Plasma Physics and Applications

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Lecture 48:Invalidity of Plasma Approximation

Hello dear students. In this lecture, we will continue our discussion on ion acoustic waves. In the last class, we have seen that we can derive the dispersion relation between omega and k like this and the omega by k which is the phase velocity of the ion acoustic wave was derived like this. So, we will take this expression ahead. We will write omega by k the angular frequency divided by the wave vector is equals to kB Te plus gamma i kB Ti divided by m i. The wave kB is the Boltzmann s constant, gamma is the ratio of specific heats for ions, Te and Ti are the temperatures of electrons and ions respectively, m i is the mass.

Now more importantly, this will be the speed of an acoustic wave travelling in plasma. Now one very important conclusion is from this expression we can realize that V P will be equal to V G which will be equal to a constant. Both the velocities will be the same. This is the unique characteristic of the acoustic wave that is travelling through plasma.

The phase velocity and group velocity will be constants. So, we can also write this expression in a more general fashion where omega by k is if electron is assumed to be travelling very fast and not making any assumption that the plasma is isothermal, we can write a generalized expression as gamma e kB Te plus gamma i kB Ti divided by m i. So, this is the speed of the acoustic wave travelling in the plasma. So, the dispersion curve for the ion waves if you see has fundamentally a different nature in comparison to the electron plasma wave. So, in electron plasma wave we have seen that the omega versus k was something like this and this was omega P.

$$-i\omega m_i n_0 u_{i1} = -\cancel{e}\cancel{n_0}(ik)\, n_{i1} \frac{k_B T_e}{\cancel{n_0}\cancel{e}} - \gamma_i \cancel{k_B T_i}\, n_0 \frac{e\phi_1}{\cancel{k_B T_i}}\, ik$$

$$\uparrow$$

$$-i\omega m_i n_0 u_{i1} = -ik\, n_{i1} k_B T_e - \gamma_i n_0 e \phi_1 ik \quad \text{—— (e)}$$

$$\uparrow$$

$$n_{i1}(-i\omega) + n_0 u_{i1}(ik) = 0$$

$$\underline{\underline{n_{i1}}}\,\omega = n_0 u_{i1} k$$

$$n_0 \frac{e\phi}{k_B T}\omega = n_0 u_{i1} k$$

$$\underline{\underline{u_{i1}}} = \left(\frac{e\phi_1}{k_B T_e}\right)\frac{\omega}{k} \quad \text{—— (f)}$$

Using (f) in (e)

The minimum frequency above which only above which electron plasma waves can exist in the plasma and omega P is nothing but the plasma frequency. Now if you say this ion acoustic wave, the dispersion relation will look something like this. So, this is Vs and this is k and this is omega. Like I said velocity will be constant. So for the ion waves exist only when there are thermal motions and in electron plasma we take the ions to be remaining at rest.

But in plasma waves electrons are not fixed and they are moving very fast. Now what is the mechanism by which this compression of the medium which you call as the propagation of the acoustic wave is being transported from one point to another point. It is basically the ions which are coming together and expanding as the pressure pattern is moving into the space. Now one very important assumption that we made while deriving this is that we took the number of electrons to be equal to the number of ions which is also called as the plasma approximation. So, you consider any plasma the total number of electrons and ions will be the same or we assume it to be the same.

$$-i\omega m_i \cancel{n_0}\left(\frac{\cancel{e\phi}}{k_B T_e}\right)\frac{\omega}{k} = -ik\,\cancel{n_0}\left(\frac{\cancel{e\phi_1}}{\cancel{k_B T}}\right)\cancel{k_B T} - \gamma_i \cancel{n_0}\,\cancel{e\phi}\; i k_1$$

$$-\cancel{i}\omega m_i \frac{\omega}{k} = -\cancel{i}k\, k_B T - \gamma_i \cancel{i} k_1 (k_B T)$$

$$-\frac{\omega^2}{k^2} = \frac{-k_B T_e - \gamma_i k_B T_i}{m_i}$$

$$v_p = \frac{\omega}{k} = \sqrt{\frac{k_B T_e + \gamma_i k_B T_i}{m_i}} = \underline{v_s}$$

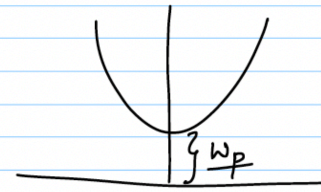
Now if you make a condition saying that the validity of plasma approximation is not imposed on the plasma for the treatment of ion acoustic wave. So, we will discuss what is called as the validity of plasma approximation. Plasma approximation is when number of electrons equal to the number of ions. So in deriving the dispersion relation for the ion acoustic wave we have made an assumption that the electron number density is equal to the ion number density. While working out the electron plasma wave we have considered the electron density not equal to the ion density where.

