Plasma Physics and Applications

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Week – 10

Lecture 46: Ion Acoustic Wave - II

Hello dear students. In continuation to our discussion in the last class we have discussed what is an acoustic wave and how we can get omega versus k for an acoustic wave. So the point is we have realized that the type of wave which propagates through the plasma in the absence of any collisions is nothing but an acoustic wave which has a velocity or let us say we call Vp is square root of gamma P0 by rho0. So in this class we will try to understand what is an ion acoustic wave and how we can get the dispersion relation for that. So the topic is ion acoustic wave. So it has been found that the ions can transmit vibration when they are vibrating because of the pressure perturbation they can transmit vibration in the absence of any collisions because of their charge.

So in the earlier picture we have made the collision terms mn U-U0 by tau to be 0 but in the absence of collisions also there is a possibility that the ions can transmit the vibration by the virtue of their charge because they have some charge. So the interaction happens in the presence of an electric field. So electric field is generated because of their charge. So since ions are heavier we can still say that the resulting waves will be very slow frequency oscillations.

Ion- Acoustic wave

$$\frac{\partial n_i}{\partial t} + \vec{\nabla}\cdot(n_i\vec{u}_i) = 0 \quad \text{—} \; (1)$$

$$\rho_i\left[\frac{\partial \vec{u}_i}{\partial t} + (\vec{u}_i\cdot\vec{\nabla})\vec{u}_i\right] = q_i n_i \vec{E}_i - \vec{\nabla}p$$

$$\vec{\nabla}p = \gamma p \frac{\nabla n}{n} \qquad p = nk_BT \Rightarrow \vec{\nabla}p = \gamma k_B T \nabla n$$

$$\rho_i\left[\frac{\partial \vec{u}_i}{\partial t} + (\vec{u}_i\cdot\vec{\nabla})\vec{u}_i\right] = q n_i \vec{E} - \gamma k_B T_i \vec{\nabla} n$$

$$\vec{E} = -\vec{\nabla}\phi$$

$$\rho_i\left[\frac{\partial n_i}{\partial t} + (u_i\cdot\nabla)u_i\right] = -\nabla\phi\, q n_i - \gamma k_B T_i \nabla n \;\longrightarrow (2)$$

So the frequency of oscillations is considered to be small because of the heavier mass of ions. We will still confine ourselves to the no magnetic field situation and we will also assume that the number of electrons will be always equal to the number of ions. So the two equations that we require are we can write so dou ni by dou t plus del dot ni Ui is equal to 0. So let us say this is equation number 1 and for the ions we can write the momentum equation as rho i the density of off ions times dou Ui by dou t plus Ui dot del times Ui is equals to Qi ni Ei minus del p. So we still have this pressure force term on the right hand side but in addition to that we have now introduced the force due to the electric field.

Where is this electric field getting created? This is representing the charge of ions or we have assumed that the information about the vibration of ions is well preserved or well represented by the electric field. We know that del p or is gamma p del n by n or using p is equals to n kBT we can write del p as gamma kBT del n. So depending on ions you can use ion temperature or for electrons you can use the electron temperature. So we can rewrite the equation for the ions as rho i times dou Ui by dou t plus Ui dot del times Ui is equals to Q ni E minus gamma kBT i delta n. We can write the electric field E as the gradient of the potential.

$$\cancelto{0}{m_e n_e}\left[\frac{\partial \vec{u}_e}{\partial t} + \cancelto{0}{(\vec{\nabla}\cdot\vec{u}_e)\,\vec{u}_e}\right] = -e n_e \nabla\varphi - \gamma k_B T_e \nabla n \quad \text{— (3)}$$

$$\Rightarrow e n_e \nabla\varphi - \gamma_e k_B T_e \nabla n = 0$$

$\gamma_e = 1$

Isothermal cond

$$e n_e \frac{\partial \varphi}{\partial x} = k_B T_e \frac{\partial n_e}{\partial x}$$

$$e \frac{\partial \varphi}{\partial x} = \frac{k_B T_e}{n_e} \frac{\partial n_e}{\partial x}$$

Integrating both sides

$$(4) \text{ — } \boxed{e\varphi = k_B T_e \ln n_e + c}$$

We know this very well. So what have I done? I have just written the two equations which are relevant the continuity equation and the momentum equation. I have rewritten delta p so that it represents temperature as well and the variation in the number density of the charged particles del n and Ti. And then I am trying to write the electric field in the units of a potential. So we can simply write it as rho i.

