Plasma Physics and Applications

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Lecture 49: Invalidity of Plasma Approximation - II

Hello dear students, we will continue our discussion on ion acoustic wave. So, in the last class we have seen how the governing equations change when we do not impose plasma approximation. So, we will continue the discussion. So, we have seen that the Poisson equation needs to be considered which is written like this epsilon naught phi 1 k square lambda d square plus 1 is e ni 1 lambda d square. So, we call it as C. And we also realize that there is no need to change the continuity equation it will remain as it is.

So, we have dou ni 1 by dou t plus n naught del dot u i 1 is equals to 0. Substituting for n i 1 is equals to n i 1 e to the power of i k x minus omega t which implies n i 1 dot is equals to n i 1 minus i omega e to the power of i k x minus omega t. Let us say we call this as equation number d. Now using equation d in this let us say we this is C 1 for example, in C 1 we have minus i omega n i 1 plus n naught i k u i 1 is equals to 0.

So, we have also used del dot u i 1 because u i 1 is supposed to be this u i 1 e to the power of i k x minus omega t. So, dou by dou x of u i 1 becomes i k times u i 1. Now what is the meaning of this? Just we have to wonder what is the meaning of this? It means that the perturbation part is oscillating around the mean value like this in a sinusoidal way. But we have an exponential i. So, we have variation in a sinusoidal fashion.

Ion Acoustic Wave

P.E $\epsilon_0 \phi_1 \left[k^2\lambda_D^2 + 1\right] = e\, n_{i1} \lambda_D^2$ — (c)

$$\frac{\partial n_{i1}}{\partial t} + n_0 \vec{\nabla}\cdot\vec{U}_{i1} = 0 \quad \text{— (c1)}$$

$u_{i1} = u_{i1} e^{i(kx-\omega t)}$

$\frac{\partial}{\partial x} u_{i1} = ik\, u_{i1}$

$n_{i1} = n_{i1} e^{i(kx-\omega t)}$

$$\Rightarrow \boxed{\dot{n}_{i1} = n_{i1}(-i\omega)e^{i(kx-\omega t)}} \quad \text{— (d)}$$

Using (d) in (c1)

C.E

$$(-i\omega)\, n_{i1} + n_0 (ik) u_{i1} = 0$$

$$n_{i1} = \frac{k}{\omega} n_0 u_{i1} \quad \text{— (e)}$$

$$n_{e1} = n_0 \frac{e\phi}{K_B T_e}$$

So, from this n i 1 can be written as n i 1 is equals to k by omega n naught u i 1. Now we in the earlier class we have derived n e 1 is equals to n naught e phi by k B T and n i 1 is derived now to be equal to k by omega n naught u i 1. Now we have this u i 1. Now if you look at the equation number C if we use n i 1 into this equation number C we can write epsilon naught phi 1 times k square lambda d square plus 1 is equals to e times k by omega n naught u i 1 lambda d square. Let us say we call this equation as e and this as f.

How did we get f? We used using equation number E in C. Now we have u i 1 which is unknown. So, let us look at the set of equations all the equations this is the Poisson equation and this is the continuity equation and then we have the same equation. Now we need to solve the momentum equation. Let us bring the momentum equation.

We can directly use the linearized momentum equation. The terms of the momentum equation will not change. So, for instance we write it as m i n i times dou u i by dou t plus u i dot del times u i is equals to e n e minus del t. So, here the advection term can be neglected because of the second order terms involved in it. We can write the linearized version like this minus i omega m i n naught u i 1 is equals to minus e n naught i k phi 1 minus gamma i k b T i times i k n i 1.

Using (e) in (c)

$$\epsilon_0 \phi_1 [k^2 \lambda_D^2 + 1] = e \frac{k}{\omega} n_0 u_{i1} \lambda_D^2 \quad \text{— (f)}$$

M.E

$$m_i n_i \left[\frac{\partial \vec{u}_i}{\partial t} + (\vec{u}_i \cdot \nabla)\vec{u}_i \right] = en\vec{E} - \nabla p$$

$$\text{(g)} \quad \text{———} \quad -i\omega m_i n_0 u_{i1} = -e n_0 i k \phi_1 - \gamma_i k_B T_i (ik) n_{i1}$$

From (f) $\Longrightarrow$

$$u_{i1} = \frac{\epsilon_0 \phi_1 [k^2 \lambda_D^2 + 1]}{e \left(\frac{k}{\omega}\right) n_0 \lambda_D^2} \quad \text{— (i)}$$

Using (i) in (g)

So, in between these 2 steps there are multiple algebra steps. Number 1 what is the first step? Number 1 the advection term is treated to be 0. Number 2 substitute the perturbation u i is equals to u naught plus u i 1 . Then what is it del P. So, del P term is this one this entire term that appears here is del P gamma n k b T i and within del P you have to use the perturbation as N or n i is equals to n naught plus n i 1.

