Plasma Physics and Applications

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Week – 10

Lecture 47: Ion Acoustic Wave - III

Hello dear students. We will continue our discussion on ion acoustic wave from the last class. So in the last class we have derived this expression which is dou ni1 by dou t plus N0 del dot U i1 is equal to 0. So which is the basic continuity equation. Now from the momentum equation we can write mi the mass of ions times N0 plus Ni1 times the convective derivative dou by dou t of U0 plus U i1. We have taken the advection term to be 0 to begin with is minus del times del phi0 phi1 times Q N0 plus N1 minus gamma gamma i basically gamma kB Ti del N0 plus Ni1.

So doing the simple algebra of reducing this expression into simpler terms or lesser number of terms we can write mi N0 dou U i1 by dou t plus mi Ni1 dou U i1 by dou t is equal to minus E del phi1 del phi0 becomes 0 del phi1 N0 minus E del phi1 del Ni1. How did I get this? This is 4 terms to 2 terms. Del of phi0 becomes 0 and product of this perturbation terms which becomes second order also becomes 0. So, this term also becomes 0.

Why because it is a product of perturbation terms. Product of perturbation terms is considered 0. We neglect it minus gamma kB T del Ni1. So, we have only one term out of these 4 terms that are being as a product and the gamma kB T term as it is and this also becomes 0 this becomes. So, this one from our earlier discussion we can write this is equal to Ni1 is N0 E phi by kB T dou U i1 by dou t because we already derived that Ni1 is N0 E phi by kB Te.

So, using this we can write Mi N0 dou U1 by dou t is equal to minus E N0 del phi1 this one minus gamma i kB Ti del Ni1. So, this is again a product of second ordered perturbed variables. This can be neglected. So, this term simply becomes 0 or not considered. So, we have this term as it is appearing here and this term and this term.

Let us write the equation for the sake of clarity. So, which is Mi N0 dou U i1 by dou t is equal to minus E N0 del phi1 minus gamma i kB Ti del Ni1. Now, we have both the linearized equation. Let us say from here we call it as A and we call this as B. So, we will assume the sinusoidal solutions for the perturbed variables Ni1 is oscillating around the mean value Ni1 itself e to the power of i.

$$\frac{\partial n_{i1}}{\partial t} + n_0 \nabla\cdot \vec{u}_{i1} = 0 \quad \text{—} \quad (a) \qquad\qquad n_{i1} = n_0 \frac{e\phi}{k_B T_e}$$

$$m_i \left[n_0 + n_{i1}\right]\left[\frac{\partial}{\partial t}(u_0 + u_{i1}) + 0\right] = -\underbrace{\nabla(\varphi_0 + \varphi_1)\, q\,(n_0 + n_1)} - \gamma k_B T_i \nabla\underbrace{(n_0 + n_{i1})}$$

$$m_i n_0 \left[\frac{\partial u_{i1}}{\partial t}\right] + \underbrace{\cancel{m_i n_{i1}\frac{\partial u_{i1}}{\partial t}}}_{\to 0} = -\underline{e \nabla\varphi_1 n_0} - \underbrace{e\nabla\varphi_1 \nabla n_{i1}}_{0} - \gamma k_B T \nabla n_{i1}$$

$$m_i n_0 \frac{e\phi_1}{k_B T_e}\frac{\partial u_{i1}}{\partial t}$$

$$m_i n_0 \frac{\partial u_{i1}}{\partial t} = -e n_0 \nabla\varphi_1 - \gamma_i k_B T_i \nabla n_{i1}$$

Let us say we consider only one dimension kx minus omega t or the k vector is to be written as kx i cap plus ky j cap plus kz k cap and R is x i cap plus y j cap plus z j cap. U i1 is using this in equation A and B. Let us say we call this entire set of equations as C using C in A and B. We can write Mi N0 U i1 times minus i omega the time derivative leaves you a factor of minus i omega is equals to minus E N0 i k phi1 minus gamma i kB Ti Ni1 times i k. So, we call this equation as D.

$$m_i n_0 \frac{\partial u_{i1}}{\partial t} = -e n_0 \nabla \phi_1 - \gamma_i k_B T_i \nabla n_{i1} \quad \text{— (b)}$$

$$\left.\begin{aligned} n_{i1} &= n_{i1}\, e^{i(kx-\omega t)} \\ u_{i1} &= u_{1i}\, e^{i(kx-\omega t)} \\ \phi_{1i} &= \phi_1\, e^{i(kx-\omega t)} \end{aligned}\right\} \quad (c)$$

$$\hat{k} = k_x \hat{i} + k_y \hat{j} + k_z \hat{k}$$
$$\vec{r} = x \hat{i} + y \hat{j} + z \hat{k}$$