So when we were discussing the electron plasma wave we have seen that within the small volume from this small volume electrons were displaced out. And while these electrons were displaced out the ions remained at rest because of their heavier mass and it is the electrons which were taken out. Now with these electrons immediately after being moved from this region will constitute an electric field and the electrons motion will be to nullify this electric field they will immediately boom back because of their mass inertia or the thermal energy if you allow it. They will execute oscillations back and forth. And this oscillation frequency is referred to the plasma frequency.

Now this is the initial picture that we have taken that is basically displacing the electrons away from this region where you are actually creating number of ions much greater than the number of electrons. This is the initial condition only this gave you a possibility to work with the Poisson equation where you have the electric field or del dot E written as Ni minus Ne times the charge. If you do not have a valid plasma approximation what is bound to happen in this electric field will not exist this will be 0. This electric field will only exist when there is unequal number of electrons and ions.

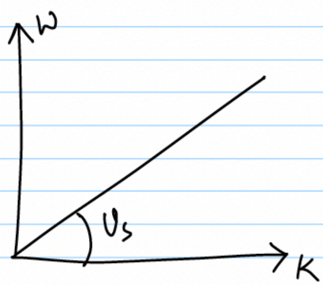
You have created it.

$$\frac{\omega}{K} = \sqrt{\frac{k_B T_e + \gamma_i k_B T_i}{m_i}} = \underline{v_s}$$



$$- \; v_p = v_g = \underline{\text{constant}}$$

$$\frac{\omega}{k} = \sqrt{\frac{\gamma_e k_B T_e + \gamma_i k_B T_i}{m_i}} = \underline{v_s} \qquad \underline{n_e = n_i}$$



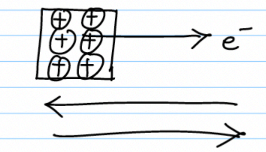
Now you realize that after doing this plasma frequency was of course there it was there plasma the oscillations were existing but we realize these oscillations when you discuss the dispersion relation so you have realized that omega pe is Ne square by m epsilon naught. What does it mean? This is the plasma frequency we have seen it many number of times. The important message here is that there is no dependence on the wave vector which means there is no information passage from one point to another point. There is no energy transfer from one point to another point by the means of wave activity. The electrons are oscillating but these oscillations or the information about these oscillations is not going into the layers of plasma.

If you consider plasma let us say this is a plasma system and you have different layers for just understanding I am drawing this. If the oscillations are happening here this information is not being transported into the all the layers that are lying here. Then we made a altered picture where we allowed the temperature of electron to be non-zero where the electron is now allowed to have thermal motion in addition to this vibrations. Then we realized when we derived the expression for omega we have seen it to be a function of k. What it means is that the electron now travels into the adjacent layers of plasma to tell the story of the vibration that are happening in the first layer.

InValidity of Plasma approximation ($n_e \neq n_i$)

$\epsilon_0(\nabla \cdot \cancel{E}) = (n_i - n_e)e$

1) $n_i > n_e$



$$\epsilon_0 \frac{\partial E}{\partial x} = e\,(n_i - n_e)$$

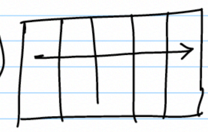
$$\left.\begin{aligned} n &= n_0 + n_1 \\ E &= E_0 + E_1 \end{aligned}\right\}$$

$$\omega_{pe} = \sqrt{\frac{ne^2}{m\epsilon_0}}$$

ω

2) $T_e \neq 0$ $\quad \omega(k)$

$$\epsilon_0 \frac{\partial E_1}{\partial x} = e\left(n_{i1} + n_{0i} - [n_{e1} + n_{0e}]\right)$$



$$\epsilon_0 \frac{\partial E_1}{\partial x} = e\,(n_{i1} - n_{e1})$$

$E = -\nabla\phi$

$$-\epsilon_0 \frac{\partial^2 \phi_1}{\partial x^2} = e\,(n_{i1} - n_{e1})$$

This is the modified picture. In this modified picture one thing is very clear number of ions is greater than the number of electrons at least in the small region. In the total plasma of course not but in the small region number of ions is greater than the number of electrons. Now if you go back and see the ion acoustic wave we have imposed a condition that number of electrons will be equal to the number of ions. We have removed the possibility of existence of an electric field because of this condition.