$$\cancel{-} i\omega m_i n_0 \left[\frac{\epsilon_0 \phi_1 (k^2 \lambda_D^2 + 1)}{e \left(\frac{k}{\omega}\right) n_0 \lambda_D^2} \right] = \cancel{-} e n_0 i k \phi_1 \cancel{-} \gamma_i k_B T_i (ik) \cancel{\left(\frac{k}{\omega}\right)} \cancel{n_0} \left[\frac{\epsilon_0 \phi_1 (k^2 \lambda_D^2 + 1)}{e \cancel{\left(\frac{k}{\omega}\right)} \cancel{n_0} \lambda_D^2} \right]$$

$$\cancel{i}\omega m_i \cancel{n_0} \left[\frac{\epsilon_0 \cancel{\phi_1} (k^2 \lambda_D^2 + 1)}{e \left(\frac{k}{\omega}\right) \cancel{n_0} \lambda_D^2} \right] = e n_0 \cancel{i} k \cancel{\phi_1} + \gamma_i k_B T_i (\cancel{i} k) \cancel{\phi_1} \frac{(k^2 \lambda_D^2 + 1)}{e \lambda_D^2}$$

Multiplying with $\dfrac{e \lambda_D^2}{\epsilon_0 \; (k^2 \lambda_D^2 + 1)}$

$$\omega m_i \left(\frac{\omega}{k}\right) = e n_0 k \left(\frac{e \lambda_D^2}{\epsilon_0 \; (k^2 \lambda_D^2 + 1)} \right) + \gamma_i k_B T_i k$$

When you do these 3 steps you will get this expression. Now the importance of this expression is this is linear directly in terms of the perturbed variables but if you are wondering how I skipped these 3 steps you can go back and see the previous lectures we have done these 3 steps in ion acoustic wave and we have obtained this relation. The only thing which has changed now is this phi 1 which appears is the change and u i 1 also carries the information about the Poisson equation. See here u i 1 carries the information about the Poisson equation. Poisson equation why is Poisson equation in this picture? Because we have treated electrons and ions to be different.

So, this equation actually you see this continuity equation carries the information of both actually the Poisson equation and the continuity. Now when we substitute for we can use this u i 1 from this equation number e into this let us say we call this equation as this equation as g. Let us not make the lecture notes clumsy so we will remove all these things. So, if you go back and see the lecture notes it should be easy for you to understand. Now we have equation number g.

Now let us look at the equation number g what is it that we are trying to do? We have phi 1 we have n i 1 we have u i 1 anything else which is a part of part which needs to be eliminated no we have only phi 1 n i 1 and we have this, this and this. So, ultimately when we derive the dispersion relation it will be in terms of omega and k and temperature k B etc. But it cannot be in terms of these perturbed variables. Now we have to eliminate this. Now using something that we have just derived we can write as u i 1 is equals to epsilon naught phi 1 k square lambda d square plus 1 divided by e times k by omega n naught lambda d square.

Where did we get this? From here from equation F. Just try to follow the mathematics the underlying physics will be explained at the end. Just try to understand how we are combining equations and trying to get the dispersion relation. What is the method? The method is the linear perturbation theory. So, from F we can write u i 1 as this.

So, let us substitute for u i 1. Let us say we call this as equation number i using equation number i in G. We will write minus i omega m i has just substituted u i 1 wherever it was appeared where here and here. So, we can cancel some things N naught can be cancelled, N naught can be cancelled, k by omega can be cancelled and we can cancel this minus minus minus. Now, we can rewrite this expression.

Rearranging the terms. & $\lambda_D^2 = \frac{\epsilon_0 k_B T_e}{n_0 e^2}$

$$\frac{\omega^2}{K^2} m_i = \frac{\boxed{e^2 n_0} k_B T_e \lambda_D^2}{\boxed{\epsilon_0 k_B T_e}[K^2\lambda_D^2+1]} + \gamma_i k_B T_i$$

$$\frac{\omega^2}{K^2} m_i = \frac{k_B T_e}{\cancel{\lambda_D^2}} \frac{\cancel{\lambda_D^2}}{(K^2\lambda_D^2+1)} + \gamma_i k_B T_i$$

$$\frac{\omega^2}{K^2} = \frac{k_B T_e}{m_i} \cdot \frac{1}{(K^2\lambda_D^2+1)} + \frac{\gamma_i k_B T_i}{m_i}$$

$$\boxed{\frac{\omega}{K} = \sqrt{\frac{k_B T_e}{m_i} \frac{1}{(K^2\lambda_D^2+1)} + \frac{\gamma_i k_B T_i}{m_i}}}$$

As this minus i omega minus i omega m i N naught times epsilon naught phi 1 k square lambda d square plus 1 divided by E times k by omega N naught lambda d square is equals to E N naught i k phi 1 minus sorry plus gamma i k b T i times i k phi 1 to k square lambda d square plus 1 divided by E lambda d square. We can cancel this iota, we can cancel this N naught which appears and we can cancel phi 1. And now, multiplying with multiply the entire expression with E lambda d square divided by epsilon naught E 1 times k square lambda d square plus 1. Multiplying the entire expression what we will get is omega m i times omega by k, all these things get will get cancelled omega by k is equals to E N naught k times E lambda d square divided by epsilon naught E 1 times k square lambda d square plus 1 plus gamma i k b T i k. Rearranging the terms and using the fact that lambda d square is epsilon naught k b T e by N naught e square where in this equation, we can write omega square by k square m i is equals to E square N naught k b T e lambda d square divided by epsilon naught k b T e times k square lambda d square plus 1 plus gamma i k b T e.

