Plasma Physics and Applications

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Lecture 50: Electromagnetic Waves in Plasma

Hello dear students. In today's class we will discuss about the propagation of electromagnetic waves in plasma. So far in our discussions we have seen electron oscillations, electron plasma frequency, dispersion relation for electron plasma waves, acoustic waves and ion acoustic waves we have derived the dispersion relation and we have discussed what are the characteristic features of such wave propagation in plasma. So, in today's class we know very well that electromagnetic waves such as radio waves propagate through plasma. So, propagation so we are discussing plasma and waves and the sub topic for today's discussion is propagation of electromagnetic waves in plasma. Electromagnetic waves in plasma.

So, for our discussion we will consider the nature of plasma to be stationary then we will assume the plasma to be cold that means there are no thermal motions of electrons and ions. We will assume the plasma to be unmagnetized what does it mean? There are no intrinsic magnetic fields but the electric field generated magnetic fields can still be present. The plasma is assumed to be homogeneous and collisionless plasma. And furthermore we will assume that the electrons and ions to be at rest.

We are considering a plasma which is stationary cold unmagnetized homogeneous collisionless and in which electrons and ions are assumed to be at rest. Now we have to see how the if the electromagnetic wave has to pass through this plasma what will be the dispersion relation for that and what are the characteristic features that will can be derived from this dispersion relation. So, let us say we can use the electron fluid equation can be expressed in the following way which is dou by dou T of Ne plus del dot Ne Ue is equals to 0. What is this? This is the continuity equation for electrons. Then we have the momentum equation which is mass times dou by dou T the convective derivative plus U dot del times Ue is equals to minus E times E plus U cross B.

Plasma & waves

EM waves in Plasma

$$\frac{\partial}{\partial t} n_e + \vec{\nabla}\cdot(n_e \vec{u}_e) = 0 \quad \text{--- } (1)$$

$$m\left[\frac{\partial}{\partial t} + \cancelto{0}{(\vec{u}\cdot\vec{\nabla})}\right]u_e = -e\left(\vec{E} + \vec{u}\times\vec{B}\right) \quad \text{--- } (2)$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{--- } (3)$$

$$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \epsilon_0\frac{\partial \vec{E}}{\partial t}\right) \quad \text{--- } (4)$$

$$J = -ne\vec{u}$$

1) Stationary
2) Cold
3) Unmagnetized $\vec{E} \Rightarrow B$
4) Homogeneous
5) Collisionless
6) e^- & ions to be at rest

Now these two equations are from the electron fluid equations which describe how the electromagnetic force will act on the plasma and how the continuity is maintained the law of conservation of mass is maintained inside the plasma. Now that we are considering the propagation of electromagnetic waves inside the plasma we should also use the Maxwell equations in this picture or in combination with this equation. So, Maxwell equations which are relevant for this discussion is del cross E is minus dou B by dou T and del cross B is equals to mu naught times the current density plus epsilon naught dou E by dou T. So, these two are the Maxwell equation that we require. The other Maxwell equations are del dot B is equals to 0 already made the magnetic field not existing in this picture and the other Maxwell equation is del dot E is equals to rho by epsilon naught.

So, let us say we call this equation as 1, 2, 3, 4. Now we can write the current density J is generally written as N e v or for electrons we will write minus e N e say u. Now what is the general approach of doing perturbation theory? We take the perturbed variables which are going to be affected by the propagation of electromagnetic wave and then we define them to be a sum of equilibrium part plus the perturbation part and then using those we will linearize the equations. So, the set of equations the set of perturbed variables are u is equals to u naught plus u 1 n the number density is n naught plus n 1 the electric field is E naught plus E 1. Now the task is simple we can use it in equation 1, 2, 3 and 4 and write down the linearized equations.

