Plasma Physics and Applications

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Week – 09

Lecture 45: Ion Acoustic Wave

Hello dear students. We will continue our discussion of electron plasma wave in which we have derived expression for phase velocity and group velocity. So, phase velocity is given as V p which is angular frequency by k and V g the group velocity is d omega by dk. So, both these expressions have been derived already and from this one very important conclusion that we have made in the last class was that group velocity being non-zero suggests that energy or the information is being transported from one point to another point by the means of these oscillations transported by the means of this oscillations of electrons. Let us try to understand few consequences of this group velocity. So, this is the dispersion relation which we call as a relation between omega and k.

Now I mentioned many number of times that dispersion relation conveys all the information about the wave. Now if you try to plot this wave, so it will look something like this. If you take k along the x axis and omega along the y axis, if you plot this the wave will look something like this. There will be a negative frequency term also which we have not considered and so this is how this dispersion relation will look or the dependence of omega with respect to k will look like.

Now you see this intercept on the y axis is at height, this height can be omega p which is the plasma frequency. Omega p is square root of n e square by m epsilon naught. Now let us say we draw a line from a point, this is point p from a point to the origin. So, the slope of this line will be equal to vp which is omega by k. If you draw a tangent from any point onto the curve, the slope of this tangent would be given by vg or d omega by dk.

$$\boxed{v_p = \frac{\omega}{k} = \sqrt{\frac{3}{2}}\, v_{th} \frac{\omega}{\sqrt{\omega^2 - \omega_{pe}^2}}}$$

$$v_p = \frac{\sqrt{\omega_p^2 + \frac{3}{2} k^2 v_{th}^2}}{k}$$

Phase velocity

1) Information transported

$$v_g = \frac{d\omega}{dk} = \frac{3}{2} \frac{k}{\omega} v_{th}^2$$

$$\boxed{v_g = \frac{3}{2} \frac{v_{th}^2}{v_p}}$$

Group velocity

So, here slope of a line from the point from p to origin gives you the phase velocity and slope of a tangent to any point on the x axis. Gives you the value of vg, the group velocity. Now you see if this is the origin, then vg the group velocity at the origin can be seen that it is 0. So, I am writing the inferences vg at the origin is 0 and similarly vp at the origin. So, if you imagine the point here, the slope would be infinite.

So, you are just looking at a vertical line along the y axis. The phase velocity at the origin would be 0 and most importantly the minimum angular frequency that is required for this plasma oscillations to sustain to form a wave is of course is omega p. That means any wave which is above the threshold of omega p can only sustain and most importantly there is no wave activity for angular frequencies less than omega p. We know what is omega p. So, let us just briefly look at this.

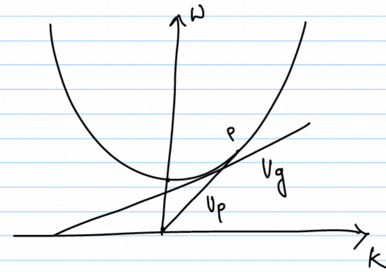
We have derived the dispersion relation for electron plasma wave. The electron plasma wave is a simple idea in which electrons are oscillating to restore the charge equilibrium and these oscillations constitute a small wave. This wave can be transported from one point to another point only if the electrons have some thermal motion. So, what we did in the beginning, we considered only the electric field term to be relevant on the right hand side. We derived a dispersion relation in which we realized the frequency of those oscillations that is electron oscillations above its position mean position will not constitute a wave.

$$\boxed{\omega^2 = \omega_p^2 + \frac{3}{2} k^2 v_{th}^2}$$

Dispersion relation $\boxed{T_e \neq 0}$

∇p

$$v_g = \frac{\omega_2 - \omega_1}{k_2 - k_1} = \frac{d\omega}{dk}$$



$$\omega_p = \sqrt{\frac{ne^2}{m\epsilon_0}}$$

1) Slope of line from $P \rightarrow (0,0) \Rightarrow v_p$

2) Slope of a tangent to any point on the x-axis $\Rightarrow v_g$.

3) v_g at $(0,0) = 0$

4) v_p at the origin $= \infty$

5) Minimum angular frequency $= \omega_p$

6) There is no wave activity $< \omega_p$

That means will not actually constitute transport of energy or information from one point to another point. But the characteristic frequency of these oscillations is the plasma frequency itself and we have also realized that there is no limit to this frequency. So, it can be any value. But when we wanted to impose the fact that electron in addition to its inertia which is by the virtue of its mass, if it has some temperature, electron temperature, so we made the electron temperature Te to be nonzero. So, by doing this we have accommodated the provision of having the delta p term to be existing on the right hand side.