I have multiplied both numerator and denominator with k b T e in the first term on the right hand side. So, if you go back what do you see here, you have all this k square E lambda d square times. So, here we wrote something by mistake. So, this E 1 is not there. Since phi 1 has already been cancelled, it should be E lambda d square by epsilon naught.

So, we are actually multiplying with this factor. So, here k b T e is multiplied onto numerator and denominator just to get that lambda d square. So, omega square by k square m i is still there on the left hand side, omega times omega, omega square by k, you take 1 k from here and bring it to the left hand side you have omega square by k square m c m i is equal to all this E square n naught by epsilon naught k b T e. This is nothing but lambda d square. This is k b T e by lambda d square times lambda d square by k square lambda d square plus 1 plus gamma i k b T i.

So, this gets cancelled. So, omega square by k square is equals to k b T e by m i times 1 by k square lambda d square plus 1 plus gamma i k b T i divided by m i or omega by k is under root k b T e by m i into 1 by k square lambda d square plus 1 plus gamma i k b T i by m i. So, this is nothing but the dispersion relation omega versus k. Now, let us compare this expression with the earlier expression. What is it? Omega by k is square root of k b T e, let us say gamma e k b T e by m plus gamma i k b T i by m. Now, the relation this is when plasma approximation is valid and in the absence of that omega by k is slightly different which is k b T e by m i times 1 by k square lambda d square plus 1 plus gamma i k b T i by m i.

$$\frac{\omega}{k} = \sqrt{\frac{\gamma_e k_B T_e}{m_i} + \frac{\gamma_i k_B T_i}{m_i}} \longleftarrow \text{When plasma approximation is Valid}$$

$$\frac{\omega}{k} = \sqrt{\frac{k_B T_e}{m_i}\left(\frac{1}{k^2\lambda_D^2+1}\right) + \frac{\gamma_i k_B T_i}{m_i}} \longleftarrow \text{Not-Valid}$$

$$\frac{1}{k^2\lambda_D^2+1} \text{ is the additional factor}$$

$$\text{Order of error} = k^2\lambda_D^2$$

$$T_i \to 0 \Rightarrow \lambda_D^2 = 0$$

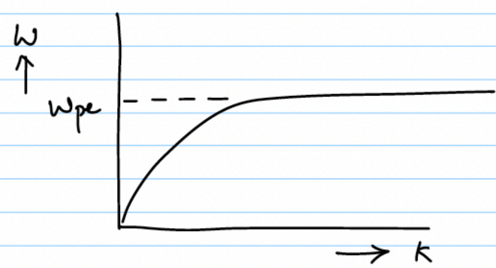
for small 'λ' $\quad k^2\lambda_D^2 \gg 1$

$$\underline{\omega^2 = k^2}$$

So, this is the difference. So, expression looks similar because we have gamma i k b T i by m i appearing on both sides. Except for the additional factor, what is the additional factor? The additional factor is 1 by k square lambda d square plus 1 is the additional factor. Why is this additional factor when plasma approximation is of course valid here, not valid here? This is the additional factor which has come into existence just by treating the number of ions not equal to the number of electrons. Now, this is the additional factor. The order of error can be those the order of error.

How much is it changing with respect to the actual value is order of error is k square by lambda d square. So, we can simply say that the error introduced is because of not following the plasma approximation. So, we know generally that lambda d is very small when we consider plasma. So, we can say that since lambda d is very small, the order of error is also going to be very small for long wavelength waves. Further, we can also say that when the temperature of ion tends to 0, we can simply write lambda d square is equal to 0.

So, at shorter wavelengths k which is 2 by lambda for shorter wavelengths k square lambda d square will become much greater than 4 small lambda k square lambda d square becomes much larger than 1 and or very large k for small value of lambda k becomes very large or we can then we can say that omega square will become equal to k square. When we plot this, let us say omega versus k, it will look something like this. This is k and this is omega and this is omega pe. So, when k square lambda d square becomes very large or for very small waves omega by k which is the phase velocity can be written as kB Te by Mi times 1 by 1 plus k square lambda d square plus gamma i kB Ti by Mi. When k square lambda d square becomes very large or when lambda is very small, we can say that this term becomes very small and can be neglected.



(or) λ is very small

$k^2\lambda_D^2 >> 1$

$$\frac{\omega}{k} = \sqrt{\underbrace{\frac{k_B T_e}{m_i}\frac{1}{1+k^2\lambda_D^2}}_{\text{very small and can be neglected}} + \frac{\gamma_i k_B T_i}{m_i}}$$

When for very small values of lambda what happens? k becomes very large because it is 2 pi by lambda and 1 plus k square by lambda square taken inwards becomes very small. So, for very small wavelengths, this factor can be neglected altogether. So, this is the difference when we compare the ion acoustic wave in the presence of plasma approximation and in the presence in the absence of plasma approximation. So, we will conclude this discussion here. Thank you.