$\omega(t)$ $(-\omega)$

Linearised governing equations

e^- fluid equations & Maxwell equations

$$\vec{u} = \vec{u}_0 + \vec{u}_1$$

$$n = n_0 + n_1$$

$$\vec{E} = \vec{E}_0 + \vec{E}_1$$

$$\frac{\partial n_1}{\partial t} + \nabla\cdot(n_1\vec{u}_1) = 0 \quad \text{— (1a)}$$

$$m\frac{\partial \vec{u}_1}{\partial t} = -e\vec{E}_1 \quad \text{— (2a)}$$

$$\vec{\nabla}\times\vec{E}_1 = -\frac{\partial B_1}{\partial t} \quad \text{— (3a)}$$

$$\vec{\nabla}\times\vec{B}_1 = \mu_0\left(\vec{J} + \epsilon_0\frac{\partial \vec{E}_1}{\partial t}\right) \quad \text{— (4a)}$$

$$\vec{J} = -n_0 e\vec{u}_1$$

1) $n = n_0 + n_1$

2) $n_1 = n_1\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$u_1 = u_1\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$E_1 = E_1\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

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Since we have done this earlier also where I am going to write the linearized equations directly dou N 1 by dou t plus del dot N 1 u 1 is equals to 0 or m times dou u 1 by dou t here the convective part of the sorry edit here. Here the advection term is removed from the convection derivative so this is made to be 0. So, we have the remaining term here and the electric field is written as E 1 and the magnetic field is not taken into account because of the initial assumptions and del cross E 1 is equals to minus dou B 1 by dou t and del cross B 1 is u naught times J plus epsilon naught dou E 1 by dou t and the current density can still be written as minus N naught E u 1. Note this these are the linearized governing equations which are a combination of electron fluid equations and the relevant Maxwell equations. Now what is the second step? The first step being writing the perturbed part, the second step being taking a solution in the sinusoidal form e to the power of i k dot R minus omega t.

Substituting (5) in (1a) to (4a)

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x}$$

$$-i\omega n_1 + i\vec{k}\cdot(n_0 u_1) = 0 \quad \text{— (a)}$$

$$\cancel{-}i\omega m\vec{u}_1 = \cancel{-}e\vec{E}_1 \quad \text{— (b)}$$

$$i(\vec{k}\times\vec{E}_1) = i\omega\vec{B}_1 \quad \text{— (c)}$$

$$i\vec{k}\times\vec{B}_1 = \mu_0\left(-en_0\vec{u}_1 - i\omega\epsilon_0\vec{E}_1\right) \quad \text{— (d)}$$

Dispersion relation

$$\vec{k}\times\vec{B}_1 = \frac{\mu_0}{i}\left(-en_0\vec{u}_1 - i\omega\epsilon_0\vec{E}_1\right)$$

$$= \mu_0\left(ien_0 u_1 - \omega\epsilon_0 E_1\right) \quad (1)$$

Using (c)

$$\cancel{i}(\vec{k}\times\vec{E}) = \cancel{i}\omega\vec{B}_1$$

Cross product with $\vec{k}$ from left

$$\vec{k}\times(\vec{k}\times\vec{E}) = \omega(\vec{k}\times\vec{B}_1) \Leftarrow$$

from (d)

We have to take u 1 as u 1 e to the power of i k dot R minus omega t and e 1 as e 1 e to the power of i k dot R minus omega t. So, using this sinusoidally varying perturbation variables we will substitute them into the governing equations and then we write the linearized equations without any derivatives. They will look like so let us say we maintain correspondence this is 1, 2, 1a, 2a, 3a and 4a and we will assume the set of equations as equation number 5 substituting equation number 5 in 1a to 4a. What we will get is this minus i omega n 1 plus i k dot n naught u 1 is equal to 0. So, just for the sake of clarity let us say we substitute n 1 into this one dou n 1 by dou t becomes n 1 e to the power of i k dot R minus omega t.