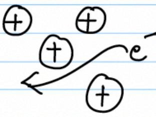
Earlier when we considered simple plasma oscillations, we have removed the pressure gradient term and also we have removed that we are considering a collision less plasma wherein the plasma particles are not colliding. And that means there is no momentum transfer that is happening. But in the modified picture, we considered the electron temperature to be nonzero. We got a pressure gradient force term in addition to the electric field term on the right hand side. Then we took those governing equations, we applied the perturbation theory and we derived an expression for omega which seemed to have the propagation constant on the right hand side.

Ions $\Longrightarrow$ Ion Acoustic Wave

$\longleftrightarrow$ Oscillation

$\longrightarrow$ ω

$$mn\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right] = \underbrace{\cancelto{0}{qn(\vec{E}+\vec{u}\times\vec{B})}}_{\text{Electromagnetic}} - \underbrace{\vec{\nabla}p}_{\text{Pressure term}} - \underbrace{\cancelto{0}{\frac{mn(\vec{u}-u_0)}{\tau}}}_{\text{Collision term.}}$$



— Collisionless

$$\rho\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right] = -\vec{\nabla}p \quad \text{——} \quad (1)$$

$$P = C\rho^{\gamma} \qquad \frac{\nabla P}{P} = \gamma\frac{\nabla n}{n} = \gamma\frac{\nabla \rho}{\rho}$$

$$\nabla p = \gamma P\frac{\nabla \rho}{\rho}$$

That means we have a dispersion relation which has a provision for a nonzero group velocity. What is the significance of nonzero group velocity you ask? It conveys that there is information transport from one point to another point. How is this information transport now being facilitated? Because electron by virtue of its temperature, kinetic temperature moves from one layer of plasma into the adjacent layer of plasma. When it is moving, it is also telling the story of what is going on in that particular layer of plasma when it moves to the next layer. This does constitute a wave.

This is the dispersion relation that we have obtained. And if you look at this dispersion relation and you plot it, you get some inferences. One inference is you draw a tangent from a point P onto the origin, the slope of that line you see V P, the slope of line is y2 minus y1 by x2 minus x1 will be equal to the phase velocity. Because you have origin, you have y2 minus 0 by x2 minus 0 which is omega 2 by k2 which is omega by k which is nothing but just the frequency. But if you draw a line onto any other value which is not 0, 0, then you will get y2 is omega 2, so y2 is omega 2 minus omega 1 divided by k2 minus k1 which is equivalent to writing d omega by dk.

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho u) = 0 \quad — ②$$

$$\frac{\partial u_0}{\partial t} = 0$$

$$\left.\begin{aligned} p &= p_0 + p_1 \\ \rho &= \rho_0 + \rho_1 \\ u &= u_0 + u_1 \end{aligned}\right\} ⓐ$$

Substitute ⓐ into ① & ②

$$(\rho_0+\rho_1)\left[\frac{\partial}{\partial t}(u_0+u_1) + \cancelto{0}{(u\cdot\nabla)u}\right] = -\gamma\,(\rho_0+\rho_1)\,\frac{\nabla(\rho_0+\rho_1)}{(\rho_0+\rho_1)}$$

$$\frac{\partial}{\partial t}(\rho_0+\rho_1) + \nabla\cdot(\rho_0+\rho_1)(u_0+u_1) = 0$$

$$\boxed{(\rho_0+\rho_1)\,\frac{\partial u_1}{\partial t} = -\gamma\,\frac{(\rho_0+\rho_1)}{(\rho_0+\rho_1)}\,\nabla\rho_1} \quad — ③$$

Thus, we call this particular slope of the line joining these two points or a tangent at any value of k equal to the group velocity. Now looking at this, we can simply say that if this point that you have taken P is at the centre of this hyperbola, you will get that the phase velocity at that point will be infinity. Because you are just forming a vertical axis whose slope is infinity and if you evaluate the group velocity at the origin, it would be 0. Because we know omega will be independent of k then and the hyperbola seems to be shifted, the positive part of this will be seems to be shifted by an intercept of omega P which is the basic frequency or the minimum frequency above which oscillations or wave activity can exist. So, we call this as the cut-off frequency and we also conclude that for any frequency which is less than omega P, we cannot expect any wave activity.