Using (c) in (a) & (b)

$$m_i n_0 u_{i1} (-i\omega) = -e n_0 (ik) \phi_1 - \gamma_i k_B T_i n_{i1} (ik) \quad \text{— (d)}$$

$$n_{i1} = n_0 \frac{e\phi_1}{k_B T}$$

Now in this we can substitute Ni1 is N0 e phi1 by kB Te. Then we can write it as minus i omega Mi N0 U i1 is equals to minus E N0 i k Ni1 kB Te by N0 e minus gamma i kB Ti N0 e phi1 by kB Ti times i k. What I have done is I have obtained equation D by substituting the perturbation into this equation into the governing equations which is the momentum equation in this case. So, all of this going into this has given me this equation and into this equation I am substituting Ni1 and U i1. What I have is this.

We can cancel few terms here E N0 kB Ti kB Ti. What will be left with is minus i omega Mi N0 U i1 is equals to minus i k Ni1 kB Te minus gamma i N0 e phi1 by k. Let us say we call this equation as in our sequence as equation number. This is the story about the momentum equation. Let us bring the continuity equation by applying the sinusoidal solutions.

We can write Ni1 times minus omega which is the time derivative dou Ni1 by dou t plus N0 U i1 times i k is equals to 0. It simply tells you that Ni1 omega is equals to N0 U i1 times k. Substituting for Ni1 which is N0 e phi by kB T is equals to times omega is equals to N0 U i1 times k. Which is U i1 is e phi1 by kB T times omega by k. This is equation F.

$$-i\omega m_i n_0 u_{i1} = \cancel{-e}\cancel{n_0}(ik)\, n_{i1} \frac{k_B T_e}{\cancel{n_0}\cancel{e}} - \gamma_i \cancel{k_B T_i}\, n_0 \frac{e\phi_1}{\cancel{k_B T_i}}\, ik$$

$$-i\omega m_i n_0 u_{i1} = -ik\, n_{i1} k_B T_e - \gamma_i n_0 e \phi_1 ik \quad \text{—— (e)}$$

$$n_{i1}(-i\omega) + n_0 u_{i1}(ik) = 0$$

$$\underline{\underline{n_{i1}}}\,\omega = n_0 u_{i1} k$$

$$n_0 \frac{e\phi}{k_B T}\omega = n_0 u_{i1} k$$

$$\underline{u_{i1}} = \left(\frac{e\phi_1}{k_B T_e}\right)\frac{\omega}{k} \quad \text{—— (f)}$$

Using (f) in (e)

Now using U i1 can be used into this equation into E. So, which is U i1 is still a variable in this equation you see here. In order to simplify or get the dispersion relation we have to eliminate these variables. So, U i1 can be substituted from equation F. So, using F in E the same.

We can write minus i omega Mi N0 e phi by kB T times omega by k is minus i k N0 e phi 1 by kB T times kB T minus gamma i N0 e phi by kB T i k1. No, it is not e phi by kB T it is e phi only. So, this N0 gets cancelled, e phi appearing on the left hand side and right hand side gets cancelled. We can now write the equation as minus i omega Mi times omega by k is equals to minus i k kB T minus gamma i i k1 kB T. So, what have we used from the continuity equation and the momentum equation.

$$-i\omega m_i \cancel{n_0}\left(\frac{\cancel{e\phi}}{k_BT_e}\right)\frac{\omega}{k} = -ik\cancel{n_0}\left(\frac{\cancel{e\phi_1}}{\cancel{k_BT}}\right)\cancel{k_BT} - \gamma_i \cancel{n_0}\cancel{e\phi}\ i k_1$$

$$-\cancel{i}\omega m_i \frac{\omega}{k} = -\cancel{i}k\, k_BT - \gamma_i \cancel{i} k_1 (k_BT)$$

$$-\frac{\omega^2}{k^2} = \frac{-k_BT_e - \gamma_i k_BT_i}{m_i}$$

$$\frac{\omega}{k} = \sqrt{\frac{k_BT_e + \gamma_i k_BT_i}{m_i}} = \underline{v_s}$$

Doing the simplification so i will again gets cancelled we have minus omega square by k square is equals to minus kB Te minus gamma i kB T i by Mi or omega square by k square is square root of kB Te plus gamma i kB T i divided by Mi. At omega by k it is supposed to be omega by k not omega by k square omega by k is this. So, this is nothing but the velocity. So, this is the dispersion relation for ion acoustic wave which is resembling the speed of sound in a medium. So, we will discuss more consequences of this dispersion relation in the next class. Thank you.