If you are talking about the time derivative the omega part will come it will be minus omega plus del dot n 1 u 1 you have to take a derivative only along the x direction then you will get this term. So, now our del operator is i cap dou by dou x that is it only along the x direction. The second equation which is the momentum equation will give you minus i omega m u 1 is equals to minus e v 1. Let us say we call this as equation a p the Maxwell equation is i k cross u 1 is equals to i omega p 1. Then we have k cross v 1 is equals to mu naught times minus e n naught u 1 minus i omega epsilon naught u 1.

Now we have to eliminate some variables from a b c and d to get the dispersion relation. So, these four equations will lead to the dispersion relation which is a characteristic of electromagnetic wave propagation in the plasma. So, using equation number C we can write k cross E equal to i omega B 1. Taking a cross product from the left with k cross product with k bar from left we can write k cross k cross E is equals to omega times k cross B 1. Now from equation number D we know that k cross B 1 is i times k cross B 1 is actually this.

$$\vec{k}\times(\vec{k}\times\vec{E}_1) = \omega\mu_0 i(en_0u_1 + i\omega\epsilon_0 E_1)$$

$$c^2 = \frac{1}{\mu_0\epsilon_0}$$

$$\vec{k}\times(\vec{k}\times\vec{E}) = \omega\mu_0 i e n_0 u_1 + i^2\omega^2\mu_0\epsilon_0 E_1$$

Using ⑥

$$= \left(\frac{eE_1}{m}\right)\mu_0 e n_0 + \left(\frac{\omega^2 i^2}{c^2}\right)E_1$$

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$$

$$= \frac{eE_1}{m}\mu_0 e n_0 - \frac{\omega^2}{c^2}E_1$$

$$= \frac{n_0e^2}{m\epsilon_0}\epsilon_0\mu_0 E_1 - \frac{\omega^2}{c^2}E_1$$

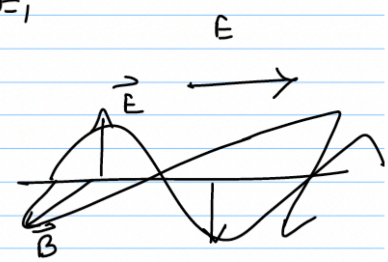
$$\boxed{\vec{k}(\vec{k}\cdot\vec{E}_1) - E_1(\vec{k}\cdot\vec{k}) = \underbrace{\frac{n_0e^2}{m\epsilon_0}}_{\omega_{pe}^2}\frac{E_1}{c^2} - \frac{\omega^2}{c^2}E_1}$$

So, all of this let us say can be substituted into this equation directly. So, we will be able to write k bar times k cross E 1 is omega mu naught i times e n naught u 1 plus i omega epsilon naught u 1. So, k cross B is this. So, we have mu naught which is appearing in the equation itself which is multiplying the hand side as it is. So, we have i which goes inside and we have i appearing here you see here.

Yeah k cross B i times k cross B is this. So, you take i to this side and then you have which is appearing right outside. So, now we can rewrite this equation taking the terms outside k cross k cross E is equals to omega mu naught i e n naught u 1 plus i square omega square mu naught epsilon naught u 1. So, simplifying this on the right hand side. So, omega times i times u 1 omega i u 1 is written as minus e E 1 divided by m from B using equation number B we can write it as e E 1 by m times mu naught e n naught plus omega square i square mu naught epsilon naught is 1 by c square times E 1 or e E 1 by m mu naught e n naught minus omega square by c square E 1 is equals to we will write n naught e square by m epsilon naught epsilon naught mu naught n naught minus omega square by c square E 1 here.