Let us say we call this form of the equation as equation number 2. Both are same but written in terms of parameters that we are going to use it. So we can write a similar expression or similar equation for the electrons which is Me Ne times dou Ue by dou t plus del dot Ue times Ue is equals to minus e Ne del phi minus gamma kBT e del n. For very small or very slow ion vibration the electron mass can be regarded as 0. What does it mean? It means the electron inertia the mass of electron is very small in comparison to the ions.

@ equilibrium $\Longrightarrow$ $n_e = n_0$

& with out any vibration $\Longrightarrow$ $\underline{\phi = 0}$

$$k_B T_e \ln n_0 = -c$$

Substituting value $\underline{c}$ into (4)

$$e\phi = k_B T_e \ln n_e - k_B T_e \ln n_0$$

$$e\phi = k_B T_e \ln\left(\frac{n_e}{n_0}\right)$$

$$\boxed{n_e = n_0 \exp\left(\frac{e\phi}{k_B T_e}\right)}$$

$$\phi \ll k_B T_e$$

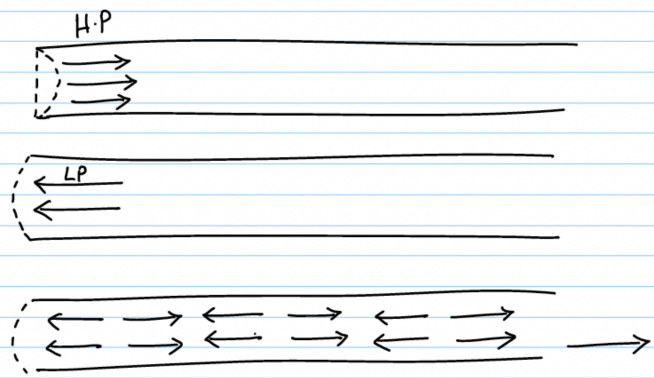
$$n_e \cong n_0\left[1 + \frac{e\phi}{k_B T_e}\right] \quad \text{— (5)}$$

Ions themselves are moving very slow. So we can neglect the electron inertia and we can also neglect the advection term which has a second order velocity in it. So we can make this to be equal to 0 and we can also make this to be equal to 0. So in that case what will be left with is e Ne del phi minus gamma kBT e del n is equals to 0. Now for slow ion waves the electrons move very fast and they equalize the temperature among the entire plasma.

Thereby you have an isothermal picture and if it is an isothermal picture you can take the gamma e or the ratio of specific heats to be gamma e to be equal to 1. When is this valid? This is valid only when you have a perfectly isothermal condition or the environment. The assumption is the electrons because of their small mass they will move around and they will equalize the temperature they will bring the temperature into an equilibrium. And if we consider only one dimensional case we can write e Ne dou phi by dou x is equals to kBT e dou Ne by dou x. And if we integrate both sides we can write e phi integrate with respect to x e phi is kBT ln Ne plus e.

So we are now getting a form of the potential that is responsible for the electric field. Now let us say at equilibrium situation we impose that Ne is equals to N0 and without any vibration and at equilibrium or at the beginning let us say without any vibration. So when things are not set into motion without any vibration we can simply write the phi is equals to 0. So these two conditions can be considered as the initial conditions for the plasma. In that case we can write putting these two things back into the equation we can get the value of C which is equals to kBT ln N0 the undisturbed number density is equals to minus C.