So, these are the major conclusions of the dispersion relation for the electron plasma wave. Now we will go ahead and try to see what is the type of wave activity in which the ions can participate. So far in our picture, we have considered the ions to be constituting a uniform or a constant background in which they are not moving because of their higher mass but rather it is the electrons which are of very small mass or inertia moving and thus creating the wave activity. So, we will now try to see what kind of waves can these ions be supportive of. In order to do that before we go and discuss what is called as the ion acoustic wave, we will study what are the properties of an acoustic wave.

$$\frac{\partial}{\partial t}\rho_1 + \nabla\cdot(\rho_0 u_1) + \nabla\cdot(u_0\rho_1) = 0 \quad \text{—— (4)}$$

$$\left.\begin{aligned} \rho_1 &= \rho_1 e^{i(k\cdot r - \omega t)} \\ u_1 &= u_1 e^{i(k\cdot r - \omega t)} \end{aligned}\right\} \quad (b)$$

$$k = k\hat{x} \; ; \; u = u\hat{x}$$

Substituting (b) into (3) & (4)

$$-i\omega\rho_0 u_1 = \frac{-\gamma P_0}{\rho_0} ik\rho_1 \quad \text{—— (c)}$$

$$-i\omega\rho_1 + \rho_0 ik\cdot u_1 = 0 \quad \text{—— (d)}$$

from (d) $\quad \rho_1 = \frac{\rho_0 \cancel{i} k u_1}{\cancel{i}\omega} = \frac{\rho_0 k u_1}{\omega}$

Using in (c)

What is an acoustic wave? Acoustic wave is a longitudinal wave which is transported by the means of alternating patterns of high pressure and low pressure. So, it is the oscillation of pressure which is transported longitudinally. So, the oscillation is in the same direction as the wave activity. So, if this is a wave or the energy transport direction and this is the oscillation that is why we call it as a longitudinal wave and let us now try to see how we can get a relation for an acoustic wave and how we can extrapolate this discussion on to something called as the ion acoustic wave. Now, let us as an introduction to these ion waves, we will briefly review the theory of acoustic waves.

Now, let us start with the basic governing equation that we are familiar with which is m n dou u by dou t plus u dot del times u is equals to q n times e plus u cross B minus del P minus m n u minus u naught by tau. Just to recall what are the terms that are there on the right hand side, you have this one which is the electromagnetic term which constitutes the effects of electric and magnetic fields on the plasma and this one is referred to as the pressure term and this one is the collision term. So, when we talk about acoustic waves, it is obvious that there is no role of electric field or the magnetic field on the acoustic waves. We are just talking about some pressure perturbations travelling in the plasma. This term is not relevant.

This term is we take it as 0, we do not consider it and we also say that the plasma that we considered is collisionless which means that there are no collisions in the plasma. We also take this term to be equal to 0. Now, let us see the consequences of having a momentum equation which has only the pressure variations on the right hand side. Let us say m is the mass of the particle and n is the number of particles per unit volume. So, which becomes m n becomes simply the density which is mass by volume, m rho times dou u by dou t plus u dot del dot times u is equals to minus del P.

$$\cancel{-}i\omega\rho_0 u_1 = \cancel{-}\frac{\gamma P_0}{\cancel{\rho_0}}\, ik\, \frac{\cancel{\rho_0} k u_1}{\omega}$$

$$c_s = \sqrt{\frac{\gamma P_0}{\rho_0}}$$

$$\cancel{i}\omega\rho_0 = \frac{\gamma P_0 \cancel{i} k^2}{\omega}$$

$$\omega\rho_0 = \frac{\gamma P_0 k^2}{\omega}$$

$$\omega^2 = \frac{\gamma P_0 k^2}{\rho_0}$$

$$\frac{\omega^2}{k^2} = \frac{\gamma P_0}{\rho_0}$$

$$\boxed{\frac{\omega}{k} = v_p = \sqrt{\frac{\gamma P_0}{\rho_0}}}$$

So, from the equation of state we can write del P by P is gamma. What is the equation of state? Equation of state is P is equals to C times rho power gamma. So, we can write gamma times del n by n or we can write it as gamma del rho by rho. So, this is one equation that we need and the second equation that we may need is the equation of continuity which is dou rho by dou t plus del dot rho u is equals to 0. Let us say we call this as equation 1 and this as equation 2.

So, earlier the third equation that we have used is the Poisson equation which is about the electric field. So, since we have not considered the electromagnetic effects in this picture, we can simply make that particular term to be 0 or we do not require that equation to begin with. Now, when we consider this pressure fluctuations to be moving in the plasma, the relevant parameters that will be perturbed when an acoustic wave propagates are the density and the pressure. So, pressure P is equals to P naught plus P 1, density rho is equals to rho naught plus rho 1 and velocity of course, u is equals to u naught plus u 1. Let us say we call this set of equations as A.