$$\vec{k}\,(\cancel{\vec{k}\cdot\vec{E}_1})^{\to 0} - \vec{E}_1(\vec{k}\cdot\vec{k}) = \omega_{pe}^2 \frac{E_1}{c^2} - \frac{\omega^2}{c^2} E_1$$

$$= \left[\frac{\omega_{pe}^2 - \omega^2}{c^2}\right] E_1$$



$$-k^2 \cancel{E_1} = \left(\frac{\omega_{pe}^2 - \omega^2}{c^2}\right) \cancel{E_1}$$

$$-k^2 = \frac{\omega_{pe}^2 - \omega^2}{c^2}$$

$$\omega^2 = \omega_{pe}^2 + k^2 c^2$$

$\omega^2 = k^2 c^2$ in vacuum

So, I have multiplied numerator and denominator with epsilon naught just to get this factor out n naught e square by m epsilon naught. So, which is n naught e square by m epsilon naught mu naught epsilon naught is 1 by c square. So, this n naught should be E 1 here it is minus omega square by c square E 1. On the left hand side we could have written k cross k cross E as k times k dot E 1 minus E 1 times k dot k. So, what how did you get this you get this by expanding the vector triple product.

So, let us just look at the algebra just to maintain some continuity. So, we have this a, b, c, d which are coming from these equations 1, 2, 3, 4, 1a, 2a, 3a, 4a. What are these equations? These are the linearized governing equations 1, 2, 3 and 4. One is continuity equation we have taken the momentum equation, we have taken the Maxwell equation for the curl of electric field and for the curl of magnetic field. Then we have used the perturbations U, N, E because we think when the electromagnetic wave passes or propagates through the plasma, these are the physical parameters which will be influenced or which will be affected.

We have substituted them into the governing equations, linearized the equations where we have gotten rid of the terms which will be the product of perturbed variables and which will be the time derivatives of the equilibrium variables. After doing that we have got this. Now, we have assumed sinusoidal solutions for the perturbed parts, substituted them into the perturbation equations and then we have got a, b, c and d. Now, these four equations have to be coupled and to get the dispersion relation. In order to do that, I started by equation number c and taken a curl of that equation with k from the left.

So, k cross k cross e becomes omega times i is already cancelled omega times k cross b1. Now, k cross b1 from this equation from equation number d is mu0 by k cross b1 is mu0 by i times minus e N0 mu1 minus i omega epsilon0 u1. So, k cross b1 is this, k cross b1 can be substituted for the right hand side. So, omega times mu0 by i. So, using equation c, we are starting from equation c, i times k cross e is i omega b1, i gets cancelled k cross e is equals to omega b1.

Now, k cross e is the angular frequency omega times b1. We take a cross product with k from the left, k cross k cross e is equals to omega times k cross b. Now, k cross b1 is appearing on the equation d on the left hand side. So, using this, so, we can write k cross b1 is equals to mu0 times mu0 by i minus e N0 u1 i omega epsilon0 u1. Now, we can simplify it as mu0 times i e N0 u1 minus omega epsilon0 u1.

If I take i out, we can write this equation, let us say we mark it as equation number 1 here. So, we can write this as mu0 taking i out, we have mu0 i e N0 u1. So, there is no i here, minus 1 can be written as plus i square. So, we should be plus 1 i has already gotten out of the bracket i omega epsilon0 u1. Now, this one, what is this equal to? This is equal to k cross k cross e.

So, I have used this as it is here, k cross k cross e is omega mu0 i e N0 u1 plus i omega epsilon0 u1. Now, I have separated this term omega times i times u1 from equation B, which is this omega i u1 is minus gets cancelled here, e by m e1. Here it is e by m u1. The rest of it is simple algebra, I have used the fact that C square is 1 by mu0 epsilon0 or C is 1 by square root of mu0 epsilon0 epsilon0. So, using this, what I have got is k is the propagation constant k or the wave vector, k times k dot e1 minus e1 times k dot k is equals to all that that appears on the right hand side.