Perturbation in e^- density is equal to Perturbation in ion density.



$$n_e = n_0 + n_1$$

$$n_{1e} = n_{1i}$$

$$n_{1e} = n_e - n_0$$

$$= n_0 + n_0 \frac{e\phi}{k_BT} - n_0$$

$$n_{1e} = n_0 \frac{e\phi}{k_BT} = n_{1i}$$

Substituting this back into this equation so we call this as this is 3 and this is 4. So substituting the value of constant C into equation number 4 what we will get is e phi is equals to kBT ln Ne minus kBT ln N0 or e phi is kBT ln Ne by N0 or we can write Ne is N0 exponential e phi by kBT e. This expression of course looks familiar. So this expression says that the electrons are distributed according to the Maxwell's distribution. Maxwell distribution is imposed on to the electrons if everything that we have considered all the assumptions that we have considered if all of them were to be true then the electrons have to be distributed according to this distribution function which is the Maxwell distribution function.

$$\left.\begin{aligned} n &= n_0 + n_1 \\ u &= u_0 + u_1 \\ \phi &= \phi_0 + \phi_1 \end{aligned}\right\} \text{ (a)}$$

Neutral Plasma in equilibrium

$$u_0 = 0, \ \phi = 0, \ \nabla n_0 = 0 \longleftarrow$$

$$\frac{\partial n_0}{\partial t} = \frac{\partial u_0}{\partial t} = \frac{\partial \phi_0}{\partial t} = 0$$

Substituting (a) in ion $\left\langle \begin{array}{l} \text{Momentum} \\ \text{Continuity} \end{array} \right\}$ equations

$$\frac{\partial n_{1i}}{\partial t} + \nabla \cdot (n_0 + n_{1i})(u_0 + u_{1i}) = 0$$

$$\frac{\partial n_{1i}}{\partial t} + \nabla \cdot \left[\cancelto{0}{(n_0 u_0)} (n_0 u_{1i}) \cancelto{0}{(n_{1i} u_0)} \underbrace{\cancelto{0}{(n_{1i} u_{1i})}}_{=0} \right]$$

Now we will make one more assumption saying that the perturbations to be very small relative to the thermal energy just to keep things simple. So perturbations or the potential that comes out because of the perturbations is very small in comparison to the thermal energy that is there inside the plasma. So in that case so this numerator e phi becomes very small in comparison to kBT e. So we can write the exponential or expand the exponential just so that only the first two terms matter to us. So Ne is N0 will be approximately equal to Ne is equal to N0 into 1 plus e phi by kBT e.

What does this mean? Just look back into what we have done. We have taken the ion momentum equation and we have made the mass of the ion negligibly small in comparison to the mass of ion. So we can neglect this term and it will be sufficient to say this but still the second order term velocity term is also neglected and so we have all the right hand side becoming equal to 0 which gives an idea that how we can evaluate del phi out of this. So this equation if it has to be true we will say that the electrons are moving randomly or with around the plasma and they are bringing the temperature to an equilibrium value. So that means that we have an isothermal condition constant temperature condition.

Isothermal condition imposes gamma e to be 1 using that we can get a form in which the potential can be written. So we have this constant C and in order to evaluate the value of that constant we consider equilibrium at equilibrium ne the number of electrons is equal to the equilibrium value of n naught and in the absence of any vibrations because the vibrations are the ones in which the information about the electric field constituted by ions is transmitted. So by the virtue of its charge because there is charge so this information is transmitted. So if without any vibration we will say that this electric field which is constituted out of potential will automatically become 0 so phi is taken to be 0 and when you take phi to be 0 you write this k B T e ln n naught is equal to C and substituting the value of C back into this equation number 4 we get this. So this will give you an empirical expression for phi which is k B T e ln n e by n naught.

$$\frac{\partial n_{1i}}{\partial t} + n_0 \nabla \cdot u_{1i} = 0 \quad \text{——} \quad (6)$$

So thereby ne can be written to be n naught times exponential e phi by k B T which says that the distribution that the electrons are following is Maxwellian in nature and to simplify things further we say that the potential that is there is very small in comparison to the thermal energy. That means if the potential is being used for to accelerate the electrons across this potential they will not gain much energy in comparison to the thermal energy that they possess already. So in that case the exponential can be truncated to the first two values which is n naught into 1 plus e phi by k B T e. Now let us say the perturbation electron density is also equal to the perturbation in ion density where is the perturbation actually coming into picture? The perturbation is coming into picture because we want a certain type of wave activity to happen. The ions are vibrating around the mean positions just so that they can transmit a pressure pattern through them.