Now, what is the procedure? The procedure is standard. Substitute A into 1 and 2. So, we can write rho naught plus rho 1 times dou by dou t of u naught plus u 1. So, I am doing the perturbation of equation number 1, u dot del dot u plus 2 minus gamma times P naught plus P 1 del plus rho 1 divided by rho naught plus rho 1. So, if we go back just look at this equation and so, this pressure which appears in the denominator goes and multiplies this on the right hand side that is why we have del P is gamma times pressure times del rho by rho.

So, therefore, we have the substituted all that and the continuity equation tells you dou by dou t of rho naught plus rho 1 plus del dot rho naught plus rho 1 times u naught plus u 1 is equal to 0. This is the continuity equation. So, we can linearize this using the simple assumption that any term which has second order perturbation variables can be neglected and any variation of the equilibrium parts with respect to time can also be neglected. So, using both of them together we can write that and for convenience we will also make this advection term which is basically a second order term anyway to be 0. So, linearizing it we will get rho naught plus rho 1 times dou u 1 by dou t because dou u naught by dou t just for the sake of quick recall dou u naught by dou t is 0 because u naught is an equilibrium part which is assumed to be a constant.

So, a derivative with respect to time would be 0 is equal to minus gamma P naught plus P 1 divided by rho naught plus rho 1 times del rho 1. So, this is the linearized momentum equation in the units of the perturbation variables. So, similarly we can also write the linearized moment the continuity equation dou by dou t of rho 1 plus del dot rho naught u 1 plus del dot u naught rho 1 is equal to 0. Go back to this equation, you expand this product rho 1 u 1 which is a second order term in perturb variables can be neglected and you have del dot the other three terms out of this rho naught plus rho 1 multiplied by u naught plus u 1. So, out of the other three terms these two terms will sustain.

So, which we can further simplify or we can use the perturbation variables to be science orally varying with respect to time. So, we can take rho 1 is equals to rho 1 times exponential i k dot r minus omega t. So, rho 1 is the mean value and it is varying with respect to rho 1. So, it is science orally varying at a mean value of rho 1 that can be meaning of this and u 1 is u 1 times e to the power of i k dot r minus omega t. Now if you look at these two equations, this one, let us say this we call this as 3 and 4.

And we consider that the wave or the oscillation is along the x direction. So, k can be taken to be k x cap and velocity u can be taken to be u x cap. So, when we substitute this, let us say we call these set of equations as B, we substitute B into 3 and 4. Substituting B into equation 3 and equation 4, we will get this equations as minus i omega rho naught v 1 is equals to minus gamma p naught by rho naught i k rho 1. So, we call this as C, the continuity equation will be i omega rho 1 plus rho naught i k dot v 1 is equals to 0.

So, we call this as equation number D. So, after substituting the sinusoidal solutions, we have these two equations. Now, we have to eliminate some variables and we can get the dispersion relation. So, from equation number D, we can write that rho 1, the perturbed density rho 1 from equation number D. Rho 1 is rho naught i k u 1 by i omega, which is equals to rho naught k u 1 by omega.

We can use this for rho 1 in equation C. What we can write is minus i omega rho naught u 1 is equals to minus gamma v naught by rho naught times i k rho naught k u 1 by omega. Now, we have eliminated rho 1 from the equation. So, cancelling all the terms, rho naught can be cancelled. And this can be cancelled. We can write i omega rho naught is equals to gamma p naught i k square u 1 divided by omega or omega rho naught is equals to gamma p naught k square by omega.

So, omega square is equals to gamma p naught k square by rho naught. Omega square by k square is gamma p naught by rho naught or omega by k is which is called as v p is square root of gamma p naught by rho naught. So, this velocity which is the velocity of wave which is being transported in the plasma. So, it is resembling something that we know very well. So, we know that speed of sound in a medium is written as gamma times p by rho naught which is pressure by density.

So, what we have realized is if you consider the plasma and if you want to know what type of wave can be transported through the plasma and if you remove the electromagnetic forces on the right hand side and the collisions also, then what you get is a wave which is nothing but an acoustic wave. So, we can conclude that an acoustic wave can be propagated through the plasma. So, these waves are the disturbances in pressure propagating from one point to another point. And in plasma as well with no neutrals and no collisions in particular, a similar type of wave propagation occurs which is called as an ion acoustic wave. So, with this background we will try to see how an ion acoustic wave can be understood.

We will take it up in the next class. Thank you.