Now, if you see the right hand side N0 e square by m epsilon0, this is something that is very familiar to us. This is the electron plasma frequency, omega pe square times e1 by C square minus omega square by C square e1. In terms of things that we know very well or we can slightly modify the equation as k times k dot e1 minus e1 times k dot k is equals to omega pe square times e1 by C square minus omega square by C square e1 which is equals to omega pe square minus omega square by C square multiplied by e1. So, the propagation constant k is this is this is a k. Now, if you see the transverse wave which is like this, now where the particles vibrate, if the direction of the wave is this, the particles will vibrate in a perpendicular direction of the wave motion.

$$\omega^2 = \omega_{pe}^2 + k^2c^2$$

$$v_p = \frac{\omega}{k}$$

$$v_p^2 = \frac{\omega^2}{k^2} = \frac{\omega_{pe}^2}{k^2} + c^2$$

$$\boxed{v_p = c\sqrt{1+\frac{\omega_{pe}^2}{k^2c^2}}} > c$$

$$\boxed{\omega > \omega_{pe}}$$

$$v_g = \frac{d\omega}{dk} = \frac{c}{\sqrt{1+\frac{\omega_{pe}^2}{k^2c^2}}}$$

$$\boxed{v_g < c}$$

Now, in electromagnetic wave, if the wave is travelling in this direction, we know very well the electric field and the magnetic field will be perpendicular to each other like this. So, this is electric field and this one the perpendicular component will become the magnetic field. Both of them will be mutually perpendicular and at the same time these two will be perpendicular to the direction of wave propagation for an electromagnetic wave, electric field and magnetic field. So, which means if k denotes the direction of propagation of the wave, k dot e this term has to be 0 because of the right angles. What will be left with is minus k square e1 is equals to omega pe square minus omega square by C square multiplied by e1.

So, let us say we get rid of e1 and what we have is minus k square is equals to omega pe square minus omega square by C square. So, we have a relation between omega and k at the end. So, we can rewrite this equation in a familiar form as omega square is omega pe square plus k square C square. Now, this is the dispersion relation that we were actually looking for. What is this? This is the dispersion relation which expresses the propagation of electromagnetic waves in plasma.

So, we have omega which is a frequency term, omega pe which is the plasma frequency, k square is the wave vector square and C square is the speed of light square. Now, what are the features or characteristics of this equation? So, to begin with if there is no plasma omega pe becomes 0. So, in vacuum we have omega square is equals to k square C square where is it in vacuum which is perfectly valid we know why omega by k or the speed itself becomes the speed of light. Now, if we evaluate the phase velocity from this expression v p, v p is omega by k. So, if you use for reference omega square is omega pe square plus k square C square.

This dispersion relation carries all the information that is required for understanding electromagnetic waves propagation in the plasma. So, v p can be written as or v p square can be written as omega square by k square which is omega pe square by k square plus C square or v p the phase velocity is C times 1 plus omega pe square by k square C square. So, this is the phase velocity. Now, if you look at this expression carefully or even at the first instance itself you will see that the phase velocity appears to be something which can be greater than the speed of light. Now, is it possible? Yes, it is possible for the phase velocity to have any value there is no limit that the phase can have the phase which is just representing a part of the wave can have any velocity which can be greater than C.

But if you go ahead and calculate the group velocity, group velocity is a more relevant parameter which expresses the speed at which the information is being transmitted. So, group velocity is d omega by dk which is C by square root of 1 plus omega pe square by k square C square. So, here you will see for yourself that v g is always less than C and even v g can reach the limit of C only when you do not have plasma that is when the plasma frequency becomes 0 that means we do not have any plasma. From the dispersion relation we can say that the prop the electromagnetic wave can only pass or can propagate through the plasma only when omega is greater than omega pe. This is the condition for the propagation of electromagnetic wave in the plasma this is the phase velocity which gave us a hint that phase velocity can exceed the speed of light and the group velocity calculated from the same dispersion relation tells you that group velocity can never be greater than C it has it should always be less than C and we already realized that in vacuum the velocity of electromagnetic wave is just C or for this group velocity to become equal to C we should have the plasma frequency omega pe to be 0 omega pe being 0 indicates that there is no plasma we are just talking about the vacuum.

So, as long as plasma is to be retained in this picture we have to make sure that the phase velocity can be anything and group velocity will always be less than the speed of light. So, we have now realized that the phase velocity can be anything even greater than C and the group velocity will always be less than C or equal to C when the wave is propagating in vacuum. you