Now what we are going to do is we are going to see if we allow the number of electrons not equal to the number of ions how will it affect the propagation of an acoustic wave through the plasma. This is the basic understanding about the invalidity of plasma approximation and how it is relevant for understanding the wave propagation especially the acoustic wave propagation in the plasma. Now let us start let us say we have this unequal number densities. So obviously we will have some electric fields. Now let us say we start with the basic is equation which is the Poisson equation.

$$\epsilon_0 \frac{\partial^2 \phi_1}{\partial x^2} = e(n_{i1} - n_{e1}) \quad \text{—} \ (1)$$

$$\phi_1 = \phi_0 e^{i(kx - \omega t)} \quad \text{—} \ (2)$$

Substituting this (2) in (1)

$$\frac{\partial \phi_1}{\partial x} = \phi_0 e^{i(kx-\omega t)} (ik)$$

$$\frac{\partial^2 \phi_1}{\partial x^2} = \underbrace{\phi_0 e^{i(kx-\omega t)}}_{\phi_1} (ik)^2$$

$$-\epsilon_0 (ik)^2 \phi_1 = e\left[n_{i1} - n_{e1}\right]$$

$$\boxed{\epsilon_0 k^2 \phi_1 = e\left[n_{i1} - n_{e1}\right]} \quad \text{—} \ (a)$$

Let us say we have epsilon0 dou E by dou x is charge E times Ni minus Ne. Now let us define the perturbation variables N is equals to N0 plus N1 the electric field E is equals to E0 plus E1 and then we have the let us substitute these perturbations into the Poisson equation. So which we will get epsilon0 dou E1 by dou x is equals to E times. It has to be Ni1 plus N0 i minus Ne1 plus N0 e. So this is also supposed to be dou by dou x of E1 plus E0.

Because E0 is a constant we have removed that term or we have not considered that term. So at equilibrium we can assume the electron density and ion density to be equal. So we can say these two terms gets cancelled. So what will be remaining is epsilon0 dou E1 by dou x is E times Ni1 minus Ne1. So these are the equilibrium terms.

This one and this one. So minus goes and these two terms gets cancelled. Now we can write the electric field in terms of a potential dou square phi1 by dou x square is equals to E times Ni1 minus Ne1 which is epsilon0 dou square phi1 by dou x square is equals to

Ne1. So we can take phi1 the perturbed potential. See what is phi1? Phi1 is responsible for the creation of that electric field which is E1 also because of the disparity in the number of electrons and ions that you have imposed. So phi1 is let us say phi0 e to the power of i kx minus omega t.

$$n_{e1} = n_0 \frac{e\phi_1}{K_B T} \quad \text{—— (b)}$$

Using (b) in (a)

$$\epsilon_0 k^2 \phi_1 = -e n_{e1} + e n_{i1}$$

$$\epsilon_0 k^2 \phi_1 = -\frac{e^2 n_0 \phi_1}{K_B T_e} + e n_{i1}$$

$$\phi_1 \left[\epsilon_0 k^2 + \frac{e^2 n_0}{K_B T_e} \right] = e n_{i1}$$

$$\epsilon_0 \phi_1 \left[k^2 + \frac{e^2 n_0}{K_B T_e \epsilon_0} \right] = e n_{i1}$$

$$\lambda_D^2 = \frac{\epsilon_0 K_B T_e}{e^2 n_0}$$

So substituting this, let us say we call this equation as 2 and this as 1. So I am doing every step just so that you follow the mathematics while I am doing it and it will be easy for you to understand the implication of this mathematical steps. So when you substitute this into that so dou square phi1 by dou x square will be so dou phi1 by dou x would be phi0 e to the power of i kx minus omega t times i k dou square phi1 by will be phi0 e to the power of i kx minus omega t times i k whole square. Substituting this, we will write epsilon0 i k square all of this will still be phi1 phi1 is equals to E. E is the charge in the Poisson equation times Ni1 minus Ne1.