Then let us say if we consider a simple case how the pressure is being transported, how the pressure fluctuation is being transported. So we can consider a fluid inside a tube and we fix a diaphragm at the end. What is the diaphragm? This diaphragm is able to oscillate back and forth. Now let us say if the diaphragm is pushed inside what happens to the fluid? The fluid is pushed in like this. Now if the diaphragm after a period of time or immediately when it comes back the diaphragm is extending out like this.

So now the fluid is being pulled here outside or outward. So what is happening in this picture? So as the time progresses, as the diaphragm is vibrating back and forth we can see that let us say after certain point of time the diaphragm is extending out then we can see that the molecular vibrations effectively transport a pattern of high and low pressures inside the fluid. So this is obviously a high pressure and this is obviously when they are extending outside you have a low pressure. So as the time progresses what are the molecules? The molecules are just vibrating around the main position they are not being transported directly into the column of this tube because you are not supplying any gas inside. So you are just vibrating it vibrating the diaphragm.

So while it vibrates alternate patterns of high pressure and low pressure are being transported into the fluid column like this. So this pattern is simply a pressure fluctuation which is being transported into the fluid. So this is what I was talking about. Now we will say that the perturbation because you consider the main position the electrons the electron density or the ion density will vary for a moment due to this vibrations. So this will make an assumption that the perturbed ion density and the perturbed electron density are equal in number.

So they will say that n e is n naught plus n 1. This is how you we write it. Now for the n 1 electron is n e minus n naught. But we know that n naught n e is n naught plus n naught e phi by k B T we have a minus n naught. What am I using? I am using this you see this is the n e I am using this n e into this equation as this minus n naught as it is.

So we can write n 1 e the perturbed electron density as simply n naught e phi by k B T. Now interestingly we see that the perturbed electron density is a function of this potential phi. Let us say we say that n 1 e because we said perturbation in electron density is equal to the perturbation in ion density. So we make it equal to n 1 i. This is because of this simple assumption perturbation in electron density is equal to perturbation in ion density.

So as a result what do we have? We have a very important equality which is n 1 e is equal to n 1 i. Let us see how we use it. Now we have established the basic equations that we require and also the variables which we think will be perturbed due to the wave activity. So we write the perturbed variables as n as n naught plus n 1 without using the suffixes for electron ion and u is u naught plus u 1 and phi is equal to phi naught plus phi 1. Now we will consider neutral plasma which is in equilibrium which imposes the fact that u is equal to 0 to begin with and phi is equal to 0 to begin with the initial values basically and delta n naught is 0.

And since we are using the linear perturbation theory we can simply write dou n naught by dou t is equal to dou u naught by dou t is equal to dou phi by dou t is equal to 0. All the initial values are constants in time so we do not have to worry about them when we linearize the equations. Now substituting these perturbations into the ion momentum equation and the ion continuity equation. So substituting let us say we call this as equation set A. Substituting equation set A in ion momentum and continuity equations.

So I am writing all these things because it will develop a sense of continuity and it will be easy for you to follow the algebra that I am doing. So we will write dou n 1 i divided by dou t plus del dot n 0 plus n 1 i times u 0 plus u 1 i is equal to 0. This is the continuity equation for ions. If we expand it we will write dou n 1 i, I have already made one term 0 before I write this and del dot n naught u naught, n naught u 1 i, n i 1, u naught, n 1 i or u 1 i.

n 1 i or n i 1 both of them are same. So this term n naught u naught which is a product of equilibrium terms can be made to be 0. And this one is a second order term in perturbation so it has to be 0 and n 1 i perturbation term multiplied by an equilibrium velocity and we have considered the initial velocity to be 0 so we have to make it 0. So you see this condition is being used here. The initial velocity u naught is 0 and things are not set into motion. So using this in combination we can now write we have only one term out of the four terms that are appearing in the right hand side of the continuity equation or the second term of the continuity equation n naught u i 1.

We will write dou n 1 i by dou t plus n naught times del dot u 1 i is equals to 0. So what is the equation number here? We call this as 5 and this is equation number 6. So we will continue this class by applying the perturbation into the ion momentum equation in the next class. Thank you very much.