau or d square x by dx square is minus 1 by dA tau x and let us say we call this equation as equation number 8. Now let us say we try to solve this equations. Now we have dT from equation number 7 we have dT by T is equals to minus 1 by tau dT.

If we integrate both sides what we have is we have ln T is minus 1 by tau T plus a constant. How do we get the value of this that constant? We can use the boundary conditions and let us say at T is equals to 0 this constant K is equals to ln T naught. At T is equals to 0 the constant becomes ln T naught you put T is equals to 0 then you will realize constant is equals to ln T naught. So substituting this we can write ln T by T naught is equals to minus T by tau or T by T naught is e to the power of minus T by tau or T is equals to T naught e to the power of minus T by tau. Now this is the solution what does it mean? If you take this form for T and substitute in equation number 7 you will get that the constant is minus 1 by tau, but this is a very important solution.

Now how about the other part? So T is equals to T naught e to the power of minus T by tau is one solution. So similarly we have to take the other part which is the d square x by dx square is minus x by dA tau. Now how can we solve this differential equation the second order differential equation there are multiple methods you can assume solution for x. A solution which is so this is the second order differential equation so you can assume a solution such that you take two derivatives of that you still have some form of the same solution appearing on the right hand side. So the solution can be of this form x of x is A cos 1 by square root of dA tau x plus B sin 1 by square root of dA tau x.

$$\boxed{n(x,t) = n_0 \cos\frac{\pi x}{2L}\, e^{-t/\tau}} \qquad n_0 = X_0 T_0$$

Co Variation of Concentration w.r.to space and time.

$$\boxed{n(x,t) = n_0 \cos\frac{1}{\sqrt{D_a \tau}}\, x\, e^{-t/\tau}}$$

Now what are A and B? A and B are constants which can be evaluated based on the geometry of slab. Where the plasma was generated. Now how did we make the geometry of the slab? This is how the slab is. Now the plasma is generated between the slabs and we are trying to see what is the role of diffusion or how plasma will decay as a result of the diffusion which also carries the information of the collisions. So this is the geometry of the plasma and this is minus L plus L the both sides of this.

Now the density of plasma is maximum between the walls this is here the density is maximum and it decays or decreases as you go towards the walls. So x the distance x can be between plus L and minus L. So minus L to L is the total distance. So now this based on the symmetry of the problem you just substitute the boundary values you will realize that the constant B has to be 0 and at x is equals to plus minus L the plasma density is 0. And at x is equals to 0 which is just in between the density is maximum.

So N of x is max but you are not calling N of x you are rather you have given a specific name to that N capital X of x which is actually the number density number of charged particles per unit volume as a function of x only. So the point is in any given volume the plasma density because of the diffusion or the collisions can change across the distance or with respect to the time this is the thing. Now with respect to time how the plasma density changes you have realized it to be something like this. Now you must remember T of t is T0 exponential minus T by tau. So this is of course a name that is given for the solution but what is T? T is actually a part of N, N of x t is what? This is x of x and T of t.

That means T is still talking about N nothing else. That means that plasma density changes with respect to time in this exponential fashion and with respect to the distance it changes like this. How does it change? At middle of the slab the plasma density is maximum N is maximum at either sides on either sides it is 0. So this type of geometry simply requires the constant B to be equal to 0. You can substitute B is x is equal to 0 and you will realize how the constant or this B will become 0.

So for the density to be maximum at the center and 0 on either sides you will require because it is sinusoidal in nature the solution that you have assumed to be sinusoidal in nature you want this condition to be maintained at plus minus L dA tau x you are putting x as L. So should be equal to pi by 2 which implies tau is equals to 2 L by pi whole square times 1 by dA simple algebra will give you this particular form. We have both the things now we know how we can get x of x and T of t. So, we know N of x t as x of x times T of t. So, we want this at L we want the density to be maximum.

So, the density can be assumed to be since B is equal to 0 the solution can become A cos 1 by square root of dA tau x is N naught and for this to be valid you would require when x is being substituted as L L by square root of dA tau is equals to pi by 2 and we can write 1 by square root of dA tau as pi by 2 L. So, you can substitute this here. Now coming back this is the original solution that we had. So, N of x t is equals to x of x into T of t. So, x of x at the middle is simply equal to the maximum density is say N naught e to the power of minus T by tau.

But we know that T of t is T naught e to the power of minus T by tau and x of x is this A cos pi x by 2 L. Now substituting all of that N of x t is x naught some value for A x naught cos pi x by 2 L times T naught e to the power of minus T by tau. Re-arranging the terms we can write N of x t is equals to N naught cos pi x by 2 L times e to the power of minus T by tau. What is N naught? N naught is nothing but x naught which is a constant that appeared in the sinusoidal solution and T naught which is another constant which appeared in the exponential solution for time dependent part and this is the space dependent part. So, this is the way the concentration changes with respect to space as well as time.

This is a very interesting solution if you see this you have both cos pi x and exponential minus T. So, concentration as a function of space and time decays exponentially as a result of diffusion. So, this is the variation of concentration with respect to space and time. So, cos pi x by 2 L is a sinusoidal term, but e to the power of minus T by t is an exponentially decaying term. So, the time constant or a unit with respect to which changes happen the smallest unit the time constant can be seen to increase with respect to L and decrease with respect to dA.

This is dA is there here. So, dA has to be there in the picture dA by x dou square by dou x. So, dA can be if you write pi by 2 L you can write it as N of x T is N naught cos pi by 2 L is nothing but 1 by dA tau under root cos 1 by square root of dA tau x e to the power of minus T by tau. So, what I was trying to say is the rate at which density changes with respect to time can be seen with increasing value of L. So, the time constant will increase with L with respect to L and will decrease with respect to the diffusion coefficient. So, this concludes the discussion on how plasma that is produced between two plates of separated by a distance of e minus L to L will decay with respect to space and time as a result of diffusion and collisions.

So, we will continue this discussion and try to understand what will be the role of magnetic field in the plasma diffusion. Because if you understand we have seen how plasmas movement as a particle or as a fluid is influenced by the magnetic field. So, if you consider even the plasma to be a fluid we have realized that it undergoes drift which is called as the diamagnetic drift. The point is plasmas motion either as a particle or as a fluid is highly influenced by the presence of magnetic field. So, this gives us enough reason to question whether the diffusion process which is very intrinsic to the plasma also gets influenced by the presence of a magnetic field.

Now, this creates two situations whether if it is weakly ionized or fully ionized. So, in weakly ionized plasma you have a lot of neutral species in fully ionized plasma you have collisions in terms of the columbic interactions. So, we will see how diffusion takes place in the presence of magnetic field for weakly ionized and fully ionized plasmas in the next lecture. Thank you.