So while writing the electric field in terms of potential I forgot to write a minus here because E is equals to minus del phi. So using that minus here so we will have this equation as minus epsilon0 i k square phi1 is equals to E times Ni1 minus Ne1 or removing the iota we can write it as epsilon0 k square phi1 is the charge times perturbed number of ions minus perturbed number of electrons. This is a very important equation that we are going to use further. So let us say we call this equation as equation A and from the last class actually we need to recall an equation Ne1 the number of perturbed electrons is equals to equilibrium number N0 times E phi by kBT. Let us say we call this equation as so this resulted from approximating the exponential saying that E phi will be much less than kBT.

So using now B in equation A what is equation A? equation A is the Poisson equation basically it is written in terms of the potential phi1. We have to write we have to substitute B into that equation. So we will write the Poisson equation first epsilon0 k square phi1 is equals to minus E Ne1 plus E Ni1 epsilon0 k square phi1 is equals to minus E square N0 phi1 by kBT plus E Ni1 just this N0 all of this N0 E phi by kBT so we get phi1 into this. So E will multiply E square N0 phi1 by kBT because it is electron we are writing Te electron temperature. So we can take phi1 common and write epsilon0 k square plus E square N0 by kBT is equals to E Ni1 or epsilon0 phi1 times k square plus E square N0 by kBT epsilon0 is equals to E Ni1.

$$\epsilon_0 \phi_1 \left[k^2 + \frac{1}{\lambda_D^2} \right] = e n_{i1}$$

$$\boxed{\epsilon_0 \phi_1 \left[k^2 \lambda_D^2 + 1 \right] = e n_{i1} \lambda_D^2} \quad \text{—— (C)}$$

	(1)	(2)
1) C.E	✓	✓
2) M.E	✓	✓
3) P.E	×	✓

$$n_{i1} \neq n_{e1}$$

$$\frac{\partial n_{i1}}{\partial t} + n_0 \nabla \cdot u_{i1} = 0$$

$$n_{i1} = n_{i1} e^{i(kx - \omega t)}$$

$$\frac{\partial n_{i1}}{\partial t} = n_{i1} (-i\omega) e^{i(kx - \omega t)}$$

Now we know that the Debye s length lambda d square is epsilon0 kBT e divided by E square N0. So this is the factor that appears here as the second term within the bracket seems to be resembling the Debye s length. So using this into this equation we can write epsilon0 phi1 times k square plus 1 by lambda d square is equals to E Ni1 or for convenience we can rearrange the terms epsilon0 phi1 is equals to k square lambda d square plus 1 is equals to E Ni1 lambda d square. Let us say we call this equation as C.

So what is A, what is B, what is C? Let us go back and see.

Equation A is the Poisson equation in terms of phi1. Equation B is the number density of perturbed electrons which we have derived in the last class. After finding it too difficult to accept this relation you can go back and see how this was derived. So we have used equation number B into this Poisson equation and then we have obtained equation number C. Now this is still the Poisson equation it carries the same information but now it carries the information in terms of the lambda d the Debye s length.

Debye s length is a characteristic parameter which conveys a lot of information about the plasma about the nature of plasma or about the constituents of plasma. Now considering what is the change that we have made from the earlier picture we have seen that number of ions is not equal to number of electrons. The perturbed number of electrons and number of ions are not equal. Now will this change the any other equation? See we have three equations. We have the continuity equation, we have the momentum equation and then we have the Poisson equation.

In the first picture the Poisson equation was not relevant because we assumed the plasma approximation to be perfectly valid so we removed it. The momentum equation was taken and the continuity equation was taken. In the second picture when the plasma approximation is not valid the Poisson equation was included and it was modified in terms of lambda d and the continuity equation will it change because of the invalidity of plasma approximation? Of course not. So it can be taken as it is and the momentum equation has to be seen how it will be modified in the in the presence of an a of an a of a electric field which is created just by not accepting the plasma approximation. We can use the momentum equation as it is so which is dou N by dou t plus N0 del dot U I1 is equal to 0 for the ion fluid.

So we know that number of perturbed electrons, number of ions can be written as Ni1 e to the power of i kx minus omega t assuming sinusoidal solution for the perturbed ions Ni1 e to the power of i kx minus omega t. So dou Ni1 by dou t is equal to Ni1 minus i omega e to the power of i kx minus omega t. What have we done? We have just taken the sinusoidal solution took a derivative with respect to time. Now we are going to substitute this back into this equation. We will continue this discussion in the next class. Thank you.